Secondary Algebra: A Quadratic Case Study

Ed Barbeau

Jennifer Hyndman (professor and chair of the Department of Mathematics and Statistics, University of Northern British Columbia) recently had a very bright fourth-year student ask what doing research would mean for her. Even though we spend our time teaching students how to solve problems, which is one stage of doing research, we often do no teaching about how to create new problems, nor do we let students understand what research is. In this article, Ed Barbeau develops the idea that creativity in mathematics can be fostered at the undergraduate level through presentation of material that allows students to formulate questions.

Too often, the mathematics curriculum is seen solely in terms of delivering to the student standard topics to be mastered. However, there is a creative side that can be accessed by students still at school; not all new results require years of study of difficult and sophisticated areas. Geometry and combinatorics are two areas where students can enter on the ground floor, but as I shall indicate by an example, it is possible for a student to obtain an original algebraic result.

While the student in question is particularly strong, I wonder to what extent it is open to students in regular classes to formulate and prove their own results (even if they may be widely known), and how problems might be composed to encourage this to happen.

The Basic Problem

Let me first reconstruct the situation that led to the problem that I posed to students in a correspondence program and an undergraduate competition, and that inspired the original research of one of them. An oblong number is any product of two consecutive positive integers. If we examine the sequence {2, 6, 12, 20, 30, 42, 56, 72, ...}, we might note that the product of two consecutive oblong numbers is also oblong. For example, 12×20 is equal to the oblong number $240 = 15 \times 16$. Adding 1 to each of the oblong numbers gives a sequence of positive integers of the form $k^2 + k + 1$, namely {3, 7, 13, 21, 31, 43, 57, 73, 91, ...}, with the same property. These empirical observations might be made by an aware student sensitive to patterns. With a little effort, they can be established by deriving the identities:

$$[(x-1)x][x(x+1)] = (x^2-1)x^2$$

and

$$\begin{aligned}
&[(x-1)^2 + (x-1) + 1][x^2+x+1] \\
&= (x^2-x+1)(x^2+x+1) \\
&= [(x^2+1)-x][(x^2+1)+x] \\
&= (x^2+1)^2 - x^2 \\
&= (x^2)^2 + x^2 + 1.
\end{aligned}$$

Noting that both the forms $x(x+1)$ and x^2+x+1 are monic quadratic polynomials, we might ask whether these observations can be generalized to numbers of the form $f(x) = x^2 + bx + c$, where b and c are arbitrary integers and x takes consecutive integer values. For example, the product of two consecutive squares is again a square.

One way to approach this problem is to experiment with various examples, and then make an inspired guess as to the value of z generated by the equation $f(x)f(x+1) = f(z)$. Then the proof amounts to just checking algebraically that you are correct.

However, there is another way to approach the problem: transformationally. Suppose that $f(x)$ is an arbitrary monic quadratic polynomial in x. Then $g(t) = f(x+t)$ is a monic quadratic polynomial in t: $g(t) = t^2 + bt + c$. Then, briefly, noting that $g(y) = f(x = y)$ for each y,

$$\begin{aligned}
f(x)f(x+1) &= g(0)g(1) = c(1+b+c) \\
&= c^2 + bc + c = g(c) = f(c+x) = f(g(0)+x) \\
&= f(f(x)+x).
\end{aligned}$$

Since this sort of manipulation is foreign to most secondary students, let us consider the aspects of the situation that students ought to be made aware of.

First of all, there is change of perspective. The problem is posed as establishing a fact for a particular quadratic and any value of the argument; the realization needs to be made that, by means of a translation of the variable, one can rather prove it for any quadratic and a particular value of the variable, namely 0.

Secondly, there is the technical problem of mediating between the two perspectives. Thirdly, it is necessary to make some interpretations: the constant coefficient as the value of a polynomial at 0, and the expression $c(1 + b + c) = c^2 + bc + c$ as the value of $g(c)$. Finally, the evolution of the solution changes the problem. Because one can actually display $f(x)f(x + 1)$ as the composition of two quadratics with integer coefficients, the property that we are dealing with f values at integers is subsumed in the more general (and interesting) representation of $f(x)f(x + 1)$ as a composite.

The General Quadratic

What would the situation be for a quadratic with an arbitrary leading coefficient? Experimentation reveals that $f(x)f(x + 1)$ need not be a later value of the quadratic when it is evaluated at integers. However, the work we have done at the end of the last section suggests that we can refocus the problem, to see whether $f(x)f(x + 1) = g(h(x))$ for suitable quadratics g and h. This turns out to be true, and this result was given as a problem to students at both the secondary and tertiary levels.

PROBLEM. Let $f(x)$ be a quadratic polynomial. Prove that there exist quadratic polynomials $g(x)$ and $h(x)$ for which $f(x)f(x + 1) = g(h(x))$.

COMMENT. One attempt might be to reduce it to the monic case, an approach that would undoubtedly be difficult for a typical secondary student to consummate but when understood should signify a pretty deep understanding of algebraic relationships. Writing $f(x) = au(x)$, where $u(x)$ is monic, we have that

$$f(x)f(x+1) = a^2u(x)u(x+1) = a^2u(x+u(x))$$

so we can take $g(x) = a^2u(x)$ and $h(x) = x + u(r)$. When $f(x) = ax^2 + bx + c$, this leads to $g(x) = a^2x^2 + abx + ac$ and

$$h(x) = x^2 + \left(1 + \frac{b}{a}\right)x + \frac{c}{a}.$$

However, as the following solutions indicate, there are at least three other ways students might tackle this problem, depending on whether they conceive of the quadratic in factored form, in descending powers of x or in terms of completing the square. In the first solution, below, note how it contains the seeds of the generalization we will discuss later. The second solution uses the method of undetermined coefficients to obtain a set of five equations in six unknowns. While this may appear formidable, the situation is tractable when the solver realizes that only one solution is needed for an overdetermined system and makes a simplifying assumption. The final solution is an adept completion of the square manipulation. Each of the solutions requires a level of sophistication that we should be encouraging in students planning to go on in science and mathematics.

SOLUTION 1. [A Remorov] Let $f(x) = a(x - r)(x - s)$. Then,

$$\begin{aligned} f(x)f(x+1) &= a^2(x-r)(x-s+1)(x-r+1)(x-s) \\ &= a^2(x^2+x-rx-sx+rs-r) \\ &\quad (x^2+x-rx-sx+rs-s) \\ &= a^2[(x^2-(r+s-1)x+rs)-r] \\ &\quad [(x^2-(r+s-1)x+rs)-s] \\ &= g(h(x)), \end{aligned}$$

where $g(x) = a^2(x - r)(x - s) = af(x)$ and $h(x) = x^2 - (r + s - 1)x + rs$.

SOLUTION 2. Let $f(x) = ax^2 + bx + c$, $g(x) = px^2 + qx + r$ and $h(x) = ux^2 + vx + w$. Then,

$$\begin{aligned} f(x)f(x+1) &= a^2x^4 + 2a(a+b)x^3 \\ &\quad + (a^2+b^2+3ab+2ac)x^2 \\ &\quad + (b+2c)(a+b)x + c(a+b-c) \end{aligned}$$

and

$$\begin{aligned} g(h(x)) &= p(ux^2+vx+w)^2 + q(ux+vx+w) + r \\ &= pu^2x^4 + 2puvx^3 + (2puw + pv^2 + qu)x^2 \\ &\quad + (2pvw + qv)x + (pw^2 + qw + r). \end{aligned}$$

Equating coefficients, we find that $pu^2 = a^2$, $puv = a(a + b)$, $2puw + pv^2 + qu = a^2 + b^2 + 3ab + 2ac$, $(b + 2c)(a + b) = (2pw + q)v$ and $c(a + b + c) = pw^2 + qw + r$. We need to find just one solution of this system. Let $p = 1$ and $u = a$. Then, $v = a + b$ and $b + 2c = 2pw + q$ from the second and fourth equations. This yields the third equation automatically. Let $q = b$ and $w = c$. Then, from the fifth equation, we find that $r = ac$.

Thus, when $f(x) = ax^2 + bx + c$, we can take $g(x) = x^2 + bx + ac$ and $h(x) = ax^2 + (a + b)x + c$.

SOLUTION 3. [S Wang] Suppose that $f(x) = a(x + h)^2 + k = a(t - (½))^2 + k$, where $t = x + h + ½$. Then, $f(x + 1) = a(x + 1 + h)^2 + k = a(t + (½))^2 + k$, so that

$$\begin{aligned} f(x)f(x+1) &= a^2\left(t^2 - \frac{1}{4}\right)^2 + 2ak\left(t^2 + \frac{1}{4}\right) + k^2 \\ &= a^2t^4 + \left(-\frac{a^2}{2} + 2ak\right)t^2 + \left(\frac{a^2}{16} + \frac{ak}{2} + k^2\right). \end{aligned}$$

Thus, we can achieve the desired representation with

$$h(x) = t^2 = x^2 + (2h+1)x + \frac{1}{4}$$

and

$$g(x) = a^2x^2 + \left(\frac{-a^2}{2} + 2ak\right)x + \left(\frac{a^2}{16} + \frac{ak}{2} + k^2\right).$$

The Generalization

One student, James Rickards of Greely, Ontario, raised the situation to a higher level when he realized that he needed to know only that $f(x)f(x+1)$ was a quartic polynomial for which the sum of two of its roots was equal to the sum of the other two. This immediately suggested the generalization that if the quartic polynomial $f(x)$ has roots r_1, r_2, r_3, r_4 (not necessarily distinct), then $f(x)$ can be expressed in the form $g(h(x))$ for quadratic polynomials $g(x)$ and $h(x)$ if and only if the sum of two of r_1, r_2, r_3, r_4 is equal to the sum of the other two.

Let us run through the proof of this statement. Without loss of generality, suppose that $r_1 + r_2 = r_3 + r_4$. Let the leading coefficient of $f(x)$ be a. Define

$$h(x) = (x - r_1)(x - r_2)$$

and

$$g(x) = ax(x - r_3^2 + r_1r_3 + r_2r_3 - r_1r_2).$$

Then,

$$\begin{aligned} g(h(x)) &= a(x - r_1)(x - r_2)[(x - r_1)(x - r_2) \\ &\quad - r_3^2 + r_1r_3 + r_2r_3 - r_1r_2] \\ &= a(x - r_1)(x - r_2)[x^2 - (r_1 + r_2)x \\ &\quad - r_3^2 + r_1r_3 + r_2r_3] \\ &= a(x - r_1)(x - r_2)[x^2 - (r_3 + r_4)x \\ &\quad + r_3(r_1 + r_2 - r_3)] \\ &= a(x - r_1)(x - r_2)[x^2 - (r_3 + r_4)x \\ &\quad + r_3r_4] \\ &= a\,(x - r_1)(x - r_2)(x - r_3)(x - r_4) \end{aligned}$$

as required.

Conversely, assume that we are given quadratic polynomials $g(x) = b(x - r_5)(x - r_6)$ and $h(x)$, and that c is the leading coefficient of $h(x)$. Let $f(x) = g(h(x))$.

Suppose that

$$h(x) - r_5 = c(x - r_1)(x - r_2)$$

and that

$$h(x) - r_6 = c(x - r_3)(x - r_4).$$

Then,

$$f(x) = g(h(x)) = bc^2(x - r_1)(x - r_2)(x - r_3)(x - r_4).$$

We have that

$$\begin{aligned} h(x) &= c(x - r_1)(x - r_2) + r_5 \\ &= cx^2 - c(r_1 + r_2)x + cr_1r_2 + r_5 \end{aligned}$$

and

$$\begin{aligned} h(x) &= c(x - r_3)(x - r_4) + r_6 \\ &= cx^2 - c(r_3 + r_4)x + cr_3r_4 + r_6, \end{aligned}$$

whereupon it follows that $r_1 + r_2 = r_3 + r_4$ and the desired result follows.

Let me allow Rickards to continue in his own words:

> I then wondered, how will this continue? What will the condition be for the composition of two third degree polynomials? I tried to construct a proof with only the assumption that the first symmetric polynomials agreed for some division into three groups of three roots of the whole ninth degree polynomial. While writing this out, it became apparent that I lacked something. I then saw that assuming the second symmetric polynomials agreed would be all that I needed. Thus I now had a good idea; I wrote out a proof for a polynomial of degree n^2. The next day or so, I realized that the fact the two polynomials being composed had the same degree was irrelevant; just minor modifications to make this as general as could be, a composition of degrees m and n.

Rickards figured out that the key property of the composite was that its roots could be partitioned into subsets for which all the symmetric polynomials agreed except for the product. This led him to a necessary and sufficient condition for a polynomial of degree mn to be the composite of polynomials of degrees m and n. He then addressed the determination of the composition factors of these degrees when the composite was given. He noted that while one could not generally know the actual roots of the polynomial, the coefficients of the composite factors depended only on knowing the value of the (equal) symmetric functions of roots in each of the partitioning sets, and these values could be retrieved from the coefficients of the given polynomial. It is convenient to give the proof for a monic polynomial, and then derive the general case; the details are found in Rickards (2011).

The relating of the monic to the general situation is a nice exercise for students. Suppose that $f(x)$ is a polynomial of degree nm and leading coefficient a, so that $f(x) = au(x)$ for some monic polynomial $u(x)$. Then we show that $f(x)$ is a composite of polynomials of degrees m and n if and only if $u(x)$ is so. Suppose that $f(x) = g(h(x))$, where $g(x)$ is of degree m with leading coefficient b and $h(x)$ is of degree n with leading coefficient c. Then, by comparison of leading coefficients, we have that $a = bc^m$. It can be checked that $u(x) = v(w(x))$, where $v(x) = (bc^m)^{-1}g(cx)$ and $w(x) = c^{-1}h(x)$.

On the other hand, suppose that $u(x) = v(w(x))$ for some monic polynomials $v(x)$ and $w(x)$ of respective degrees m and n. Then $f(x) = g(h(x))$ with $g(x) = au(x)$ and $h(x) = v(x)$.

Is Rickards's Result New?

I was enchanted by the elegance of Rickards's result. While the determination of a different criterion for a quartic to be the composite of two quadratics actually appears in my book (Barbeau 2003, problem 1.9.8, 44, 266), it is from the more pedestrian standpoint of a condition on the coefficients. Specifically, $ax^4 + bx^3 + cx^2 + dx + e$ is a composite of two quadratics if and only if $4abc - 8a^2d = b^3$. I was completely unaware of this new result, and a check of colleagues, the literature and the Internet did not reveal that it was previously known.

Whether it is actually new is open to question. While the composition of polynomials does not appear to have received much attention, it is conceivable that over the past 300 years, someone might have addressed the issue. However, such a result, if published, could have appeared in an obscure place and be impossible to track down. It seemed pretty enough to warrant appearing in a widely circulated current journal, regardless of its status.

Conclusion

It seems clear that if a curriculum is to be successful in preparing mathematics students for later study, it has to go beyond a straight presentation of results. Students require material that engages them, so that they acquire facility with the conventions and distinctions of mathematics and are able to make judgments about how a situation might be approached. Therefore, we need to be on the lookout for investigations and problems that encourage different perspectives and the search for connections.

I have presented one situation and mentioned issues that might arise. I hope that teachers may be able to present other examples, and that eventually exercises and problems that might lead to open-ended investigations by students might be more prominent in textbooks. As educators, we need to develop other case studies and then encourage teachers to try them out in their own settings. I have not had the opportunity to attempt this in a regular classroom situation. Its evolution is probably highly dependent on the context; it may happen that the discussion goes in a completely different direction and other questions emerge.

There are important issues pertinent to the preparation of students bound for science, technology and mathematics. Should such students be able to negotiate the subtleties of algebra usage illustrated by this example? If so, what are the implications for teacher training, the syllabus, the classroom experience and examinations? What preparation should be occurring all through the algebra sequence so that students attain both the perspective and the skills to manage it? What is the appropriate balance between presentation of such material by the teacher and investigation by individual students and groups? I invite teachers to try this example with their own classes and circles.

References

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