Alberta High School Mathematics Competition 2011/12

The Alberta High School Mathematics Competition is a two-part competition taking place in November and February of each school year. Book prizes are awarded for Part I, and cash prizes and scholarships for Part II. Presented here are the problems and solutions from the 2011/12 competition.

Part I

November 15, 2011

1. If $2^{2012} + 4^{1006} = 2^n$, then n is
(a) 2013 (b) 2014 (c) 3018 (d) 4024
(e) not an integer

2. Mini-marshmallows are cubes of 1 cm on each side, while giant marshmallows are cubes of 3 cm on each side. The number of mini-marshmallows whose combined surface area is the same as the surface area of one giant marshmallow is
(a) 3 (b) 6 (c) 9 (d) 27 (e) 54

3. The number of customers in a restaurant on Tuesday is 20% more than the number on Monday, the number of customers on Wednesday is 50% more than the number on Monday, and the number of customers on Wednesday is n% more than the number on Tuesday. The value of n is
(a) 20 (b) 25 (c) 30 (d) 50 (e) none of these

4. Sawa starts from point S and walks 1 km north, 2 km east, 3 km south and 4 km west. At this point, her distance, in kilometres, from S is
(a) $\sqrt{5}$ (b) $2\sqrt{2}$ (c) 4 (d) 8 (e) 10

5. A millennium number is a positive integer such that the product of its digits is 1000. The number of six-digit millennium numbers is
(a) 60 (b) 120 (c) 140 (d) 180 (e) 200

6. Let $x_1, x_2, \ldots$ be a sequence of positive rational numbers such that $x_1 = 16$, $x_2 = 32$ and
$$x_n = \frac{x_{n-1} + x_{n-2}}{2}$$
for all positive integers $n \geq 3$. Then the value of x_6 is
(a) 24 (b) 25 (c) 26 (d) 27 (e) 28

7. The number of real solutions of the equation
$2x^2 - 2x = 2x\sqrt{x^2 - 2x} + 1$ is
(a) 0 (b) 1 (c) 2 (d) 3 (e) 4

8. Let $f(x)$ be a quadratic polynomial. If $f(1) = 2$, $f(2) = 4$ and $f(3) = 8$, then the value of $f(4)$ is
(a) 12 (b) 14 (c) 15 (d) 16 (e) 18

9. A lucky number is a positive integer n such that 7 is the largest divisor of n that is less than n. The number of lucky numbers is
(a) 1 (b) 2 (c) 3 (d) 4 (e) more than 4

10. The perimeter of a square lawn consists of four straight paths. Annabel and Bethany started at the same corner at the same time, running clockwise at constant speeds of 12 and 10 km/h, respectively. Annabel finished one lap around the lawn in one minute. During this minute, the number of seconds that Annabel and Bethany were on the same path was
(a) 36 (b) 42 (c) 48 (d) 50 (e) none of these

11. ABCD is a quadrilateral with AD = BC and AB parallel to DC. It is only given that the lengths of AB and DC are 20 and 15 cm, respectively. Adrian puts n copies of this tile together so that the edge BC of each copy coincides with the edge AD of the next, and the edges DC of all copies together form a regular n-sided polygon. The value of n is
(a) 6 (b) 8 (c) 12 (d) 20
(e) not uniquely determined

12. The sum of 20 positive integers, not necessarily different, is 462. The largest possible value of greatest common divisor of these numbers is
(a) 21 (b) 22 (c) 23 (d) 33 (e) 42

13. The largest real number m such that $(x^2 + y^2)^3 > m(x^3 + y^3)^2$ for any positive real numbers x and y is
(a) ⅓ (b) ½ (c) 1 (d) 2 (e) none of these

14. Of the 49 squares of a 7 × 7 square sheet of paper, two are to be coloured black while the others remain white. Two colourings are called *distinct* if one cannot be obtained from the other by rotating the sheet of paper about its centre. The number of distinct colourings is
(a) 288 (b) 294 (c) 296 (d) 300 (e) 588

15. The lengths of the sides of triangle ABC are consecutive positive integers. D is the midpoint of BC, and AD is perpendicular to the bisector of ∠C. The product of the lengths of the three sides is

(a) 24 (b) 60 (c) 120 (d) 210 (e) 336

16. For any real number r, $\lfloor r \rfloor$ is the largest integer less than or equal to r. For example, $\lfloor \pi \rfloor = 3$. Let n be a positive integer. Let $a_1 = n$, $a_2 = \left\lfloor \frac{a_1}{3} \right\rfloor$, $a_3 = \left\lfloor \frac{a_2}{3} \right\rfloor$ and $a_4 = \left\lfloor \frac{a_3}{3} \right\rfloor$. The number of positive integers n from 1 to 1000 inclusive such that none of a_1, a_2, a_3 and a_4 is divisible by 3 is

(a) 144 (b) 192 (c) 210 (d) 280
(e) none of these

Solutions

1. We have $2^{2012} + 4^{1006} = 2^{2012} + 2^{2012} = 2^{2013}$. The answer is **(a)**.
2. The surface area of a mini-marshmallow is 6 sq cm while that of a giant marshmallow is 54 sq cm. Thus, the desired number of mini-marshmallows is $54 \div 6 = 9$. The answer is **(c)**.
3. Suppose there are m customers on Monday. Then there are $1.2m$ on Tuesday and $1.5m$ on Wednesday. The increase of $0.3m$ from Tuesday to Wednesday is 25% of $1.2m$. The answer is **(b)**.
4. Sawa is 2 km south and 2 km west of S. Her distance from S, by the Pythagorean theorem, is $\sqrt{2^2 + 2^2} = \sqrt{8} = 2\sqrt{2}$ km from S. The answer is **(b)**.
5. Since $1000 = 2 \times 2 \times 2 \times 5 \times 5 \times 5$, the digits can only be 1, 2, 4, 5 and 8. Three of them must be 5s, and they can be placed among the six digits in $\binom{6}{3} = 20$ ways. The product of the other three digits is 8, and they are (1, 1, 8), (1, 2, 4) or (2, 2, 2). They can be placed in three, six and one ways, respectively. Hence, the total number of six-digit millennium numbers is $20(3 + 6 + 1) = 200$. The answer is **(e)**.
6. We have $x_3 = 24$, $x_4 = 28$, $x_5 = 26$ and $x_6 = 27$. The answer is **(d)**.
7. Squaring both sides of $2x^2 - 2x - 1 = 2x\sqrt{x^2 - 2x}$, we have $4x^4 - 8x^3 + 4x + 1 = 4x^4 - 8x^3$, which simplifies to $4x + 1 = 0$. Hence, the only solution is $x = -\frac{1}{4}$. Indeed,

$$2\left(-\frac{1}{4}\right)^2 - 2\left(-\frac{1}{4}\right) = \frac{5}{8}$$

and

$$2\left(-\frac{1}{4}\right)\sqrt{\left(-\frac{1}{4}\right)^2 - 2\left(-\frac{1}{4}\right)} + 1 = \frac{5}{8}.$$

The answer is **(b)**.

8. Note that $g(x) = f(x + 1) - f(x)$ is a linear polynomial. Since $g(1) = f(2) - f(1) = 2$ and $g(2) = f(3) - f(2) = 4$, we have $g(3) = 6$. Hence, $f(4) = f(3) + g(3) = 8 + 6 = 14$. The answer is **(b)**.
9. Since a lucky number n is divisible by 7, it has the form $n = 7k$ for some positive integer k. If k is not a prime number, then it has a divisor h where $1 < h < k$, and $7h$ is a divisor of n larger than 7 but not equal to n. Hence, k must be a prime number. Moreover, it cannot be greater than 7. Hence, there are only four lucky numbers—namely, 14, 21, 35 and 49. The answer is **(d)**.
10. Annabel spent 15 seconds on each path, and Bethany 18 seconds. On the first path, Bethany was with Annabel all 15 seconds. On the second path, Bethany joined Annabel 3 seconds late, and was with her for 12 seconds. On the third path, Bethany was with Annabel for 9 seconds. On the fourth path, Bethany was with Annabel for 6 seconds. The total is $15 + 12 + 9 + 6 = 42$ seconds. The answer is **(b)**.
11. We can draw a regular polygon of any number of sides such that the side length is 20 cm. We can then draw a regular polygon of the same number of sides but with side length 15 cm, placed centrally inside the larger polygon. Then, a tile can be chosen that can pave the ring-shaped region inside the larger polygon but outside the smaller one. Hence, the answer is **(e)**.
12. Let $x_1, \ldots, x_{20}$ be the given numbers. If d is the greatest common divisor of these numbers, then

$$x_1 + \cdots + x_{20} = d\left(\frac{x_1}{d} + \cdots + \frac{x_{20}}{d}\right) = 462 = 21 \cdot 22.$$

The value $d = 22$ is obtained if $x_1 = \cdots = x_{19} = d$ and $x_{20} = 2d$. For each i, $x_i/d \geq 1$. Hence, $d \leq 462/20 = 23.1$. Since d divides 462, the largest value for d is indeed 22. The answer is **(b)**.
13. Dividing throughout by y, we have $(z^2 + 1)^3 > m(z^3 + 1)^2$, where $z = x/y$. This is equivalent to

$$(1 - m)z^6 + 3z^4 + (3 - 2m)z^3 + 1 - m > 0$$

for any positive real z. Hence, it is necessary to have $1 - m \geq 0$ (ie, $m \leq 1$). If we take $m = 1$, the inequality $(x^2 + y^2)^3 \geq (x^3 + y^3)^2$ is equivalent to

$$x^2y^2((x - y)^2 + 2x^2 + 2y^2) > 0,$$

which is clearly true. The answer is **(c)**.
14. There are $\binom{49}{2} = 1176$ colourings. The number of symmetrical colourings with respect to the middle square is

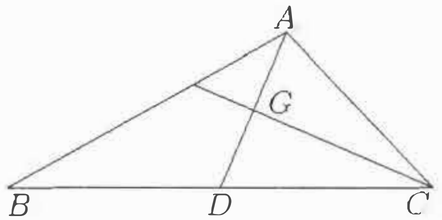
$$\frac{49 - 1}{2} = 24.$$

These colourings are counted twice. All the other colourings are counted four times. The desired number is

$$\frac{24}{2} + \frac{1176 - 24}{4} = 300.$$

The answer is **(d)**.

15. Let AD intersect the bisector of ∠C at G. Then ∠CGA = 90° = ∠CGD, ∠GCA = ∠GCD and GD = GD. Hence, triangles GCA and GCD are congruent, so that AC = DC. It follows that we have BC = 2DC = 2AC. Now among three consecutive positive integers, one is double another. This is only possible if the integers are 1, 2 and 3, or 2, 3 and 4. The former does not yield a triangle. Hence, AC = 2, AB = 3 and BC = 4, so that AB · BC · CA = 24. The answer is **(a)**.

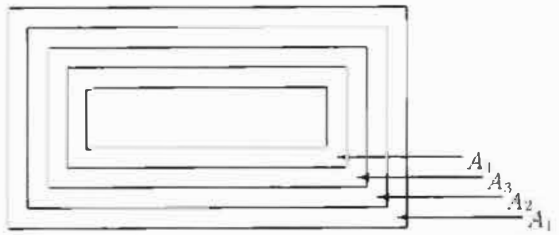


16. Note that if we write a positive integer m in base 3, then the base 3 representation of $\left\lfloor \frac{m}{3} \right\rfloor$ is simply the base 3 representation of m with the rightmost digit removed. Also, a positive integer m is divisible by 3 if and only if the rightmost digit of m is 0. Hence, in order that none of a_1, a_2, a_3 and a_4 is divisible by 3, the rightmost four digits of the base 3 representation of n are all non-0. Note that $1000 > 2(3^5 + 3^4 + 3^3 + 3^2 + 3 + 1)$. If n has at most six digits in its base 3 representation, the first two can be any of 0, 1 and 2, while the last four cannot be 0. There are $3^2 \times 2^4 = 144$ such numbers. Clearly, n cannot have more than seven digits as otherwise $n \geq 3^7 > 1000$. Suppose n has exactly seven digits. As before, the last four cannot be 0. Since $1000 < 3^6 + 3^5 + 3^3 + 3^2 + 3 + 1$, the first one must be 1; the second must be 0; and the third can be any of 0, 1 and 2. Hence, there are $3 \times 2^4 = 48$ such numbers. The total is 192. The answer is **(b)**.

Part II

February 1, 2012

1. A rectangular lawn is uniformly covered by grass of constant height. Andy's mower cuts a strip of grass 1 m wide. He mows the lawn using the following pattern. First he mows the grass in the rectangular "ring" A_1 of width 1 m running around the edge of the lawn. Then he mows the 1 m wide ring A_2 inside the first ring, then the 1 m wide ring A_3 inside A_2, and so on until the entire lawn is mowed. Andy starts with an empty grass bag. After he mows the first three rings, the grass bag on his mower is exactly full, so he empties it. After he mows the next four rings, the grass bag is exactly full again. Find, in metres, all possible values of the perimeter of the lawn.



2. In the quadrilateral ABCD, AB is parallel to DC. Prove that

$$\frac{PA}{PB} = \left(\frac{PD}{PC}\right)^2,$$

where P is a point on the side AB such that ∠DAB = ∠DPC = ∠CBA.

3. A positive integer is said to be *special* if it can be written as the sum of the square of an integer and a prime number. For example, 101 is special because 101 = 64 + 37. Here, 64 is the square of 8, and 37 is a prime number.
   (a) Show that there are infinitely many positive integers that are special.
   (b) Show that there are infinitely many positive integers that are not special.

4. In triangle ABC, AB = 2, BC = 4 and CA = $2\sqrt{2}$. P is a point on the bisector of ∠B such that AP is perpendicular to this bisector, and Q is a point on the bisector of ∠C such that AQ is perpendicular to this bisector. Determine the length of PQ.

5. Determine the smallest positive integer n for which there exist real numbers $x_1, \ldots, x_n$, $1 \leq x_i \leq 4$ for $i = 1, 2, \ldots, n$, which satisfy the following inequalities simultaneously:

$$x_1 + x_2 + \cdots + x_n \geq \frac{7n}{3}$$

and

$$\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n} \geq \frac{2n}{3}.$$

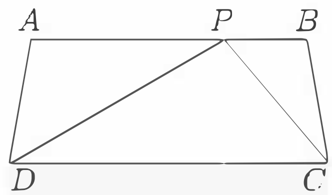
Solutions

1. Let the dimensions of the lawn (in metres) be a by b. The area of the first three rings is given by $ab - (a - 6)(b - 6) = 6(a + b) - 36$. Similarly, the area of the next four rings is given by $(a - 6)(b - 6) - (a - 14)(b - 14) = 8(a + b) - 160$. These two regions contain the same amount of grass, so they must be the same area. Thus, $6(a + b) - 36 = 8(a + b) - 160$. It follows that the only possible value of the perimeter of the lawn is $2(a + b) = 124$ m.

2. Since ∠DAB = ∠CBA and AB is parallel to DC, we have AD = BC. Since AB is parallel to DC, ∠BPC = ∠PCD. It foll●ws that triangles BPC and PCD are similar. A similar argument shows that triangles ADP and PCD are also similar. Hence,

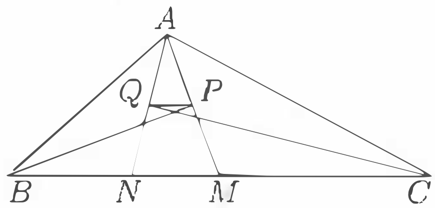
$$\left(\frac{\text{PD}}{\text{PC}}\right)^2 = \frac{\text{BC}}{\text{BP}} \cdot \frac{\text{AP}}{\text{AD}} = \frac{\text{PA}}{\text{PB}},$$

as desired.



3. (a) For any positive integer n, $n^2 + 2$ is special.
   (b) We claim that for infinitely many positive integers n, n^2 is not special. Suppose $n^2 = m^2 + p$ for some integer m and some prime number p. Then, $p = n^2 - m^2 = (n - m)(n + m)$. We must have $n - m = 1$ and $p = n + m = 2n - 1$. If we let $n = 3k + 2$ for any positive integer k, then $2n - 1 = 6k + 3$ is not a prime number. This justifies the claim.

4. Extend AP and AQ to cut BC at M and N, respectively. Then, ABM and ACN are isosceles, so that BM = 2 and NC = $2\sqrt{2}$. Hence, MN = $2\sqrt{2} - 2$. Now PQ is the segment joining the midpoints of AM and AN. Hence, PQ = MN/2 = $\sqrt{2} - 1$.



5. Suppose the real numbers $x_1, \ldots, x_n$, $1 \le x_i \le 4$ for $i = 1, 2, \ldots, n$, satisfy the two given inequalities. Then, $(x_i - 1)(x_i - 4) \le 0$ so that $x_i + 4/x_i \le 5$. Equality holds for $x_i = 1$ or $x_i = 4$. From these inequalities and the given ones, we obtain

$$5n = \frac{7n}{3} + \frac{8n}{3} \le x_1 + x_2 + \cdots + x_n + 4\left(\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}\right) \le 5n.$$

Hence, $x_i = 1$ or 4, $i = 1, 2, \ldots, n$. Suppose $x_1 = x_2 = \cdots = x_k = 1$ and $x_{k+1} = x_{k+2} = \cdots = x_n = 4$ for some index k. Then $x_1 + x_2 + \cdots + x_n = k + 4(n - k) = 7n/3$. Hence, $5n = 9k$ so that 9 divides n. It follows that the smallest value of n is 9, with the numbers 1, 1, 1, 1, 1, 4, 4, 4 and 4.