delta-k

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OF THE ALBERTA
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Guidelines for Manuscripts

delta-K is a professional journal for mathematics teachers in Alberta. It is published twice a year to

- promote the professional development of mathematics educators, and
- stimulate thinking, explore new ideas and offer various viewpoints.

Submissions are requested that have a classroom as well as a scholarly focus. They may include

- personal explorations of significant classroom experiences;
- descriptions of innovative classroom and school practices;
- reviews or evaluations of instructional and curricular methods, programs or materials;
- discussions of trends, issues or policies;
- a specific focus on technology in the classroom; or
- a focus on the curriculum, professional and assessment standards of the NCTM.

Suggestions for Writers

1. *delta-K* is a refereed journal. Manuscripts submitted to *delta-K* should be original material. Articles currently under consideration by other journals will not be reviewed.
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5. All manuscripts should be submitted electronically, using Microsoft Word format.
6. Pictures or illustrations should be clearly labelled and placed where you want them to appear in the article. A caption and photo credit should accompany each photograph.
7. References should be formatted consistently using *The Chicago Manual of Style*'s author-date system or *The American Psychological Association (APA)* style manual.
8. If any student sample work is included, please provide a release letter from the student's parent/guardian allowing publication in the journal.
9. Articles are normally 8–10 pages in length.
10. Letters to the editor or reviews of curriculum materials are welcome.
11. Send manuscripts and inquiries to the editor: Gladys Sterenberg, 195 Sheep River Cove, Okotoks, AB, T1S 2L4; e-mail gladyss@ualberta.ca.

MCATA Mission Statement

Providing leadership to encourage the continuing enhancement of teaching, learning and understanding mathematics.

delta-K

Volume 49, Number 2 June 2012

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From the Editor's Desk

Gladys Sterenberg

Throughout my teaching career, at both school and postsecondary levels, I have often integrated stories into my lessons as a way of helping my students engage in processes of communication and connections as they build conceptual understanding of math. I am beginning to understand better how the development of mathematical concepts occurs in a contextual and relational manner. Moreover, I believe that mathematics learning is affected by the stories we tell and the stories we hear about mathematics.

Many of the articles in this issue reflect the importance of story and present both theoretical considerations (see Sherry Matheson's article) and practical examples (see lesson ideas written by Sarah Danielsen, Petra Nagtegaal and Caitlin Dickinson) for integrating literature into math lessons. In recent years, I have been exploring how to further enrich learning by incorporating historical and cultural stories of mathematics into my teaching. Regina Panasuk and Leslie Bolinger Horton, researchers based in the United States, present arguments for considering the incorporation of the history of mathematics at the high school level that are worth considering for a Canadian context. This merging of stories, histories and mathematics offers a creative way of enhancing the program of studies.

Another insightful way of enhancing the program of studies is to reframe the prescribed content. Tim Sibbald and Jerry Ameis prompt us to look at cosine laws and fractions in a deeper way and offer ideas for incorporating alternative relational approaches for teaching such concepts in our classrooms.

As I approach the summer break, I look forward to long days of reading while camping in the foothills of southern Alberta. In slower-paced days of rest and relaxation, I hope you will find time to read this issue of *delta-K*. These innovative articles will provide both insight into curriculum change and motivation for incorporating new teaching strategies and ideas to enhance the current program of studies in mathematics.

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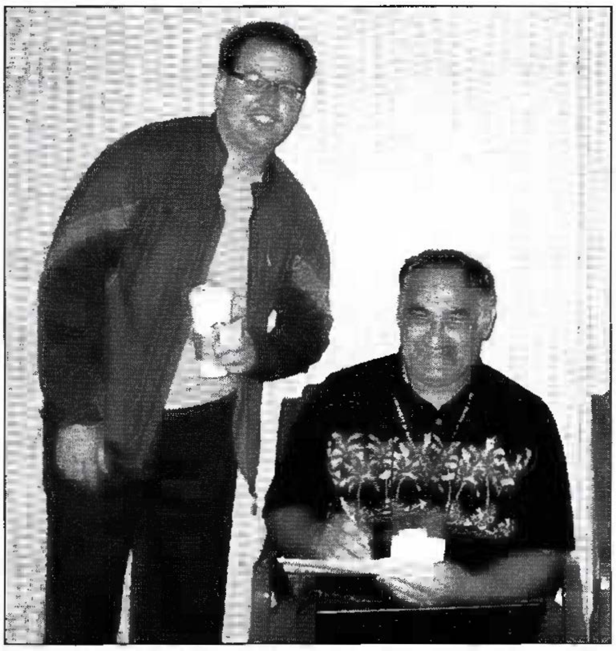

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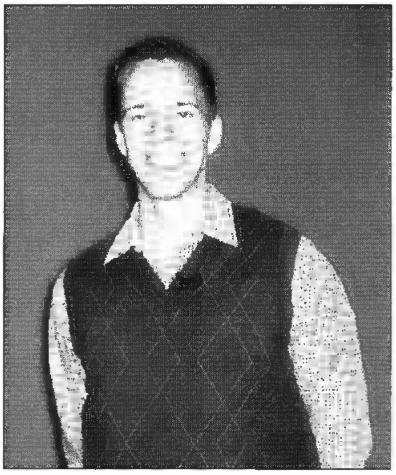

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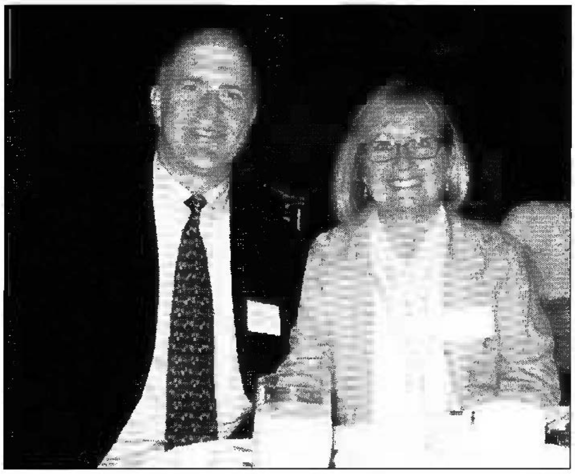

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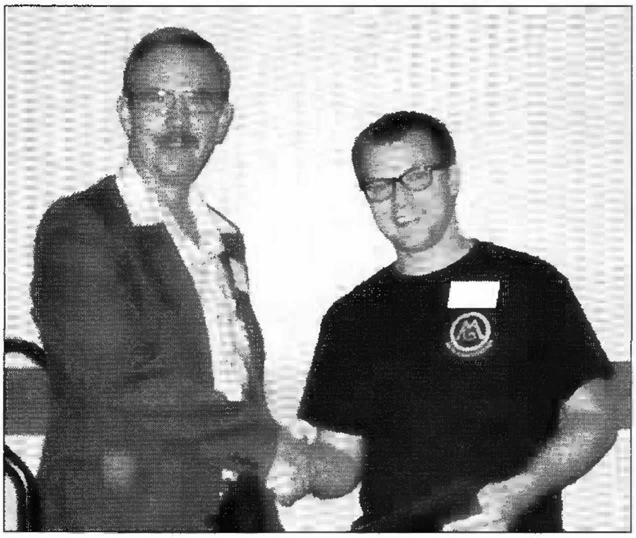

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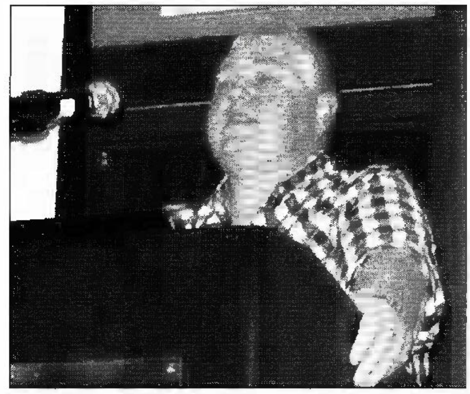

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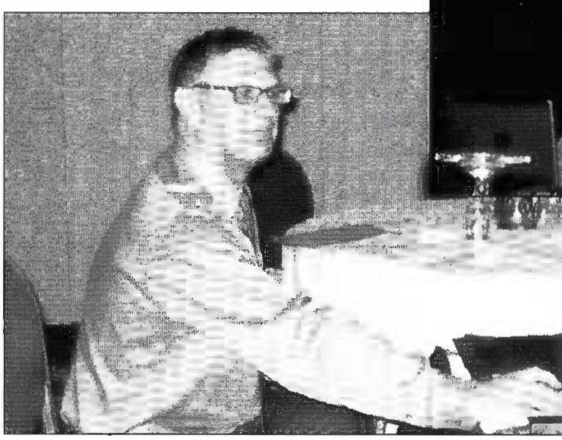
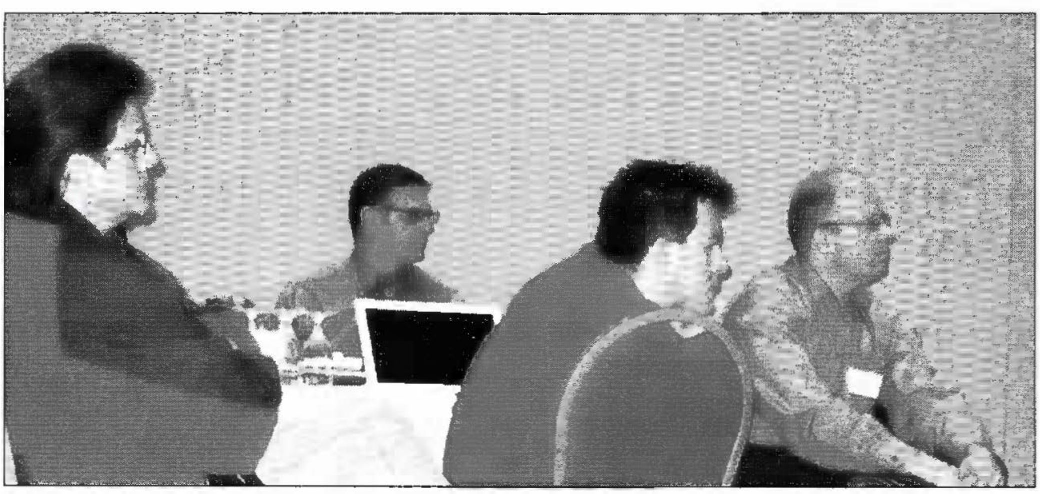
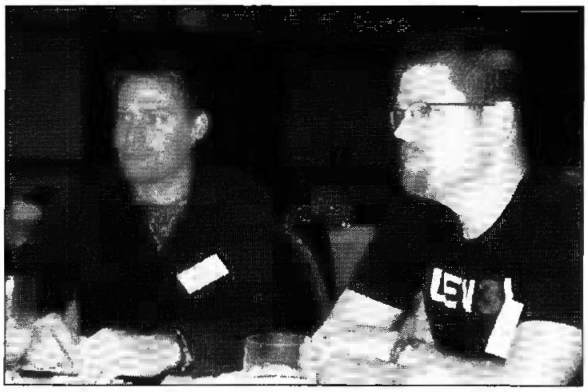




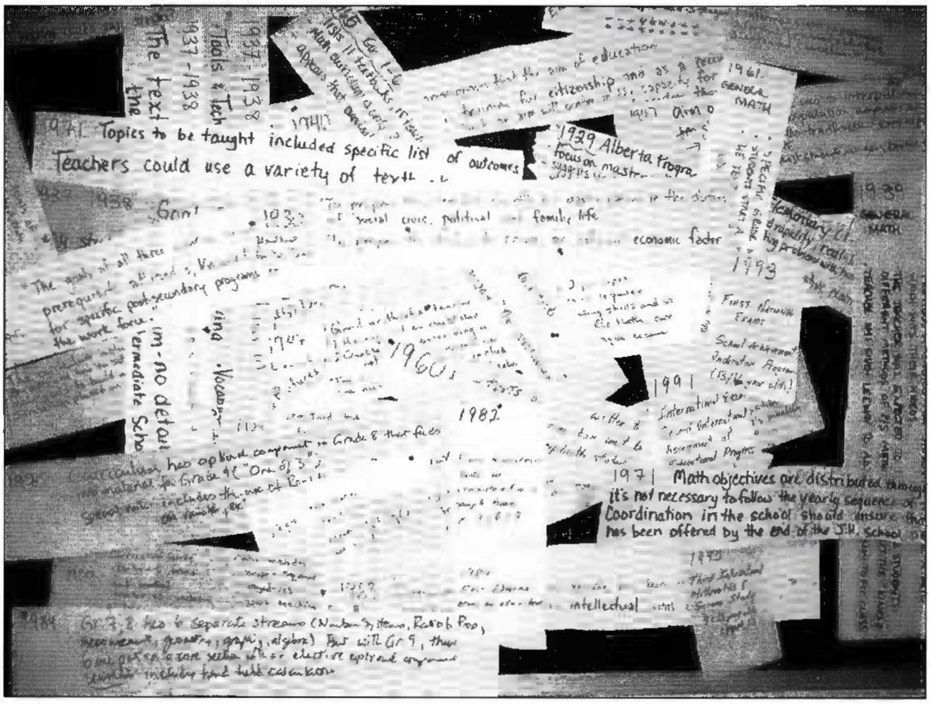


History of Mathematics Curricula Change in Alberta

Elaine Simmt

Mathematics curriculum reform is once again a hot topic of discussion among educators, parents, and concerned citizens. Practising mathematics teachers find themselves at the centre of those conversations with little more than their own histories to inform their response. Postgraduate studies provide an opportunity for teachers to investigate not only current curriculum but the history of curriculum, enabling them to be more fully prepared to contribute their professional knowledge to the debates about mathematics curriculum, teaching and learning.

In course work last fall at the University of Alberta, a group of teachers studied the history of curriculum change in Alberta from the early 20th century to the present. By creating timelines we were able to identify some patterns in curriculum reform. One pattern was to review student performance in mathematics, call for curriculum reform, change the curriculum, review student performance, call for curriculum reform, change the curriculum ... With each review there was a concern that students "lacked something"—arithmetic and algebra skills, problem-solving skills, or the ability to apply their arithmetic and problem-solving skills. Not until we begin to understand such patterns will we be able to do anything about them. The bulletin board shown below gives viewers a peek into the mathematics curricula of Alberta's past. Hopefully, it will remind us to ask ourselves what we really know of the past and how we can learn more about it in order to incorporate lessons from our history into today's process of curriculum reform.



This bulletin board was created by the 2011 EDSE 540 class: Tom Asquith, Mona Borle, Carolyn Bouchard, Cathy Campbell, Simon Christou, Madeleine Escobar, Dan Vandemeer and Dean Walls.

Making Connections: Mathematics and Literature

Sherry Matheson

When teachers begin their careers, they often do not stray far from the same path of instruction that they themselves followed as students. In the late 1950s, *mathematics* was not a curricular focus for primary students—*arithmetic* was. Computations completed quickly and correctly earned high praise and high marks for students. Teachers believed that in order to acquire high marks, successful students sat quietly at their own desks and practised addition and subtraction facts and memorized multiplication tables. If a student was not achieving at a satisfactory level, more practice was assigned.

For the first 25 years of my career, in my own primary classroom, I taught my students in the manner that I had been taught in the late 1950s. I was probably not alone—other teachers might also have continued to follow methodology from an earlier era.

When I focused on creating yearly plans of instruction in mathematics for my students, I considered the order of the concepts that I would teach. I ensured that, at the Grade 3 level, the students had opportunity to learn or perhaps actually memorize the basic addition and subtraction facts to 18 before they were introduced to double-digit addition and subtraction. Adding and subtracting without regrouping were to be successfully accomplished before regrouping was introduced. Other areas of mathematics instruction, such as measurement, geometry and graphing, were separate units of instruction that would be inserted into the final months of the school year, if time allowed. Problem solving, too, was a separate unit, taught in isolation just before the provincial achievement tests in June.

I had not considered that the problems that were presented should reach the students at the level at which they were currently constructing their own understanding of the mathematics being taught. I had not thought through the connection between the computation and the contemplation of the solution. I had not considered that the problems must scaffold from the students' prior knowledge and move into the next level of investigation or that they must represent what is important about mathematics and be part of real-world situations. Because the correct answer was the ultimate prize, I did not consider that successful mathematicians engage and delight in the opportunity to ponder, discuss and work strategically through solutions, often finding success because of the failures they experienced.

In the last five years, in an attempt to create a more child-centred and problem-based mathematics program in my own primary classroom, I began to attend professional development opportunities that brought the curriculum changes into focus and highlighted how the mathematical concepts to be taught were intertwined and should scaffold into new learning at higher grade levels. In the workshops, I discovered activities that could be incorporated into all my classes. When a presenter read aloud a children's story and demonstrated how the language of the story could introduce a rich problem into the mathematics being discussed with the students, I began to make connections myself. I recognized the power of story to clarify the language needed in the understanding of mathematics. The area of greatest comfort and strength throughout my teaching career had always been language learning. Through stories of any and every kind, long and short, imaginative, fanciful, endearing or bewitching, I could bring to life any concept that I was meant to teach. Perhaps I could use story in my mathematics classroom.

Using the children's story from the workshop in my own classroom, I marvelled at the students' conversations and attempts to collaboratively try to solve the problem posed. The air was electric with the students' excitement and engagement. The challenge that now faced me was not *whether* to change my teaching methodology to reflect a major shift in understanding of how students best engage in mathematics learning. Instead, the challenge was *how* to engage the students in meaningful investigations and bring meaningful problems and rich mathematical language into the classroom.

The use of literature-based instruction in classrooms to support young children's literacy is relatively new, having been introduced into schools after a shift away from basal reading programs. The availability of high-quality literature; the whole-language movement, in which students are read to every day; the use of literature as the basis of reading instruction; and collaborative reading, writing and discussions about the learning have been identified as characteristics of literature-based instruction (Morrow and Gambrell 2001).

Moyer (2000) suggests that language and mathematics instruction for young children not be separated: "Both literature and mathematics help us to organize and give order to the world around us" (Moyer 2000, 248). Literature assists in the development of mathematical communication as students negotiate what they are learning through talking to each other, talking to their teacher and talking to themselves.

Literature in the mathematics classroom provides a meaningful context for students to explore number sense, problem solving and real-world applications of mathematical concepts and to make connections between mathematics and problems that might be experienced in their own lives (Whitin and Wilde 1992). The use of children's literature in a mathematics classroom gives students opportunities to demonstrate how mathematics is meaningful in the everyday things they do, whether it be purchasing a treat at a store or setting the table for extra guests at dinner. The power of literature-based instruction is that the story becomes a shared experience, a jumping-off place to start their conversation. The language of mathematics can be experienced and meaning attached to it more readily through the context of story.

Teachers of mathematics can employ a variety of strategies that are commonly used in literature-based instruction—for example, using large-format books to point out features to the entire class, using shared reading to show how to read a text and understand its features and vocabulary, or using guided reading to give small-group differentiated instruction on the use of a strategy are all strategies to bring children's literature to the forefront in mathematics learning. Elementary classrooms usually dedicate a portion of a wall or bulletin board to display a collection of words being learned. The words are organized, usually alphabetically, and are displayed in large letters so that students can easily read them. A separate area of the classroom could be dedicated to a math word wall; the words could be organized by mathematical concepts being studied. If words that have already been studied are visible, the students are reminded of the terms and more likely to use them in their discussions and writing.

Many teachers, especially in elementary school classrooms, have a strong literature-based background for teaching the language arts curriculum. Using reading and comprehension strategies when teaching mathematics will focus the learning on the comprehension and fluency of mathematics:

- In *reading,* teachers often ask students to make predictions about what might come next; in *math,* they can ask students to make estimates before solving a problem.
- In *reading*, writing and oral communication are important aspects of instruction; in *math*, having students write down and discuss their ideas can help them develop, cement and extend their understanding.
- In *reading*, teachers do not expect children's writing to be identical, even when writing about the same topic; in *math*, teachers can encourage different methods for reasoning, solving problems and presenting solutions.
- In *reading*, vocabulary instruction is integral; in *math*, teachers can start a word chart for math terminology, consistently use correct math vocabulary and encourage children to do the same.
- In *reading*, read-aloud books provide students with common experiences from which they can learn; in *math*, many children's books provide stimulus for problem-solving. (Burns 2005, 2–3)

Knowing that my own area of comfort and strength throughout my teaching career has always been language learning, I feel empowered by the potential to use children's literature to bring literature-based instruction into mathematics. Students can come to recognize and understand mathematical terms through children's literature and the rich problem-based potential of story.

I recognize that, when looking for stories to use in mathematic instruction, I should consider the notion that "mathematical responses emerge naturally" (Shih and Giorgis 2004, 328). When I reflect on the story *Beezus and Ramona* (Cleary 1955), which I read aloud to my class this past school year, I realize that the story contains opportunities to bring mathematics alive through the problems that Ramona created in her everyday antics with her sister. In the chapter

"A Party at the Quimbys'," Ramona invites her entire playschool group to her house for a party, but she has not told her mother! The students arrive in the rain. How many boots would that have been? They ate applesauce and fig newtons for snack. How many fig newtons are in a carton and how many cartons would it have taken to feed the entire group? Mother was curling her hair when the children arrive. Only half her head was in pincurls. What would Mother's head have looked like? The children made a parade and blew whistles, tooted horns, banged drums and waved flags. How many of each would be needed? When Mother asks Ramona what she should do with her for being naughty, she answers that she should be locked in a closet for a million years. How old would she be when she was allowed to come out?

The illustrations in children's literature tell the story in the same manner as the words. I recognize that I could challenge students to tell their own story through the use of pictures to which mathematical language is added. This is an area where technology, which is one of the tools students use to learn and problem solve, could assist with visualization. If students are exploring the concepts of whole numbers to 10, they can use a digital camera to capture images that correspond to the number; the images could include words, numerals or groups of objects. The story created with the use of images and the language of mathematics could be shared with the class on a Smart Board or with parents by e-mail.

Journal entries, written explanations and poetry are potential avenues for exploring the relationships of different mathematical concepts through writing. Using Brown's (1949) *The Important Book* as a model, students could share their own understandings of a mathematical concept using the language of mathematics. For example, in a lesson on money, the first line written would emphasize the most important aspect of money. The following lines would share other aspects of the concept and the final line would reiterate the most important aspect as defined by the student. Writing in the mathematics classroom creates the potential for assessing concepts in both language arts and mathematics.

At the beginning of the 20th century, John Dewey stated, "If we teach as we taught yesterday, we rob our children of tomorrow."[1] The words ring true today, at the beginning of the 21st century. I realize that the use of children's literature in the mathematics classroom does not eliminate the need for specific mathematics instruction, but there is a place for the use of children's literature in the mathematics classroom. The two methodologies can be balanced in the mathematics classroom in the same way that whole-language and phonics-based instruction now coexist in the balanced literacy approach to language learning instruction. I can use the knowledge I possess in language arts to serve the needs of students learning mathematics.

Note

1 Quoted at http://powpak.lakeview.k12.oh.us/powpak/data/tech/files/techquotes.pdf (accessed January 10, 2012).

References

Brown, M W. 1949. *The Important Book*. New York: Harper.

Burns, M. 2005. "Building a Teaching Bridge from Reading to Math." *Leadership Compass* 3, no 2: 1–3. Available at www.mathsolutions.com/documents/2006_Building_Bridge.pdf (accessed January 10, 2012).

Cleary, B. 1955. *Beezus and Ramona*. New York: Harper.

Morrow, L, and L Gambrell. 2001. "Literature-Based Instruction in the Early Years." In *Handbook of Early Literacy Research*, ed S Neuman and D K Dickson, 348–60. New York: Guilford.

Moyer, P S. 2000. "Communicating Mathematically: Children's Literature as a Natural Connection." *Reading Teacher* 54, no 3: 246–55. Available at www.jstor.org/pss/20204901 (accessed January 10, 2012).

Shih, J C, and C Giorgis. 2004. "Building the Mathematics and Literature Connection Through Children's Responses." *Teaching Children Mathematics* 10, no 6: 328–33.

Whitin, D J, and S Wilde. 1992. *Read Any Good Math Lately? Children's Books for Mathematical Learning, K–6*. Portsmouth, NH: Heinemann.

Sherry Matheson is a Grade 2/3 teacher at Jean Vanier Catholic School in Sherwood Park, Alberta. She has recently completed a master of education in elementary mathematics education at the University of Alberta.

An Alberta Rose in the New Curriculum?

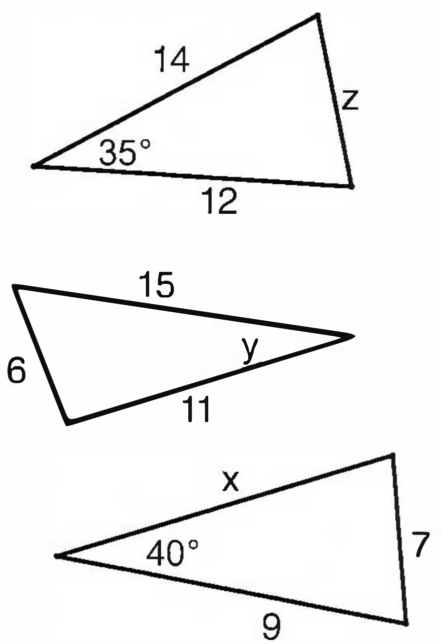
Tim Sibbald

Over the years I have had occasion to look at the mathematics curricula for different provinces of Canada. It is not surprising that the expectations for student learning are often similar and the regional differences subtle. This observation is consistent with the implication of the NCTM [National Council of Teachers of Mathematics] standards document (NCTM 2000) that there is broad consensus about the key topical areas of mathematics. This article examines one small difference that may help Alberta teachers make connections between the cosine law and quadratics in Mathematics 20-1 (Alberta Education 2008) that teachers in Ontario (and, doubtless, other jurisdictions that aren't home to the author) would find difficult to make.

The cosine law has three types of questions, shown in Figure 1. Usually, only the first two shown—finding the angle given the three side lengths and finding the length of the side opposite a given angle that is between two side lengths—are used when teaching the cosine law. It is not unusual for teachers to omit having the unknown side adjacent to the given angle for two reasons: first, it can be solved using a double application of the sine law and, second, if the cosine law is used it generates a quadratic equation. The purpose of this article is to demonstrate that using the cosine law to develop the quadratic case has benefits. The equivalent method of applying the sine law twice is also explored to see what trigonometric interpretations of the quadratic formula might come to light.

The curriculum distinction that allows this approach in Alberta, but not as easily in Ontario, arises because the quadratic equation has two solutions. The double application of the sine law generates two solutions when the ambiguous case of the sine law arises. It is this last point that is an option in Alberta, whereas in Ontario, the ambiguous case of the sine law is taught a year after the sine law for acute angles, the cosine law and quadratics are introduced. While Ontario teachers could re-examine quadratics after the ambiguous case is discussed, it would be after the horse has left the barn. In Alberta, however, the option is an interesting possibility because the ambiguous case is included when the sine law, cosine law and quadratics are introduced.

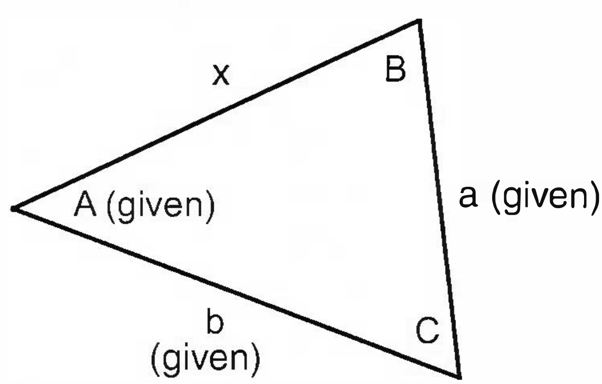
Figure 1: Cosine law questions



The Cosine Law

The following general analysis is based on Figure 2, in which x is the unknown and A, a, and b are specified values. The general derivation enables the teacher to choose values suitable to teaching, which will be considered later in this article. We start by examining the cosine law approach and then examine the double application of the sine law.

Figure 2: General setup



Setting up the cosine law using the given values and rearranging to highlight the quadratic in x leads to

$$x^2 - 2bCos(A)x + (b^2 - a^2) = 0$$

Using the quadratic formula, this can be solved directly as

$$x = \frac{2bCos(A) \pm \sqrt{4b^2Cos^2(A) - 4(b^2 - a^2)}}{2}$$

The common factor of 4 is extracted from the square root, leading to

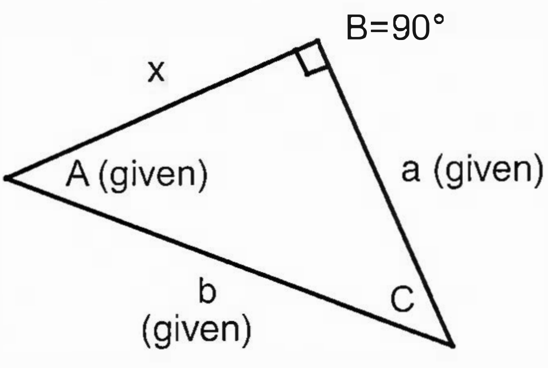
$$x = bCos(A) \pm \sqrt{b^2Cos^2(A) - b^2 + a^2}$$

Under the square root, the first two terms have a common factor that leads to a trigonometric simplification: $b^2(Cos^2(A) - 1) = -b^2Sin^2(A)$. This gives

$$x = bCos(A) \pm \sqrt{a^2 - b^2Sin^2(A)}$$

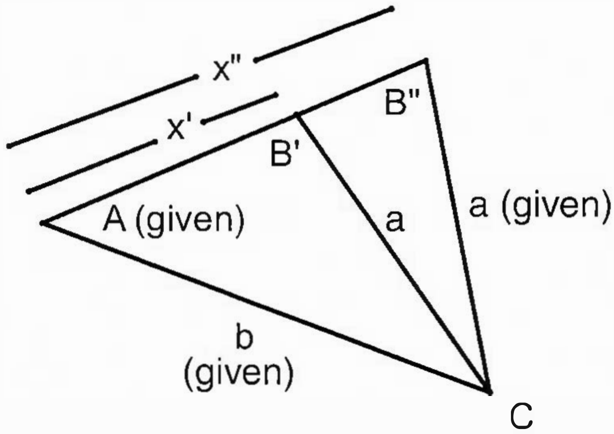
The discriminant, which appears under the square root, indicates whether the quadratic equation has two, one, or no real solutions. In the form rendered here, it has a value of zero, indicating a single solution to the quadratic, if side a has a length such that sin(A) = a / b—in other words, if the triangle shown in Figure 3 has angle B=90°. This is the solution that corresponds to the minimal value of a that allows a solution. When side a is made shorter than the right angle distance (ie, the minimal a distance), it is not long enough to connect the vertex to the other side and the triangle cannot physically be constructed.

Figure 3: The minimal *a* solution



If a is made longer, as shown in Figure 4, the discriminant is positive and there are two solutions corresponding to a being rotated about point C and intersecting at two values x = x' and x = x". These two solutions correspond to acute (ie, B") and obtuse (ie, B') values of the supplementary angles such that sin(B") = sin(B'). This is a much more satisfying interpretation of the discriminant than algebraic interpretations because it gives a geometric meaning.

Figure 4: Case of a positive discriminant



This analysis shows that the cosine law generates a quadratic where the discriminant has a geometric interpretation. The geometric interpretation is consistent with whether or not the triangle can be constructed. However, it requires an understanding of the ambiguous case of the sine law. Specifically, it requires that students recognize that sin(B) = sin(180° – B).

Double Application of the Sine Law

If the sine law is applied to Figure 2 twice, then the acute and obtuse solutions must be considered. In the following section, B is taken to be acute, thereby defining the acute case, and the obtuse case uses the supplementary angle 180° – B. (Since there are two solutions, some readers may find Figure 4 useful for visualizing both cases simultaneously, provided that one remembers that B" and B' are supplementary.)

$$\frac{Sin(A)}{a} = \frac{Sin(B)}{b} \qquad \frac{Sin(A)}{a} = \frac{Sin(180° - B)}{b}$$

Using the given A, a, and b allows one to solve for angle B. This is then used to find angle C so that a second application of the sine law can determine x. In the acute case, C = 180° – A – B, while in the obtuse case C = 180° – A – (180° – B) or C = B – A. Note that in the acute case, sin(180° – A – B) = sin(A+B) so that C = A+B can be used for the argument of the sine function, and it is this form that is used below.

Applying the sine law again with these two possible C values across from x gives

$$\frac{Sin(A)}{a} = \frac{Sin(A+B)}{x} \qquad \frac{Sin(A)}{a} = \frac{Sin(B-A)}{x}$$

Rearranging this gives

$$x = \frac{aSin(A+B)}{Sin(A)} \qquad x = \frac{aSin(B-A)}{Sin(A)}.$$

These are the two solutions to the quadratic. On one hand, this is so very different from the quadratic formula you wonder if you have lost your way; on the other hand, it is interesting to see that the ± component of the quadratic formula has been reduced to the sum and difference of angles.

A short proof that this result is consistent with the quadratic formula

$$x = bCos(A) \pm \sqrt{a^2 - b^2 Sin^2(A)}$$

is now provided. This would not be used in teaching, but is provided so that teachers can have confidence in the result. The method of proof is to take the trigonometric solution, use the sum and difference of angles formulas, and interpret the resulting algebraic form geometrically. Following this approach,

$$x = \frac{aSin(B \pm A)}{Sin(A)}$$

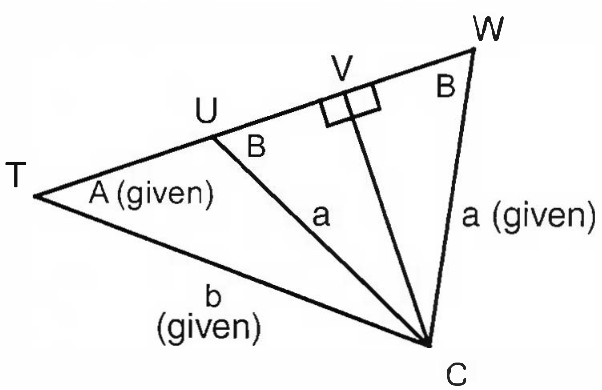
$$= \frac{aSin(B)Cos(A) \pm aCos(B)Sin(A)}{Sin(A)}$$

$$= \frac{aSin(B)Cos(A)}{Sin(A)} \pm aCos(B)$$

It is the last algebraic form that is interpreted geometrically using Figure 5. This figure is essentially the same as Figure 4, but has had letters added to the four vertices at the top as well as a perpendicular line segment VC. The two terms are considered separately by replacing the trigonometric terms with the equivalent ratios of sides. The first term is

$$\frac{aSin(B)Cos(A)}{Sin(A)} = \frac{a\left(\frac{VC}{a}\right)\left(\frac{TV}{b}\right)}{\left(\frac{VC}{b}\right)} = TV \cdot$$

Figure 5: Geometric interpretation



The second term, aCos(B), is equal in magnitude to VU and VW. The formula as a whole shows that x = TV – VU = TU or x = TV + VW = TW.

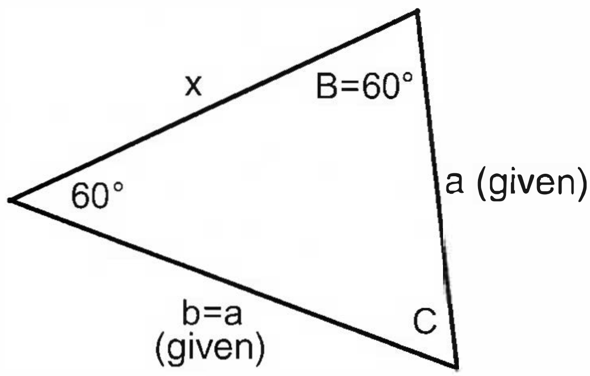
Instructional Examples

In terms of instruction, the connection highlighted here is considerable and many teachers may be hesitant to develop the connection explicitly. A useful option is to use specific instructional cases that are developed from the theory, without explicit instruction about the theoretical aspect. A few examples are provided to whet the reader's instructional appetite.

As a first example (shown in Figure 6), if $a = b$, then the constant term in the quadratic formula is zero. In addition, if A = 60° so that cos(A) = ½, the quadratic equation becomes

$$x^2 - bx = 0.$$

Figure 6: Instructional example.



This has solutions $x = 0$ and $x = b$, which correspond geometrically to a triangle with side length zero (ie, a line segment) and to the equilateral triangle.

As a second example, if A = 30°, $b = 2\sqrt{3}$ and $a = 2$, then the quadratic equation becomes

$$x^2 - 6x + (12 - 4) = 0 \quad \text{or} \quad x^2 - 6x + 8 = 0.$$

This can be factored, drawn as a geometric triangle and solved using the sine law. To confess, I used a pattern in which for some value of k, A = 30°, $b = k\sqrt{3}$ and $a = k$ gives

$$x^2 - 3kx + 2k^2 = 0, \text{ leading to } (x-k)(x-2k) = 0$$

for whatever value of k one chooses.

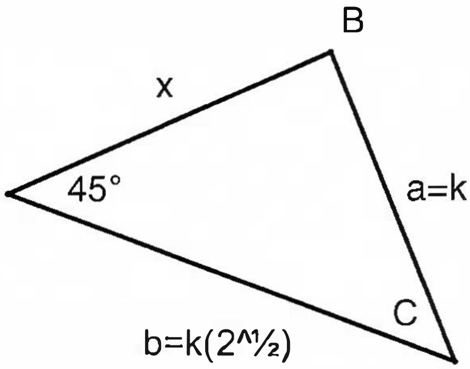
Using k again to highlight a family of solutions, one can choose A = 45°, $b = k\sqrt{2}$ (or numerical approximation), and $a = k$, as shown in Figure 7, to get

$$x^2 - 2kx + k^2 = 0 \text{ leading to } (x-k)(x-k) = 0.$$

This has instructional value because the Pythagorean theorem can be used to show, geometrically, that there is only one solution. Specifically, in the absence of knowing angle B, a solution for x can be found under the assumption that B is 90°. The solution is then observed to be unique, which implies that this is the minimal a solution and therefore B is a right angle. There is no need to use the sine law for this particular example, although it would support student understanding of the logic used.

Last, examples with no solution can be generated by making a smaller than the minimal value provided by bSin(A). Using Figure 7, one could use A = 45°, $b = 2\sqrt{2}$ and $a = 1$.

Figure 7: A Pythagorean example



Conclusion

The utility of this overall approach is that it can be used as a demonstrative bridge that brings together the algebra of quadratics and geometry of triangles. It facilitates multiple instructional approaches—some teachers may simply want to use instructional examples that support the connection, while other teachers may explore the connection as a pedagogical approach. In either case, there is a geometric rationale for the number of solutions to a quadratic equation that rivals the algebraic completion of the square. The only caveat in the entire approach is that it is instructional and not a replacement for factoring or the quadratic formula. Rather, its value is in the introduction of the quadratic and factoring using trigonometry that may provide a much deeper connection than other instructional approaches.

References

Alberta Education. 2008. *Mathematics Grades 10–12*. Program of studies. Edmonton, Alta: Alberta Education.

National Council of Teachers of Mathematics (NCTM). 2000. *Principles and Standards for Mathematics*. Reston, Va: NCTM.

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Integrating History of Mathematics into High School Curriculum: To Be or Not to Be?

Regina Panasuk and Leslie Bolinger Horton

Introduction

A decade ago, the National Council of Teachers of Mathematics (NCTM) created principles and standards for school mathematics, called *Standards 2000*, as an effort to provide benchmarks of achievement as well as consistency in content and rigour for all mathematics students in the United States. These documents were designed to give students a well-rounded, enriched and useful mathematics education that would foster conceptual understanding, appreciation and achievements in mathematics for all students (NCTM 2000, 13–14). While there was a strong emphasis on values and appreciation of mathematics, the history of mathematics (HOM) was ignored. Recently, the new Common Core Mathematics Standards[1] have been issued and adopted by many states. Not unlike the NCTM principles and standards, the goal in the adoption of the Common Standards is to ensure that students across the United States are provided with a curriculum that is unified in rigour and content. And again, unfortunately, nowhere in the Common Core Standards is there any mention of learning HOM.

This continuing omission is puzzling. It can be demonstrated that many of the goals outlined in the principles and standards would be better met by incorporating HOM into the curriculum. There is significant evidence that including HOM in math instruction is not only worthwhile but also an enrichment to the mathematics curriculum and can help students reason critically.

Does the absence of HOM from a national document not send a political message to teachers that learning HOM plays no role in learning the subject? If at the state and national levels there exists the implicit message that learning HOM is not important, why would a teacher consider the contrary?

It is fair to say that most teachers make every effort to stimulate their students' interest in mathematics and try to teach mathematics as a social construct—an activity that makes sense through its usefulness and utility. To appreciate mathematics, students should have a variety of experiences related to the cultural and historical aspects of the evolution of mathematics so they can value the role of mathematics in the development of our society. Unfortunately, the historical dimension of teaching mathematics is either totally absent, ignored or viewed as an "exotic luxury," as Whitrow (1932) suggested.

While HOM may not be appreciated in the United States, its importance is recognized in many other countries. According to Fasanelli et al (2000), "the experience of many mathematics teachers across the world (is) that the history of mathematics makes a difference: that having history of mathematics as a resource for the teacher is beneficial" (p 1). If mathematics teachers around the world see value in the inclusion of HOM, why doesn't the United States? Research into teachers' perspectives on the inclusion of HOM in the classroom is scarce, but some studies (Philippou and Christou 1998; Schram et al 1988; Siu 2004; Smestad 2009; Stander 1989) indicated that teachers' interest in and valuing of mathematics increased when they were introduced to HOM.

An interesting dichotomy revealed by the work of Fasanelli et al (2000) is that, while the importance of HOM is widely recognized, these studies emphasized that teachers found no interest in *using* HOM with the mandated curriculum. This failure of inclusion is important for us to explain and contradicts what our study shows; some enthusiast teachers do include HOM in their instruction and therefore must see benefit in doing so.

In light of these considerations, the purpose of our large-scale study was to gain understanding of high school mathematics teachers' perceptions and beliefs about the integration of HOM into their instruction. While we documented multiple findings, this paper addresses some of the factors that affect high school teachers' decision whether to include HOM as a systemic part of their mathematics courses.

Background

History of Mathematics

Several key assumptions served as a foundation of our research. First, by learning about the evolution of at least some mathematics concepts—which are inescapably linked to mathematicians, who, through years of sacrifice, trials and tribulations, created the mathematical concepts and the language to communicate them—a student looks into the past and can trace the intellectual development of humankind. Second, HOM provides a background for gaining a rich and deep understanding of the development of mathematical concepts. Third, there may exist an implicit relationship between the learning of HOM and students' attitude toward mathematics. Although there is no empirical proof for such assumptions related to mathematics, certainly the teaching of historical development of basic concepts is fundamental in the natural and social sciences. Since the work of Thomas Kuhn (1996), the history of science has been a rapidly growing field of research and publication. The popularity of the science histories of Stephen Hawking, in physics, and Stephen Jay Gould, in biology, suggests that history is useful even in abstract and complex disciplines. If the history of a discipline is important in the natural and social sciences, why should it not also be in mathematics? In the words of J W L Glaisher (1848–1928), uttered more than 100 years ago, "I am sure that no subject loses more than mathematics by any attempt to dissociate it from its history" (Glaisher 1890).

The importance of HOM in the school curriculum has been supported by research (Siu 2004; Ho 2008) and scholarly writings (Kline 1972; Wilder 1968; Katz 1997). There is even a debate about whether the notion of "using history of mathematics" should be replaced with "integrating history of mathematics" to encourage the view that the history of mathematics is inseparable from the subject itself (Siu and Tzanakis 2004).

It seems common sense to believe that it is beneficial for all mathematics learners to be aware of the purposes and intellectual struggles of those who created mathematics and to appreciate the process of invention. Further, as Swetz (1994) said, "The history of mathematics supplies human roots to the subject. It associates mathematics with people and their needs. It humanizes the subject and, in doing so, removes some of its mystique" (p 1).

We support the position that mathematics is a "cultural phenomenon" (Wilder 1968, xi), and that meaningful learning of school mathematics must be facilitated by studying the cultural significance of mathematics, the role of the evolution of mathematical concepts and scientific thought. According to Wilder (1968), "The anthropologist George P Murdock listed 'numerals' as one of 72 items that occurred, so far as was known, in every culture known to history or ethnography" (p 33).

Swetz et al (1995) suggested that exposure to HOM at the high school level can have "... a profound effect! For it is at the secondary or high school level that students first experience the power of mathematics and begin to realize the wide scope of its application and possibilities" (p 1). Barbin et al (2000) posited that

> the conviction that the use of history improves the learning of mathematics rests on two assumptions about the process of learning. The more a student is interested in mathematics, the more work will be done; and the more work that is done, the greater will be the resulting learning and understanding. (p 69)

Fasanelli et al (2000) reported that many countries have a magistrate of education who outlines the educational goals for the entire country. Countries such as Austria, Brazil, China, Denmark, France, Greece, Italy, New Zealand, Norway and Russia have a national set of frameworks that explicitly incorporate HOM into the learning standards. Fasanelli et al (2000) asserted, "These decisions are ultimately political, albeit influenced by a number of factors including the experience of teachers, the expectations of parents and employers, and the social context of debates about the content and style of the curriculum" (p 1). It is highly probable that a primary influence on a teacher's decision about whether or not to include HOM in the classroom is the set of government-created learning objectives provided in the national or state curriculum requirements.

Clearly, there are gaps between what is espoused in the professional and scholarly arena about the possible benefits of students learning HOM; the curriculum standards, which have no trace of HOM; and teachers' views on the integration of HOM in curriculum.

About the Nature of Mathematics

The historical dimension of mathematics and its evolution as a living organism suggest that mathematics is viable, ever changing and socially constructed. There are numerous examples supporting the value of understanding the evolution of mathematics (for example, the act of counting goes back to primitive civilizations and their predecessors; the need to record the process of division led to the creation of new type of numbers [that is, fractions and their symbolic representation]; the advancement of the numeration system led to the development of a whole new language to communicate newly invented and discovered mathematical ideas in a more precise and elegant way). We emphasize that referring to mathematics as a language is critical to accentuate that mathematics is a result of social practices of people, rather than an objective realm that is metaphysical and superhuman.

Due to humans' natural tendency to "symbolic initiative" (Wilder 1968, 5), it is logical to accept that, historically, mathematics was born during the process of the invention of a language to communicate the patterns observed in the real world and the internal structures in mathematics.

From an anthropological perspective, such invention was possible because of humans' natural ability, borne out of necessity, to think abstractly, manipulate mental objects at a conceptual level, and establish connections between ideas and communicate them. "Mathematics is something that man himself creates, and the type of mathematics he works out is just as much a function of the cultural demands of the time as any of his other adaptive mechanisms" (Wilder 1968, 4).

The dual nature of mathematics and its evolution is subtle but critical to understand. A lack of understanding might cause misconceptions about the nature of mathematics as a discipline, which could lead to further misconceptions about the teaching and learning of mathematics.

Method

To investigate the factors that influence whether teachers include HOM into their classroom instruction, we used SurveyMonkey, a web-based software program (www.surveymonkey.com) to design and administer a comprehensive survey scale instrument. Having weighed the advantages and the disadvantages of both online and postal mail, we chose to disseminate a web-based survey and had the participants send responses to the company's server.

Participants

This study took place in a US state that had 372 operating public high schools, including charter schools, and about 2,909 mathematics teachers. To encourage participation, we sent e-mails to all public and seven private high school principals (total 379) requesting them to forward to their mathematics teachers an invitation to participate in an anonymous, online survey. The number of the private high school mathematics teachers remains unknown. Only six principals declined the request. We also used our extended personal contacts and sent e-mail messages to high school mathematics teachers, encouraging them to participate. The invitation letters sent to the principals and directly to the teachers included a description of the study, detailed instructions on how to access it and contact information should they have any questions. The survey was available on the Web for several weeks. A total of 367 teachers participated in the online survey, which is about 12 per cent of all high school mathematics teachers in the state.

Instrument

The survey's six parts—(1) Attitudes Toward Mathematics as a Discipline, (2) Philosophical Perspectives of Mathematics, (3) Philosophical Perspective on the Nature of Mathematics, (4) Perceptions of History of Mathematics, (5) Reasons For or Against the Inclusion of History of Mathematics and (6) Teacher Background Variables—included 110 items that would require about 25 to 30 minutes to complete. Some of the items were designed by the researchers. Others were adapted from surveys of previous studies (Tapia and Marsh 2004; Dutton 1962; Shulman 1986; Alken 1974; Charalambous, Panaoura and Philippou 2009; Tzanakis et al 2000), and some were extracted from the National Assessment of Educational Progress Mathematics Teacher Background Questionnaire (National Center for Education Statistics 2009).

We established a Likert scale consisting of five declarative sentences, with choice responses varying from strongly disagree, with value of 1, through disagree, neutral and agree, to strongly agree, with value of 5. We used skip logic to direct the teachers to a custom path of the survey that, depending upon their response of yes or no to the inclusion of HOM into their teaching, led them to the statements pertaining to reasons for or against.

Reliability Analysis and Scales Formation

Our analysis of the data focused on teachers' perceptions about HOM and the factors that affect their decision to integrate HOM into their curriculum. Of the 110 questions, 74 Likert (ordinal) questions were used to run a reliability analysis, which yielded a Cronbach's alpha value of 0.94.

Subsequently, a total of seven constructs were formed as the result of a factor analysis of all 74 Likert (ordinal) data items. For each construct, scales were developed by averaging the responses (1—strongly disagree to 5—strongly agree) for the variables that loaded into the corresponding construct. Three of the seven constructs, Perceptions of HOM, Reasons for Including HOM and Reasons for Not Including HOM, and their relationship are described below.

Data Analysis

Our primary purpose was to evaluate whether teachers' perceptions of the utility, benefits and importance of HOM influence their decision to incorporate HOM into their instruction.

We analyzed three sets of data, shown in Table 1.

Table 1: Summary of yes/no to inclusion of HOM

	Frequency	Per cent
Valid .00	30	8.2
No	133	36.2
Yes	204	55.6
Total	367	100

One set included all teachers (N = 337) who responded to the questions of whether they included or did not include HOM. The other two sets consisted of those who included HOM (N = 204; we will refer to this group of teachers as the Yes group) and those who did not include HOM (N = 133; we will refer to this group of teachers as the No group).

Factor analysis of the survey items showed that the perceptions that were most relevant were encapsulated in five fundamental statements, the responses to which were measured on a Likert scale of 1 (strongly disagree) to 5 (strongly agree). These five statements and the mean scores for the entire sample, All Teachers, and for both Yes and No groups are shown in Table 2.

The mean scores for all teachers are significantly greater than 3 (neutral); in fact, they are very high for Likert scale surveys, with the exception of the last statement. Also, the mean score for the first statement is extraordinarily high for a Likert scale survey. There was little disagreement with the statement "All students should be taught some history of math." The fifth statement, on the other hand, suggests a possible incongruity with the first statement. Obviously, many participating teachers believed that HOM has significance and value, yet there was not an agreement with the statement that understanding mathematics would be easier if HOM were taught. Responses were mid-neutral at best. It is likely that teachers could view HOM as a time-consuming burden rather than a valuable teaching tool for capturing students' attention and focusing their learning. More discussion on this issue is below.

Nonetheless, the high mean scores for the entire sample would suggest that if teachers' perceptions of the utility and importance of HOM influence their decision to incorporate HOM in their instruction, the vast majority of our sample should have been employing HOM. Unfortunately, as shown in Table 1, this is

Table 2: Teachers' perceptions of HOM

	All students of mathematics should be taught some history of mathematics.	History of mathematics is worthwhile and necessary to the understanding mathematics.	Knowledge of history of mathematics is valuable to nonscientists or nonmathematicians.	Knowing history of mathematics may assist students in learning mathematical concepts.	Understanding mathematics would be easier if history of mathematics were taught.
All teachers					
N Valid	347	347	347	346	346
Missing	20	20	20	21	21
Mean	3.90	3.46	3.48	3.81	3.21
Yes group of teachers					
N Valid	203	203	203	202	202
Missing	1	1	1	2	2
Mean	4.14	3.74	3.78	4.03	3.43
No group of teachers					
N Valid	132	132	132	132	132
Missing	1	1	1	1	1
Mean	3.54	3.06	3.05	3.48	2.90

not true—only a small majority (55 per cent) were using HOM.

A comparison between Table 1 and Table 2 shows that there are many teachers in the No group with a positive perception of HOM (3.54 mean for statement 1). Interestingly, many teachers in the Yes group think that HOM may not necessarily make understanding mathematics easier (3.43 mean for statement 5). The comparison between groups Yes and No is shown in Table 3.

Perceptions of history of mathematics is a construct that averages all of the statement rankings presented in Table 2. The data ($p < .001$) suggest that teachers who viewed HOM as valuable, necessary or worthwhile would have tended to include it in their classroom lessons.

Teachers in the No group (see Table 2) scored the items lower than the means of the group of all participants, which is consistent with their decision to not include HOM in their instruction. Apparently, among many factors that affect teachers' decisions, there is notable evidence that teachers' beliefs about the importance of HOM and their views on the benefits and values of HOM play an important role.

We also observe a significant difference in the Yes and No groups (see Table 2). Teachers in the Yes group appeared to be more likely to agree that all high school mathematic students should be taught some HOM, that HOM is worthwhile and necessary to the understanding of mathematics, that HOM is valuable to nonscientists or nonmathematicians, that knowing HOM may assist students in learning mathematical concepts, and that understanding mathematics would be easier if HOM were taught. That was anticipated and expected. It is just as important to note that the No group, while not as strongly supportive of these statements, is nonetheless moderately favourably disposed to HOM except for the last statement. "Understanding math would be easier if the history of mathematics were taught."

There is an interesting inconsistency in the ranking of a few very closely related statements. The teachers in the No group do not see that "Understanding mathematics would be easier if the history of mathematics were taught" (mean 2.90). However, the statement "Knowing the history of mathematics may assist students in learning mathematics concepts" was rated at 3.48. But an even more prominent result is that the teachers in the Yes group ranked the statement "All students of mathematics should be taught some history of mathematics" very high (mean of 4.14), but the statement "Knowing the history of mathematics may assist students in learning mathematics concepts" was ranked with a mean of 4.03. What would seem to be one of the major reasons for inclusion of HOM (that is, to make understanding mathematics easier), the statement "Understanding mathematics would be easier if the history of mathematics were taught" was ranked with a mean of 3.43. This would imply that the Yes group also did not view the learning of HOM as an aid to understanding mathematics.

A question can be raised as to why there was such a varied rating for these two statements. Either the teachers in this sample did not associate these statements or they did not think that the teaching of HOM would indeed assist the students to learn mathematics; that is, *make it easier.*

We suggest that teachers would benefit from more guidance and training about the values and use of HOM as a means to help students understand mathematics.

Factors That Affect Teachers' Decision

The factors that affected the Yes group's decision to include HOM are shown in Figure 1. The top two responses indicate that the Yes group enjoyed teaching HOM and believed that their students enjoyed learning it.

Enjoyment seems a very important factor to this group. The ratings of the third and fourth responses revealed that teachers view HOM as a means to help students see the development of and connections between mathematical concepts and that HOM affects their attitude toward mathematics.

However, the responses to the last statement prompt an idea that the teachers in the Yes group yet again separated mathematics from its history. They may not have seen that helping students understand the development of and connections between mathematical concepts and enhancing students' interest in mathematics are closely linked and that they make it easier for students to understand mathematics.

The factors that affect the No group are shown in Figure 2 in descending order.

Table 3: Comparison between groups Yes and No

	Perceptions of History of Mathematics	N	Mean	Std Deviation	Std Error Mean
I include History of Mathematics in my lessons	No	132	3.206	.684	.060
	Yes	201	3.824	.573	.040

Figure 1: Teachers' reasons for including HOM

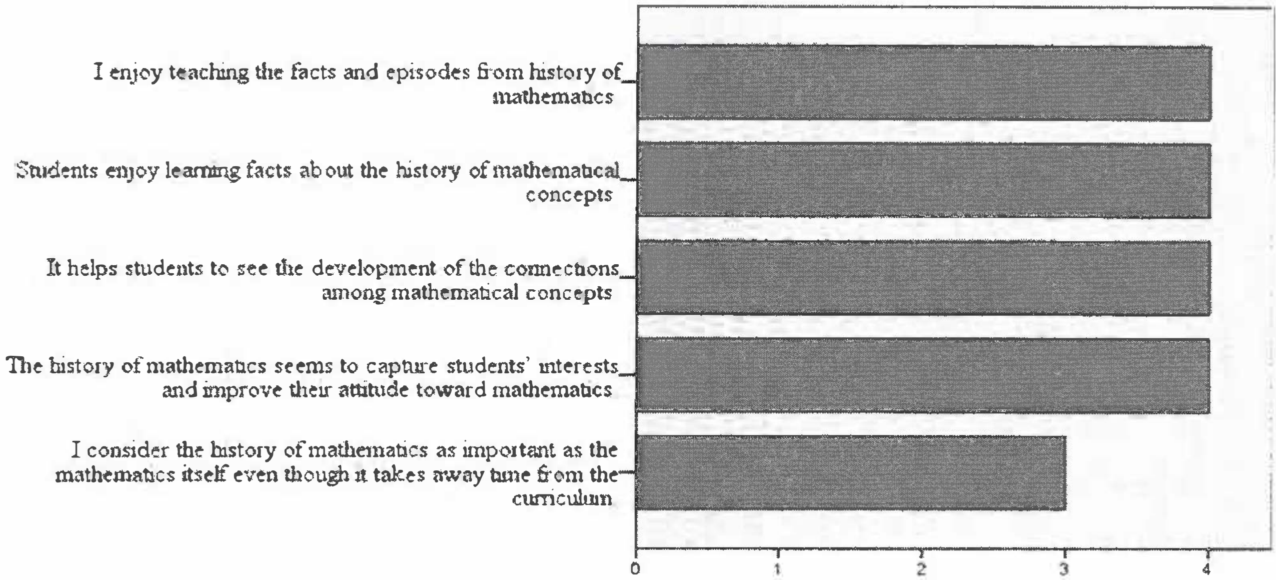
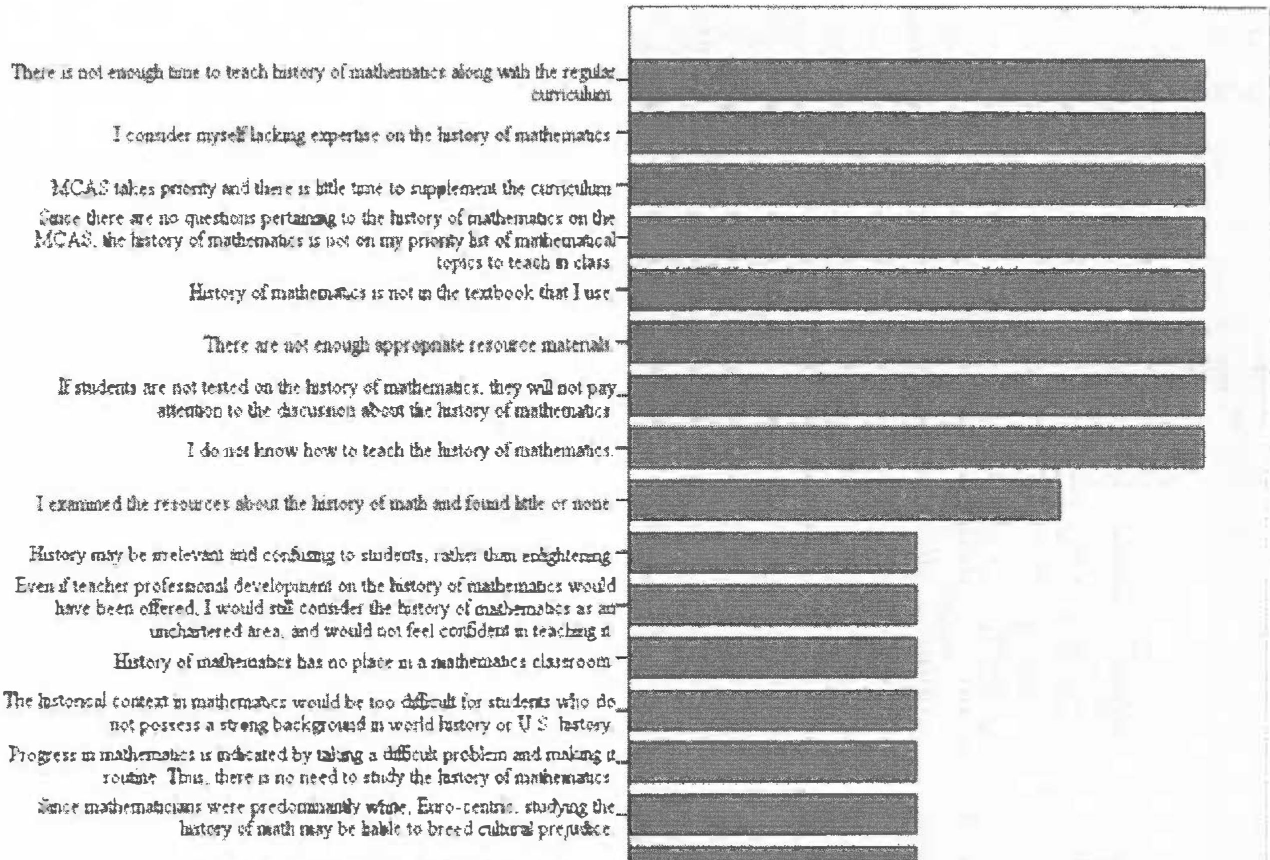


Figure 2: Teachers' reasons for not including HOM



The ranking of each statement indicates the extent to which each factor would deter teachers from including HOM. All top eight factors were notable and are addressed in the next section. However, two of the factors that suggest a lack of confidence (ie, "I consider myself lacking experience in the history of mathematics" and "I do not know how to teach the history of mathematics") prompted us to examine the relationship between a teacher's decision to include HOM and the number of courses on HOM taken by the teacher (see Table 4).

The results of the chi-square test of independence analysis ($p < .005$) indicated significance between the two variables, and thus a possible relationship between the number of HOM courses a teacher took and his or her decision to incorporate HOM into the curriculum. We have presented several factors that may influence a teacher's decision whether or not to include HOM in classroom instruction. Detailed analysis and discussion follow in the next section.

Discussion

After analyzing all constructs and factors examined so far, we offer several speculations and suggestions.

Time and High-Stakes Testing

Mathematics teachers in the United States are expected to teach (not just "cover") and their students are expected to learn (not just "go over") a mile-long curriculum. Thus, the shortage of class time is certainly a serious concern to many teachers when it comes to supplementing the curriculum. This concern is amplified by the advent of the new Common Core Standards for classroom instruction (www.corestandards.org). Adjustment to the new curriculum and accommodation of the new mandate, as it is anticipated, will take a substantial amount of time. And since nowhere in the Common Core Standards is there a mention of students learning HOM, it is unlikely that teachers will be able to consider the inclusion of HOM as a supplement to their curriculum, except perhaps for some true enthusiasts. One participant wrote

> I assign my students a brief presentation on the biography of the mathematicians we talk about in class. Many [students] like it; particularly those who are liberal-arts inclined; they write nice essays and show more interest in math. (David K, seven years of teaching Grades 9–12)

Until education policy makers understand the value of learning HOM at all grade levels and supplement the common core standards with their own requirement that students become familiar with HOM, it is doubtful that it will be widely and consistently incorporated into an already full curriculum.

States' high-stakes testing was also indicated as one of the primary reasons that teachers decide to not teach the history of mathematics. Obviously, high-stakes testing is a priority, and unless HOM is included as an essential practice standard in state frameworks, most teachers would either not include HOM or would struggle with how to sacrifice or alternate the curriculum to include it.

The highly ranked statement (mean of 4, see Figure 2) "If students are not tested on HOM, they will not pay attention to the classroom discussion about the HOM" reveals a pedagogical misconception. Such a pragmatic view is probably based on a teacher belief that all learning experiences must necessarily be directly related to performance and immediately and

Table 4: Summary of the number of courses on HOM for groups Yes and No

			How many courses on history of mathematics have you taken?			
			0 courses on history of mathematics	1 course on history of mathematics	2+ courses on history of mathematics	Total
I include the history of mathematics in my lessons	No	Count	80 (80/331)	43 (43/331)	7 (7/331)	130
		% of Total	24.2%	13.0%	2.1%	
	Yes	Count	90 (90/331)	79 (79/331)	32 (32/331)	201
		% of Total	27.2%	23.9%	9.7%	
Total		Count	170	122	39	331
		% of Total	51.4%	36.9%	11.8%	100%

unswervingly measurable. However, one cannot expect a straightforward and instantaneous relationship between learning HOM and performance in mathematics. Factors such as motivation, appreciation and enjoyment are critical to facilitate engagement and interest in mathematics, which are, in turn, prerequisites for better performance.

> When I teach distributive law in algebra, I always tell the students about Euclid's geometric algebra and the ways to multiply binomials with a diagram, and other diagrams that show algebraic identities, which I found in Euclid's *Elements, Book II*. It is amazing how many useful diagrams that help students visualize math can be found in Euclid. (Linda M, five years of teaching Grades 9 and 10).

Teacher Knowledge of History of Mathematics

"Confidence in knowledge and teaching history of mathematics" is another reason why teachers chose to not include HOM. The statement "I consider myself lacking expertise on the history of mathematics" received a mean rating of 4 (Agree), and presents an important issue worth addressing. We offer two accounts that may be associated with the issue. First, teaching a subject requires content knowledge of the subject (Shulman 1986). HOM content knowledge does not need to be saturated with complex mathematics concepts; yet, to effectively include HOM into their instruction, teachers must have pedagogical content knowledge of how to make it understandable to others, including what makes it simple or difficult to learn. Second, research (eg, Ball 1988; Cooney, Shealy and Arvold 1998; Dutton 1962; Furinghetti, 2007; Philippou and Christou 1998) suggests that teachers teach in the manner in which they were taught. They could have subconsciously built in images of pedagogy from their instructors. Thus, the teachers' presentations of mathematical concepts are largely affected by their past experience as students. This implies that teachers who have had relatively little or no exposure to HOM as high school and higher education students may perceive themselves as lacking expertise and, consequently, do not expose their students to HOM.

We support the requirement of at least one course on HOM during mathematics teachers' preparation program. It would serve two important purposes to the teacher: (1) a course on HOM would contribute to the expansion of the repertoire of mathematical content knowledge and would increase teachers' confidence in their ability to teach HOM, and (2) a teacher who is aware of at least some mathematical history will likely be inclined to integrate stories about the twists, turns, trials and tribulations of math into the curriculum. Students will probably be excited to learn about the development of our numeration system, development of geometry, inventions of major constants in mathematics and so on. Thousands of years of HOM provide a massive number of fascinating tales that can be discussed in class to enrich learning:

> I like to tell my students stories about numbers and the origin of their names. For example, it was during the 1700s that π and e both got their names ... π is an abbreviation of the Greek word for perimeter ($\pi\varepsilon\rho\iota\mu\varepsilon\tau\rho o\varsigma$), or an abbreviation for periphery/diameter ... the first appearance of π began with William Oughtred in 1647 when he used π for circumference ... Then, in 1697, David Gregory made π represent the ratio of circumference to radius. In 1706, William Jones defined π as the ratio of circumference to diameter as is done today. The story of e getting its name is equally interesting. Leibniz originally named it b in 1690, but the name did not catch on. The name e was granted by Euler in 1731 and managed to spread. In this case, Leibniz usually goes unnoticed since the name b was never commonly recognized. However, some of his notation did last, especially in calculus. When I was planning lessons on slope, I learned that the earliest known use of m [slope] appears in 1757. It is not exactly known why the letter m was chosen for slope. However, it has been suggested that m could stand for *modulus of slope*. There are remarks that that the French word for *to climb* is *monter* ... Mathematicians like to use the first or the last letter of the words for symbols ... for example, x is the last letter in the word *radix*, which is *root* in Latin. (John B, novice high school teacher)

One can also assume that with increased knowledge of mathematics and its history, teachers will be able to present mathematics as a coherent system of knowledge, rather than a collection of disjointed concepts.

According to Victor Katz (1997), the majority of the certification requirements for teachers at secondary schools in the United States include a course in HOM. However, the evidence presented in Table 4 indicates that over half (51.4 per cent) of the 331 teachers who responded to the survey question "How many courses on the history of mathematics have you taken?" had never taken a course on HOM. This prompts the question, How and why were the majority of the participating teachers certified with no courses on HOM? We offer one possible answer to this question. In an open-response question, a participating teacher commented, "... even though one of the requirements for my graduate degree included the history of math [*sic*]. That was conveniently wived [*sic*] for expediency." Allowing a preservice teacher to opt out of a course on HOM for the sake of expediency suggests that HOM is not important and is not going to be used in the classroom. We have no intention to generalize such a case; yet, the ramifications of such a decision need to be examined. No one can ascertain the extent of the benefits of teachers and students learning HOM. However, to date, there exists no empirical research that indicates that HOM is *not* beneficial to teachers' and students' construction of mathematical content knowledge. This returns us to the point made in the introduction that the history of sciences has become increasingly important to the teaching of the basic sciences.

From another point of view, it is interesting to note that 90 teachers (27 per cent of the total $N = 331$ who responded) who indicated that they had never taken a course on HOM included it in their classroom instruction. This group of teachers is representative of a point discussed previously. We speculate that teachers who believe that HOM is important and beneficial to students' construction of mathematical knowledge will likely include HOM regardless of their formal education.

> I have never taken any courses on the history of math but always include it in my classes. My students love it! I tell them how math has evolved through thousands of years from Babylonian to modern times, about the struggle and success of people who made math. They [students] like to hear about Pythagoras, how and why irrational numbers were discovered and not accepted at first. I think they learn irrationals better when [the material is] presented in a historical context. (Patrick S, 12 years of teaching in K–12)

Resources

The availability of resources is another factor influencing the teachers' decision to teach HOM. Two of the most highly ranked reasons for not including HOM were "History of mathematics is not in the textbook that I use" and "There are not enough appropriate resource materials." Time is a precious commodity for a teacher, and not having readily available grade-appropriate resources may explain why teachers do not include HOM. However, current technological advances provide amazingly easy access to an abundance of information related to practically any historical overview on any mathematics topic studied in the K–12 curriculum.

> I encourage my students to ask where a theorem or a rule came from. We talk about the observations that initiated the theorem and the mathematician who associated with it. (Tapasya D, five years of teaching Grades 9–11)

> I established a Mathematician of the Month to highlight accomplishments in math. Students like it. (Richard H, three years of teaching Grades 9 and 10)

> I use some of the online lectures by Judith Grabiner [professor of math at Pitzer College; eg, www.infocobuild.com/education/learn-through-videos/mathematics/math-philosophy/lecture-16.html] ... lectures about Euclid and non-Euclidean geometry are great. (Jenn S, eight years of teaching Grades 9–11)

We argue that professional development on the history of mathematics may serve as an effective learning experience by which mathematics teachers at all levels can be introduced to the teaching of HOM. Seventy-seven per cent of the participating teachers responded positively to the question about whether they would participate in professional development on HOM. Offering practical information on methods of integrating HOM will likely help answer the question of whether all students of mathematics should be taught some history of mathematics.

Conclusion

Teaching and learning HOM may benefit both teachers and students, and there are significant reasons for a requirement that teachers learn the history and nature of mathematics. The impact of the factors mentioned above may be far reaching. It is hard to argue against the humanizing benefit of HOM and its potential to strike a chord with the affective domain of learning (Krathwohl, Bloom and Masia 1973). Our data analysis shows that teachers who enjoyed teaching HOM believed their students enjoyed learning it. Many researchers (Fauvel 1992; Furinghetti 1997; Siu 2004; Smestad 2009) believe that HOM provides students with the opportunity to construct a personal, visual and emotional connection to the development of concepts. Following are recommendations that may lead to systematic integration of HOM in the classroom.

Teaching mathematics with no knowledge of its history is tantamount to a lawyer being allowed to practise law with no knowledge of the history of the judicial system, or scientists immersed in science without knowing its history. Higher education must not waver on the importance of knowing HOM—to do so would suggest that awareness of the evolution of mathematics concepts is not essential to understanding mathematics itself, and to teaching and learning of mathematics in particular.

There is strong need for students to view mathematics as a human creation that began thousands of years ago as an ever-changing body of knowledge. Students who view mathematics as a set of discrete, disconnected topics may have difficulty in understanding the relational worth of each mathematical concept or its attachment and value to human life.

In addition to undergraduate and graduate education in HOM, we suggest that systematic professional development focused on HOM would be an excellent way to introduce teachers at all levels to the resources for teaching HOM and a great opportunity for them to acquire the necessary pedagogical content knowledge and to build upon subject content knowledge. The National Council of Teachers of Mathematics (NCTM) has initiated a professional development scholarship emphasizing the history of mathematics (NCTM 2011); its goal is to

> provide financial support for (1) completing credited course work in the history of mathematics, (2) creating and field-testing appropriate classroom activities incorporating the history of mathematics and (3) preparing and delivering a professional development presentation to colleagues. (NCTM 2011)

In summary, including HOM as part of classroom instruction requires community effort. It requires teachers to reconceptualize the importance of learning the evolution of the mathematics concepts. It requires higher education instructors, administrators and parents to expect that teachers are entering the classroom with strong background and confidence in their knowledge of HOM. For HOM to be included in the classroom, publishers will need to provide teachers with resources that allow for the inclusion of HOM. Finally, it will require the inclusion of HOM as a standard of mathematical practice for state frameworks. If teachers were supported by the states' mandates or recommendations, they would not be alone in their efforts to integrate the history of mathematics into their classrooms.

Note

1 Available at www.corestandards.org (accessed January 11, 2012).

References

Alken, L R. 1974. "Two Scales of Attitude Toward Mathematics." *Journal for Research in Mathematics Education* 5, no 2: 67–71.

Ball, D. 1988. "Unlearning to Teach Mathematics." *For the Learning of Mathematics* 8, no 1: 40–48.

Barbin, E, G T Bagni, L Grugnetti, M Kronfellner, E Lakoma and M Menghini. 2000. "Integrating History: Research Perspectives." In *History in Mathematics Education: The ICMI Study* vol 6, ed J Fauvel and J Van Maanen, 63–90. Dordrecht, Netherlands: Kluwer.

Charalambous, C Y, A Panaoura and G Philippou. 2009. "Using the History of Mathematics to Induce Changes in Preservice Teachers' Beliefs and Attitudes: Insights from Evaluating a Teacher Education Program." *Educational Studies in Mathematics* 71, no 2: 161–80.

Cooney, T J, B E Shealy and B Arvold. 1998. "Conceptualizing Belief Structures of Preservice Secondary Mathematics Teachers." *Journal for Research in Mathematics Education* 29, no 3: 306–33.

Dutton, W H. 1962. "Attitude Change of Prospective Elementary School Teachers Toward Arithmetic." *Arithmetic Teacher* 9, no 8: 418–24.

Fasanelli, F, A Arcavi, O Bekken, J Carvalho e Silva, C Daniel, F Furinghetti et al. 2000. "The Political Context." In *History in Mathematics Education: The ICMI Study* vol 6, ed J Fauvel and J Van Maanen, 1–38. Dordrecht, Netherlands: Kluwer.

Fauvel, J. 1992. "Using History in Mathematics Education." *For the Learning of Mathematics* 11, no 2: 3–6.

Furinghetti, F. 1997. "History of Mathematics, Mathematics Education, School Practice: Case Studies in Linking Different Domains." *For the Learning of Mathematics* 17, no 1: 55–61.

———. 2007. "Teacher Education Through the History of Mathematics." *Educational Studies in Mathematics* 66, no 2: 131–43.

Glaisher, J W L. 1890. "Opening Address in Section A (Mathematics and Physics) at the British Association." *Nature* 42, 464–68.

Ho, W K. 2008. "Using History of Mathematics in the Teaching and Learning of Mathematics in Singapore." In *Proceedings of 1st Raffles International Conference on Education*, 1–38. Available at http://math.nie.edu.sg/wkho/Research/My%20publications/Math%20Education/hom.pdf (accessed January 12, 2012).

Katz, V J. 1997. "Some Ideas on the Use of History in the Teaching of Mathematics." *For the Learning of Mathematics* 17, no 1: 62–63.

Kline, M. 1972. *Mathematical Thought from Ancient to Modern Times*. Vol 1. New York: Oxford University Press.

Krathwohl, D R, B S Bloom and B B Masia. 1973. *Taxonomy of Educational Objectives: The Classification of Educational Goals. Handbook II: Affective Domain*. New York: McKay.

Kuhn, T S. 1996. *The Structure of Scientific Revolutions*. 3rd ed. Chicago: University of Chicago Press.

National Center for Education Statistics. 2009. National Assessment of Educational Progress Mathematics Teacher Background Questionnaire. Available at http://nces.ed.gov/nationsreportcard/pdf/bgq/teacher/BQ09_teach_math_g08.pdf (accessed January 12, 2012).

National Council of Teachers of Mathematics (NCTM). 2000. *Principles and Standards for School Mathematics*. Reston, Va: NCTM.

———. 2011. "Professional Development Scholarship Emphasizing the History of Mathematics for Grades 6-12 Teachers." Available at www.nctm.org/resources/content.aspx?id=23623 (accessed January 12, 2012).

Philippou, G N. and C Christou. 1998. "The Effects of a Preparatory Mathematics Program in Changing Prospective Teachers' Attitudes Towards Mathematics." *Educational Studies in Mathematics* 35, no 2: 189–206.

Schram, P. S Wilcox, G Lappan and P Lanier. 1988. "Changing Preservice Teachers' Beliefs About Mathematics Education." In *Proceedings of PME-NA 11* vol 1, ed C A Maher, G A Goldin and R B Davis, 296–302. New Brunswick, NY: Rutgers University.

Shulman. L S. 1986. "Those Who Understand: Knowledge Growth in Teaching." *Educational Researcher* 15, no 2: 4–14.

Siu, M-K. 2004. "No, I Do Not Use History of Mathematics in My Class. Why?" In *History and Pedagogy of Mathematics: Proceedings of HPM 2004*, ed S Kaijser, 375–76. Uppsala, Sweden: Uppsala University.

Siu, M-K, and C Tzanakis. 2004. "History of Mathematics in Classroom Teaching—Appetizer? Main Course? Or Dessert? *Mediterranean Journal for Research in Mathematics Education* 3, no 1–2: v–x.

Smestad, B. 2009. "Teachers' Conceptions of History of Mathematics." Available at http://home.hio.no/~bjorsme/HPM2008paper.pdf (accessed January 12, 2012).

Stander, D. 1989. "The Use of the History of Mathematics in Teaching." In *Mathematics Teaching: The State of the Art*, ed P Ernest. 241–46. Philadelphia, Pa: Falmer.

Swetz, F J. 1994. *Learning Activities from the History of Mathematics*. Portland, Maine: Walch.

Swetz, F, J Fauvel, O Bekken, B Johansson and V Katz. eds. 1995. *Learn from the Masters*. Washington, DC: Mathematical Association of America.

Tapia, M, and G E Marsh. 2004. "An Instrument to Measure Mathematics Attitudes." *Academic Exchange Quarterly* 8. no 2. Available at www.rapidintellect.com/AEQweb/cho25344l (accessed January 12, 2012).

Tzanakis, C, A Arcavi, C Correia de Sa, M Isoda, C-K Lit, M Niss, J Pitombeira de Carvalho et al. 2000. "Integrating History of Mathematics in the Classroom: An Analytic Survey." In *History in Mathematics Education: The ICMI Study* vol 6, ed J Fauvel and J Van Maanen, 201–40. Dordrecht, Netherlands: Kluwer.

Whitrow, G J. 1932. "The Importance of the History of Mathematics in Relation to the Study of Mathematics Technique." *Mathematical Gazette* 15, no 220: 225–27.

Wilder, R L. 1968. *Evolution of Mathematical Concepts: An Elementary Study*. New York: Wiley.

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Learning Through a Penny Jar

Lee Makovichuk

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Having worked with communities of children for years, I know how competent and capable young children are. Recently, the work of the infant and preschool programs in Reggio Emilia has attracted the attention of educators across the globe to the notion of a competent and capable child—a strong child who is able to interpret his or her knowledge and experiences of the world (Edwards, Gandini and Forman 1993). As well, the North American Education for Young Children (NAEYC) association describes young children as having informal experiences that are the foundation for school learning when we help them to connect previous experiences with curricular outcomes (NAEYC/NCTM 2002, 2008). Yet, it is through our action as teachers that we communicate and make decisions to create a learning environment that either recognizes or ignores the capabilities and competencies of young children as contributors to their learning communities.

Underlying the tangible objects that adorn the classroom walls and shelves is a belief system about who the teachers are and who the learners are. Perceptible or not, this belief system is foundational to the creation of a learning environment. The Reggio Emilia environments are noteworthy in their use of natural materials: woven baskets, light, shadow, and spacious, airy environments that extend beyond the classroom walls and invite children to explore, create, invent and interpret the world (Edwards, Gandini and Forman 1993). Eagerly, these elements are transplanted into classrooms and elsewhere, assuming that the materials reflect a strong and capable image of the child. When materials are chosen without understanding our own beliefs about teachers and learners, the materials clutter our space and time, and consume our energy and budget dollars. The materials do not communicate our understanding of teachers and learners. Rather, it is in our knowing that the roles of the learner and the teacher evolve into dynamic, collaborative and unconventional ways that determine the intentional selection of meaningful materials that provoke engagement and interaction between those who live in learning spaces together. Through this belief I begin with groups of children, as a teacher and as a learner, to create an environment that reflects the knowledge that each child is competent and capable in the creation of our learning experience together. I do this through the process of pedagogical documentation. This process positions me as a learner working to understand what children already know and what the possibilities are, and as a teacher striving to deepen the children's creative invention and interpretation of their experiences in the learning community through asking questions that will invite children to make meaningful inquiries.

"Will We Count the Pennies?" Children's Initial Counting Strategies

In September, as I placed a penny jar on a window ledge at the children's height, I wondered, How will the children use these pennies? What kinds of ideas will be prompted? Where will those ideas take us in our learning together? Then I waited. It was December before I noticed that the children were interested in the jar. It was Sophie's question that awakened me to the children's interest in the pennies. She asked, "Will we count the pennies?" Her question came just days before the winter break—a busy time in the classroom as we prepared for a community celebration. I held on to the question, not wanting to dismiss it in the flurry of activity.

In January the regular pace of classroom life resumed and, with it, Sophie's question echoed in my mind. I decided to bring the question back to the children.

"Sophie, do you remember before winter break when you asked me about this penny jar?"

She remembered. "Yes, will we count the pennies?" With the question restated, a conversation developed with the morning class about how we could count the contents of the jar.

"You can skip count. Count by twos: two, four, six, eight," offered Rachel.

"We can each take some and count it," added Esther.

"We can make a pile and move the counted into another pile and call it the counted pile," reasoned Flynn.

"We can count all of the jar!" Garrett exclaimed.

"We can count some and then he (pointing to Cole) can count some and he (pointing to Pablo) can count some until it's all counted," confirmed Errol.

"How many do you think are in the jar?" I asked, wanting to extend this conversation further.

Flynn: "Two million."

Nikos: "A kazillion."

Pablo: "One thousand."

Rachel: "One hundred."

Garrett: "Nine."

Following this initial conversation, I considered possible next steps. I was surprised that the conversation engaged few children. MacNaughton and Williams (2009, 116) explain, "When you listen to someone ... you concentrate on what is said and what is not said; you note what they are saying and not saying and think about it carefully." With this in mind, I prepare counting mats (11 × 14-inch construction paper), a camera and note paper to gather further ideas about counting from the group. I thought that perhaps the children could demonstrate their ideas for counting more easily than articulating their strategies in words. The following day, I invited them to talk about their counting strategies once again.

Devon: "Count one, two, three, four until they are counted."

Nikos: "Share the pennies and everyone count them."

Flynn: "Move the pennies that you count."

Rachel: "Give the same to each kid and everyone count."

Garrett: "One, two, three, four, five, six, seven, eight, nine, ten."

Pablo: "Count to a hundred."

Errol: "Split into groups and move your pennies when you count them."

Malak: "Each take a penny and take turns until all the pennies are counted."

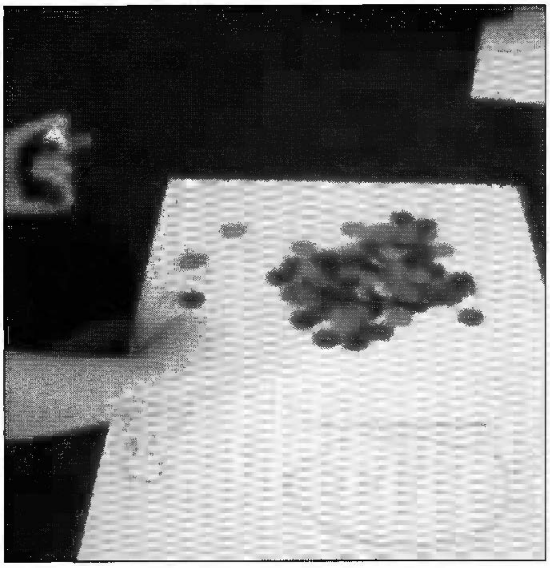
Tatum: "Take them out and put them in a line, then count."

Sophie: "Take groups of pennies. Then count them. Then add them up."

After a few more children expressed their ideas, I asked them to show me how they count. I assured them that there was no wrong way and that I was really interested to learn about their way of counting.

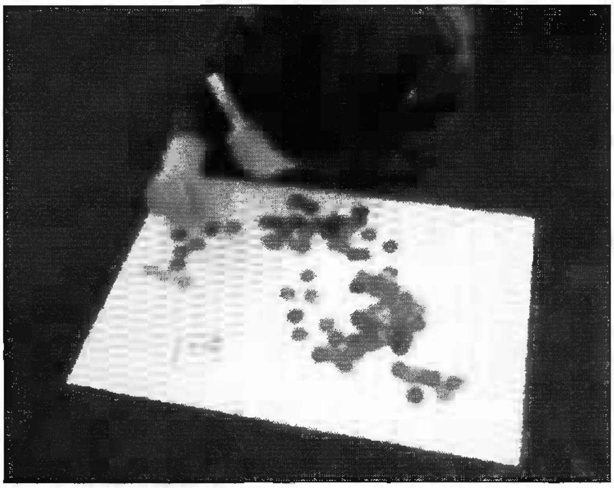
Eagerly, the children began. As promised I photographed and took notes, recording how each child approached the task. The following examples demonstrated a variety of approaches children used to count an unknown group of pennies:

Figure 1. Counting in 10s.



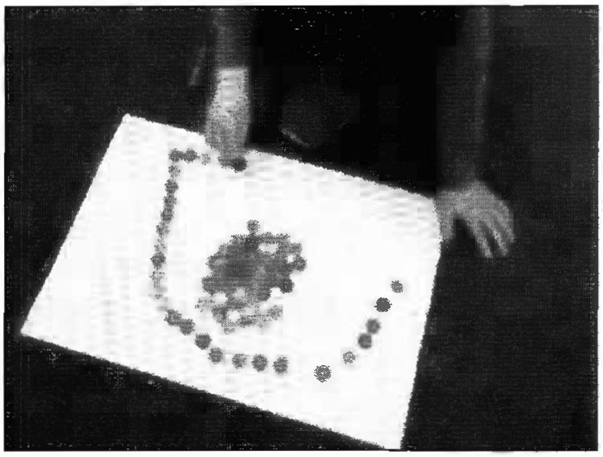
Jared explained, "I can count to 10." He drew the number 10, and then drew 10 circles (Figure 1). He placed a penny in each circle, sliding them across the paper into a pile on the right side. He then repeated his "counting 10" process until all the pennies were in the pile on the right side.

Figure 2. Using estimation.



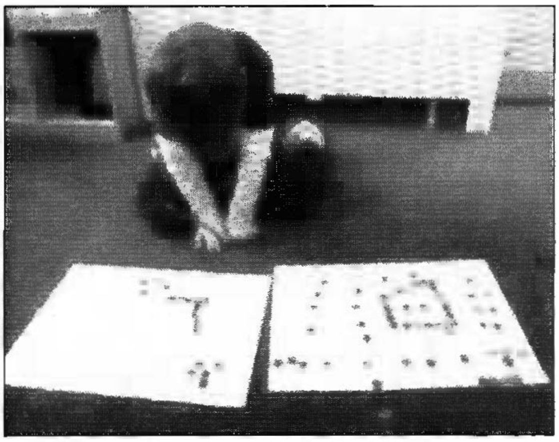
Mark estimated how many pennies he had in his pile. He recorded 100 on his counting mat (Figure 2). He counted each penny as he slid them across his paper. He ended his count at his estimated number.

Figure 3. A solution to a problem.



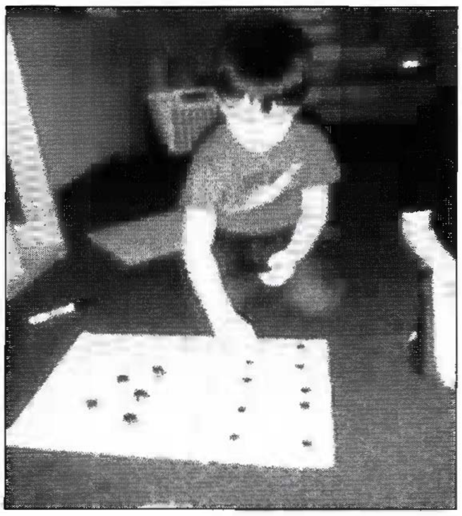
Tatum began by numbering each penny as she slid them into a line (Figure 3). Getting to number five, she recognized, "I don't know how to write five." I encouraged her to think of a way she could solve her problem. After several moments, she began a new strategy by lining the pennies around her paper. Once she had the pennies lined up, she began touch counting. As she neared her starting point, she stopped, thoughtfully placed her other hand across the approximate place she started and continued her count to 33.

Figure 4. Drawing and counting.



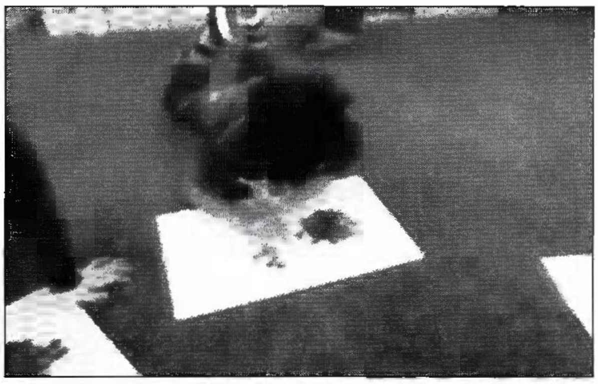
Garrett made a square shape with some pennies and drew a square around those pennies (Figure 4). He placed two pennies at the bottom edge of the square and drew a circle around each. He named it, "penny car." He drew another shape he named, "a tower" and placed pennies inside the tower. He drew an airplane and placed pennies inside the airplane. He repeated this process, drawing many shapes, filling each shape with pennies until he had a very small pile of pennies. He counted the pennies in his pile, 1 to 11, drew an "island" and placed all 11 pennies on the island.

Figure 5. Expanding on another's idea.



After listening to Jared explain his approach to counting 10 (Figure 1), Mark was excited. He drew 10 circles on one side of the counting mat he had divided in half (Figure 5). He placed a penny in each circle, collected the group of 10 and placed them on the other half of his mat. He drew a circle around the group of 10. He repeated this process making 9 groups of 10 pennies. He asked, "How many pennies do I have?" Together, we counted by 10s.

Figure 6. Counting in Portuguese.



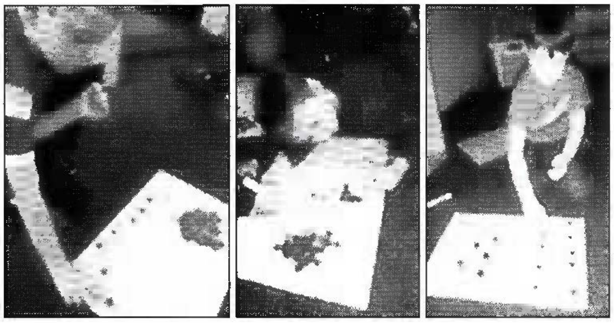
Nikos slid each penny from one pile into another pile and counted in Portuguese (Figure 6).

To move forward with the children in their meaning making, I reflected on their engagement with materials and ideas. I was reminded, "When we document we are co-constructors of children's lives, and we also embody our implied thoughts of what we think are valuable actions in a pedagogical practice" (Dahlberg, Moss and Pence 2007, 147). My reflections on what I have listened to—hearing and seeing how the children have counted (shown in the Figures 1–6)—contribute to my understanding of the big ideas that will frame further learning. I know that what I have observed and listened to is limited by my own lens. I am looking and listening to the children's mathematical thinking. Through another lens, I might see and hear something else. In the context of pedagogical documentation Forman and Fyfe (1999, 240) explain that "the curriculum is child originated and teacher framed."

Exploring Mathematical Possibilities with the Children

I framed several mathematical possibilities for further exploration, as shown in the photos and children's documented words in Figures 7–10.

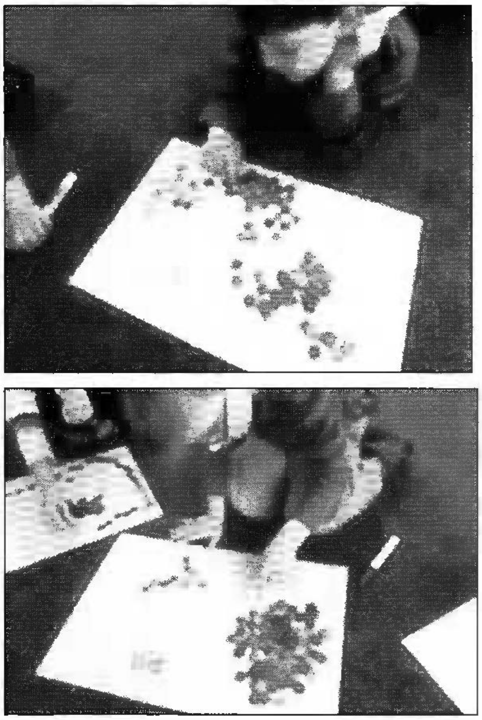
Figure 7. Emerging ideas of skip counting.



Rachel: "I counted by twos: two, three, four, five, six, seven."

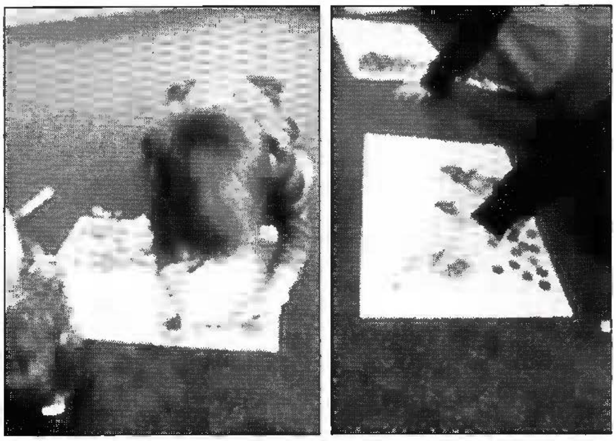
Josh: "Skip count, like two, four, six, seven, eight, ten."

Figure 8. Emerging ideas of estimation.



Sophie: "One million."
Sophie: "One hundred."
Safi: "Twenty."
Tatum: "Two million."
Malak: "One million, one hundred."
Errol: "Nine hundred."
Spencer: "One million."

Figure 9. Emerging ideas of place value thinking.



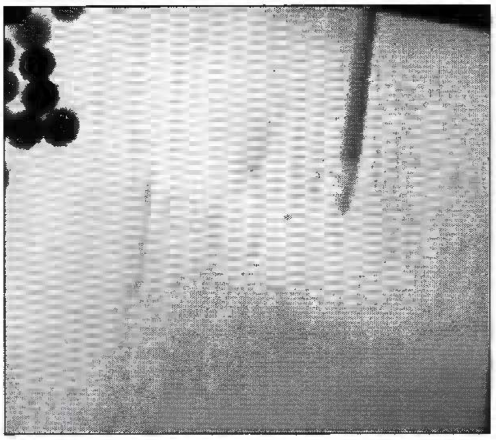


Figure 10. Story of 12.

While ordering the numbers on the February calendar, the children searched each other's calendar number cards to locate the number 12. To support their search, I explained as I wrote on the whiteboard, "Twelve is a ten (1) and a two (2)." Devon laughingly commented, "Lee, you said a ten and a two, not a one and two."

In addition to my learning about each child's approach to counting, my learning through a master's level math course had implications for deepening the learning experiences in the classroom. My awareness of how mathematical ideas are communicated between teacher and learner has been heightened;

Moseley (2005, 385) uses the term *math-mediated language* to describe the process that occurs between teacher and student(s) in creating and communicating mathematical understanding. Through understanding the children's mathematical theories and my knowledge of pedagogical documentation, I form a question. The pedagogical question sets the stage for children to explore, create and think aloud as they make meaning and build personal knowledge in meaningful ways. Although framing questions guide further exploration and learning paradoxically, the questions limit the lenses through which we observe and listen. Therefore, as I ask a question that furthers children's mathematical learning, I recognize that I might not hear beyond the boundaries of my question.

Pedagogical activity can be seen as a social construction by human agents in which the child, the pedagogue and the whole milieu of the early childhood institution are understood as socially constituted through language. However, this perspective also implies that this activity is open to change; if we choose to construct pedagogical activity in one way, we can also choose to reconstruct it in another (Dahlberg, Moss and Pence 2007, 144).

By intentionally listening to the children's engagement of counting strategies and revisiting the collection of data (Figures 7–9), I became aware that the pedagogical question can take many avenues. I might have formed a question that focuses on children's knowledge of object counting, skip counting or estimation; however, as the children ordered the numbers for the February calendar (Figure 10), my curiosity was ignited. I decided to proceed with further learning in the pedagogical question, "What do children know about place value?" In doing so, I considered the possibilities for learning that can occur for the whole group:

- Object counting and skip counting are concepts that the children will explore as we investigate their understanding of place value.
- Experience counting groups of objects will help the children develop an understanding of estimation.
- Alberta Education's program of studies (2009) does not include place value formally until Grade 3.
- Alberta Education program of studies for kindergarten (2009) focuses on number and spatial sense through developing children's personal meaning and competencies in "communication, connections, mental math and estimation, problem solving, reasoning, technology and visualization" (p 17).

As well, children arrive in kindergarten having had an abundance of mathematical experiences that we can and should build upon:

> Children's confidence, competence, and interest in mathematics flourish when new experiences are meaningful and connected with their prior knowledge and experience. At first, young children's understanding of mathematical concepts is only intuitive. Lack of explicit concepts sometimes prevents the child from making full use of prior knowledge and connecting it to school mathematics. Therefore, teachers need to find out what young children already understand and help them begin to understand these things mathematically. (NAEYC 2008, 4)

Teachers must understand the complexities of the concepts that they explore with children. My own investigation of place value helps me to understand that many children "fail to differentiate between the face value of each symbol in a number and the complete value of the same symbol" (Varlas and Becker 1997, 265). As well, Clements and Sarama (2009) highlight language as a factor in understanding base 10 numbers. Whereas English language users use the suffixes "teen" and "ty" to identify 10, Chinese language users read numbers 10-1, 10-2 and so forth, which is more helpful for children's conceptual understanding of numbers beyond 10.

Can Kindergarten Children Understand Place Value? Challenges Presented by Numbers 11 and 12

Understanding the conceptual difficulties of place value learning and language meaning, I considered a way to invite children to explore the number 12. With Devon's approval to share the story—the exchange between him and me during the building of the February calendar (Figure 11)—I gathered the penny jar, counting mats, audio recorder and cameras.

I wrote the number 12 on the whiteboard. "What is the one in 12?" I asked.

Many of the children called out, "One."

I pointed to the two and explained that two means two, holding up two fingers. "If two means two and this one means one—we know that one plus two equals . . ." I paused.

The children confirmed, "Three."

I wondered aloud, "What is the one in 12, then?"

"It's one of something," Cole offered.

"Yes it is, but what is the something?" I wondered.

"We can discover what the one is. Let's each take 12 pennies," I explained as I modelled with the pennies. "We know that the two is two, so I am going to move these two pennies to one side. Now I have some pennies left; what do you think I should do?"

"You can count them," Devon offered.

"Okay, let's all try this." With a counting mat and 12 pennies in hand, the children set about to think and explore the 1 in 12. I take on the role to facilitate children in their exploration, observe their processes, record what they do and say using digital photography, audio and video.

Reviewing a recorded videoclip of Errol and Mark working beside each other, I see that each has two pennies off to one side and another group of pennies in the centre of his mat. Mark told Errol, "Count them" (referring to the group of pennies in the centre of the mat). Errol touch counted as Mark looked on, "One, two, three, four, five, six, seven, eight, nine, ten."

Mark remarked, "Let me count them." He counted, "One, two, three, four, five, six, seven, eight, nine and ten."

Continuing to take the video footage, I prompted, "What does the one mean?"

Mark paused momentarily, "Hum, 10! It's 10!"

He reasoned, "One means 10 and two means two."

I asked, "What is 10 + 2?"

Mark concluded, "Twelve!"

Videos, photos and recorded dialogue archive the children's exploration. The images and recorded children's words in Figure 12 allow me to see that some children had been exploring number composition. A few children continued to explore object counting and a few children were beginning to understand that the 1 in 12 was, in fact, a 10.

Spencer: "Look five and seven make 12."

Sophie: "It makes 10 if I take two."

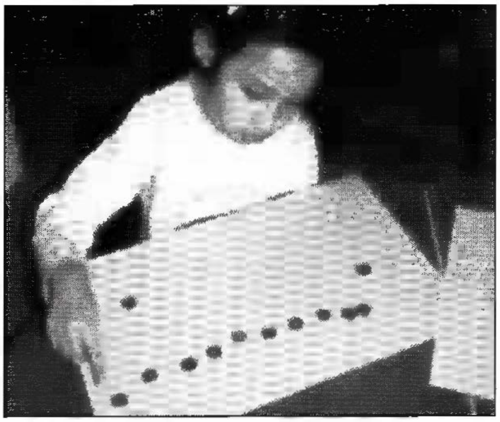
Shivani: "Six and six are 12."

Flynn: "I took two away and made 10. The one is first."

Malak: "I put all the pennies in a row and counted them. They were 10."

Garrett: "I counted 12."

Figure 11. Exploring the 1 in 12.



I knew that another opportunity to explore two-digit numbers was important. The following day, I asked the children about number 11. In much the same way as I posed the problem of 12, I wondered what the ones in 11 represented. Through this exploration, only one child demonstrated an understanding that 11 is a 10 and a 1. The responses of many children, as shown in Figure 12, left me wondering if place value was an appropriate direction for their mathematical learning.

Figure 12. Confusion with 11.

What Is Two in Twenty?

Reflecting on the children's confusion over number 11, I considered that the double 1s in 11 were problematic. It is confusing to think about the numeral 1 as both a 10 and a 1 this early in the exploration. With this in mind, I considered exploring number 13 with the afternoon class. Before we could begin, Isaac declared, "I want to do all the numbers up to 20!" As I introduced the number 13, Minh confidently explained, "The 1 is 10 and the 3 is 3!" I then wrote 14 and the children chanted, "The 1 is 10 and the 4 is 4." I wrote 15 and again the children chanted, "The 1 is 10 and the 5 is 5." I wrote 20 and asked, "What is the 2 in 20?" The children paused. I explained that like the 2 in 12, which means 2, the 0 in 20 means 0. I suggested that the 2 in 20 means 2 of something the same, and that if they drew a line down the middle of their counting mat that might help them to think about 2 numbers the same.

Once again, I am observer, recorder and facilitator of the children's engagement with the challenge of 20. This time, I collect the children's thinking processes in video, and they record their ideas on their counting mats as seen in Figure 13.

In the recorded videoclips of the children's exploration of 20, the following interactions are documented:

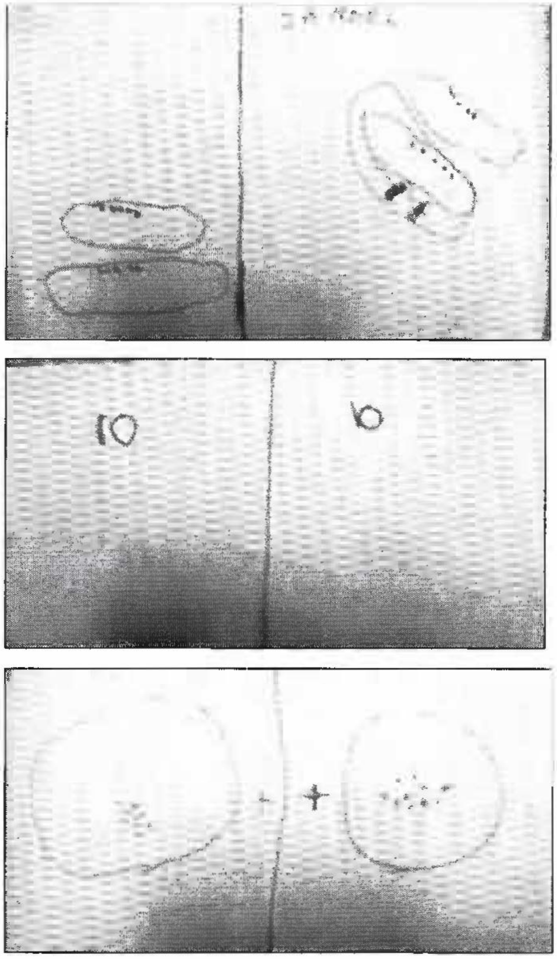
Josh draws my attention: "I figured it out."

I asked him to explain his strategy to another group working at an adjacent table. He explained, "I tried to make 10 on each side."

I prompted, "Can you tell Jared?"

Jared explained, "I got nine plus nine."

Figure 13. Exploring 20.



I asked, "What is nine plus nine?"

Jared replied, "I don't know." He begins to count the pennies on his mat.

Josh interjected, "Eighteen."

Josh explained to Jared, "If you put 10 on each side, you make 20."

Jared explained, "No if I put 10 on this side then they aren't the same. It will be 10 and 1, 2, 3, 4, 5, 6, 7, 9. It would be 10 and 8."

Realizing that perhaps Jared was only working with 18 pennies rather than 20, Anh (a colleague) helped him to adjust his pennies to 20, and I moved over, asking Mark what he has discovered.

Mark explained, "I got 10 on each side."

I asked, "What is the 2 in 20?"

Mark searched, "Two zeros, two tens, two pennies?"

I wondered along with him, "Two pennies?"

Daveed exclaimed interrupting us, "Ten here and 10 here."

I inquired, "What does 10 plus 10 make?"

Daveed replied, "Twenty."

I prompted, "What does the 2 in 20 mean?

Daveed and Josh, confirmed, "Ten!"

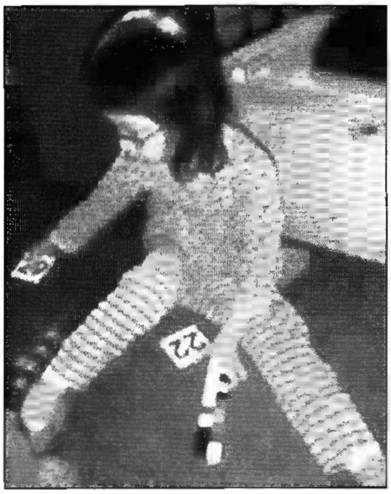
In another videoclip I see Daneel working. He has two circular shapes drawn on each side of his counting mat. He explains, "Five plus here, five plus here, five plus here, five plus here. Five plus five makes ten. Ten plus ten makes twenty. Only two tens and no more."

After I revisited the collected documentation, I came to believe that the afternoon group of children had developed an understanding for place value 10. The morning group of children had been exploring several concepts, including place value, composition of numbers and object counting. In both groups, I felt that my question (What do children know about place value?) and the manner in which we explored those queries engaged every child in a way that challenges them and engages their thinking within the scope of meaningful learning. My next thoughts were on generalizing the idea of base 10. Could the children recognize tens in other numbers? I introduced the hundreds chart, which prompted Minh to explain his theory about two-digit numbers. As he pointed to the number 83, he explained, "The first one has some tens in it and the second number doesn't—it's the regular number."

Building Numbers

To further explore this idea, I introduced an idea to build numbers. Using familiar materials explored in previous contexts, the Unifix blocks and calendar numbers 11 to 31, I proposed a game called building numbers. The rule of the game is that you cannot build a tower taller than 10. I imagined that children would want to build towers as tall as possible, so I stipulated a limit to promote the concept of place 10. Many of the children explored this game as I imagined, as seen in Figure 14.

Figure 14. Building numbers.



A videoclip of the activity revealed that the teacher–child interaction challenged the tension between what we intend and what children interpret.

I directed my attention to Sophie, "Can you show us what you have?" Sophie counted a tower of nine. In response, I asked the whole group, "What should Sophie do?"

Cole said, "Add one more." Removing a single block from another tower, Sophie added the block to her tower in question, making a tower of 10 blocks. I explained, "Sophie's number is 31. Sophie, tell us what your next tower is."

She counted, "Eight." She then counted her next tower, "One, two, three, four, five, six, seven. Seven." She counted her last tower, "One, two, three, four, five, six." She looked at me.

I summarized for the whole group, "Sophie made a tower of ten, a tower of eight, a tower of seven, and a tower of six." I probe further, "Can she make any more towers of ten?"

Sophie responded, "No, because it would make more (than 31 blocks). She then counted the blocks in each tower to confirm her count.

I concluded, "Sophie made her number 31 a different way. Cole made three towers of 10 for the number 30, and Sophie made 4 towers, 10, 8, 7 and 6 for 31."

Sophie commented, "Almost the same."

In another videoclip, Isaac explained, "I got 31." Together we described, "Ten plus 10 plus 11 makes 31."

Although not documented in the penny jar experience, through dialogue with colleagues I noticed my use of evaluative language as I described the children's engagement with the pennies. Unintentionally, I used evaluative language to describe the children who were exploring number composition or object counting and the children involved in working through the problem of place value. "The afternoon children are showing a strong understanding of place value, but most of the morning children are only exploring composition of number and object counting." Evaluative language places importance on one in relation to others. Taguchi (2008, 272) explored "deconstructive talk as a tool in the displacement of dominant or taken-for-granted ways of thinking and doing." Her goal was to "search for ways to understand childhood and learning that work with and make use of—rather than muting—the complexities, diversities and multiplicities arising from different contemporary theoretical perspectives on childhood, child development and learning." Listening to myself in dialogue with colleagues gave me an opportunity for learning and challenged me to attend to my use of evaluative language and clarify my ideas—my perspective reflected in my practice with children.

Interactions with Children: Reflections on My Teaching

The opportunity to revisit documentation gave me a backward glance to listen to my interaction with Sophie and her construction of the number 31. Moments of teacher–student interaction captured in video allowed me to see another hidden bias. I saw myself working with a young girl who had built 4 block towers that amounted to 31. In this learning experience, I had proposed a game for the children to build numbers using connecting blocks, with the stipulation that the towers could not be more than 10 blocks high. My intention was that children would build as many towers of 10 as each number allowed. This particular child had built towers of 6, 7, 8 and 10 blocks to construct the number 31. I saw in my tone and probing questions that I viewed her solution as lacking, even asking the whole group, "Can she construct any more towers of 10?" By reviewing this example I recognized that, in fact, she had achieved an original solution to the game that I had posed to the group, yet my response had not celebrated her creativity. Looking back on the video I saw my bias. I valued the children's responses that correlated with my question and undervalue a solution that was different from the one I had in mind. Upon further reflection, I have come to understand that the pedagogical question is the teacher's question. When teachers ask pedagogical questions, we cannot necessarily assume that the children will take up the question along with us.

Returning to Sophie's question, we counted the pennies in the jar. Interestingly, as we began this phase of the work with the penny jar, the children gave estimates, no longer fantastical in nature, but rather predicative, such as, 1,000, 600 and 960.

In the months that followed the penny jar exploration, I continued to notice the presence of mathematical language in the children's conversations each day. Upon our return to school in April after spring break, Nikos mused, "What is the 3 in 31?" Cole and Flynn took up the challenge and reported back at the end of the morning. On another day, Sophie and Flynn watched as Devon recorded the day's temperature on the whiteboard. As he wrote +11, Sophie commented, "One, one." Flynn reminded her, "No, remember it's a 10, 1, it's 11. Plus 11." Sophie replied, "That's right. It's 10 and 1. That makes 11."

Taguchi (2008) reports that the teachers with whom she worked to explore deconstructive talk realized that there was no going back to old ways; rather, they were "ethically obligated to re-examine [their] practices, always looking for better ways to 'do good' for the particular children with whom [they] were working" (p 280). Deep in my knowledge of who I am as a classroom teacher, this is true for me. Remarkably, in 1963, Sylvia Ashton-Warner described a teaching practice we continue to strive for—a practice that begins with and focuses on the cultural experiences of children. At a time when dominant Western European views were imposed on cultures considered less developed, less desirable, Ashton-Warner was a teacher of the five-year-olds of the Maori infant rooms. I am inspired by Ashton-Warner's approach to generating a "key vocabulary" inviting each child to contribute personally significant words toward creating a classroom vocabulary list. These words became the material for developing printing skills, handwriting and reading skills, and eventually created Maori readers. I consider Ashton-Warner to be both skilled and courageous—a teacher who listened intently and created a curriculum that connected children, their lived experience and their learning experience. As well, Paley (1997) speaks of one particular kindergarten student, Reeny. Through developing a deep connection for Leo Lionni's story character Frederick, Reeny breathes life into each newly introduced story character through a class study of the author's work. Reeny's passion propels each class member to take up the journey along with her, which transforms the class identity. Paley observes,

> I too require passion in the classroom. I need the intense preoccupation of a group of children and teachers inventing new worlds as they learn to know each other's dreams. To invent is to come alive. Even more than the unexamined classroom, I resist the uninvented classroom. (p 50)

I share Paley's sentiments—a classroom that breathes originality and creativity is a listening classroom—listening with intention to learn, to create and to invent. It is in these classrooms that the would-be readers that Bruner (2000) speaks of find themselves in the world of possibilities. When we attend to children's experiences in school and to their experiences in the world, we create together that which cannot be packaged and duplicated elsewhere.

Teaching and learning are highly complex processes, and it is through closely attending—through listening with intention to learn that which is not yet known—that it becomes an art, complex and evocative, mindfully open to possibilities. It is this notion that I take with me, into the classroom with children, listening for possibilities, with a keen awareness of how the context, materials, learners and teachers are entwined in the process of creating multiple interpretations. Pedagogical documentation situates me, the teacher, as a learner—willing to reflect on and refine my practice toward understanding what each child knows and can do and how I determine further learning experiences for children, with children.

Upon opening this article I focussed on the potential superficiality of transplanting the Reggio Emilia context into a North American classroom. This is a real hazard when our goal is understanding our own beliefs about children and learners and how classroom practice communicates our beliefs. I hold the highest regard for the work of the teachers and children of the infant and preschool programs of Reggio Emilia—not for what I can duplicate, but for what I can learn about my own practice through what they have shared about theirs.

References

Alberta Education. 2009. *Alberta Education Curriculum: Kindergarten Curriculum Handbook*. Alberta Education website. www.lrsd.ab.ca/programs/curriculum/Pages/default.aspx (accessed July 18, 2011).

Ashton-Warner, S. 1963. *Teacher*. New York: Simon & Schuster.

Bruner, J. 2000. "Reading for Possible Worlds." In *National Reading Conference Yearbook* 49, ed T Shanahan and F V Rodriguez-Brown, 31–40. Chicago: National Reading Conference.

Clements, D, and J Sarama. 2009. *Learning and Teaching Early Math: The Learning Trajectories Approach*. New York: Routledge.

Dahlberg, G, P Moss and A Pence. 2007. *Beyond Quality in Early Childhood Education and Care: Languages of Evaluation*. 2nd ed. New York: Routledge.

Edwards, C, L Gandini and G Forman. 1993. *The Hundred Languages of Children*. New Jersey: Ablex.

Forman, G. and B Fyfe. 1999. "Negotiated Learning Through Design, Documentation, and Discourse." In *The Hundred Languages of Children: The Reggio Emilia Approach–Advanced Reflections*, ed C Edwards, L Gandini and G Forman, 239–60. Westport, Conn: Ablex.

MacNaughton, G, and G Williams. 2009. *Techniques for Teaching Young Children: Choices for Theory and Practice*. 3rd ed. Frenchs Forest, NSW: Pearson Education Australia.

Moseley, B. 2005. "Pre-Service Early Childhood Educators' Perceptions of Math-Mediated Language." *Early Education and Development* 16, no 3: 385–98.

NAEYC/NCTM. 2002, 2008. *Early Childhood Mathematics: Promoting Good Beginnings*. A joint position statement of the National Association for the Education of Young Children (NAEYC) and the National Council for Teachers of Mathematics (NCTM).Washington, DC: NAEYC.

Paley, V G. 1997. *The Girl with the Brown Crayon*. Cambridge, Mass: Harvard University Press.

Taguchi, H L. 2008. "An 'Ethics of Resistance' Challenges Taken-for-Granted Ideas in Swedish Early Childhood Education." *International Journal of Educational Research* 47, 270–82. doi:10.1016/j.ijer.2008.12.006.

Varlas, M, and J Becker. 1997. "Children's Developing Understanding of Place Value: Semiotic Aspects." *Cognition and Instruction* 15, no 2: 265–86.

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Sharing Cake with King Lion: A Literature-Based Lesson for Grade 4

Sarah Danielsen

This article is intended as a lesson plan; it addresses several outcomes in the number strand for Grade 4 mathematics through the use of a children's story, *The Lion's Share: A Tale of Halving Cake and Eating It, Too*, by McElligott (2009). There is ample research currently available that emphasizes the effectiveness of using children's literature in mathematics classes across all grades, and the natural progression of moving from stories to specific skills and concepts is evident to professionals who work with students (Martinez and Martinez 2000, 55). The lesson plan concludes with a detailed description of the mathematical processes addressed.

Synopsis

Ant has been invited by the king, Lion, to join him and his animal kingdom guests for a special dinner party. Once there, Ant is shocked at the other guests' behaviour; they are acting like atrocious animals! When the guests take more than their fair share of dessert, Ant offers to bake the king a delicious cake for the next day, an offer that the animals see as an opportunity to doubly outdo one another.

Mathematical Concepts

This book explicitly and effectively addresses the mathematical concepts of multiplication and division. The story begins by the animal guests taking half of the cake placed in front of them, no matter how little. For students who are being introduced to the concept of division, this would be an effective story to use for a discussion regarding fair sharing—"the idea of evenly distributing a quantity is the conceptual basis for division" (Baroody and Wilkins 1999, 59). The discussion would direct students' attention to the animals' misconceptions about taking half the cake without taking into consideration the number of animals left that must also have a piece.

The concept of multiplication is also clearly addressed, as illustrated in the following lesson plan. It is an effective story to introduce the process of multiplying one-, two- and three-digit numerals by 2, as each animal wishes to outdo the last by baking twice as many cakes. This lesson could easily be adapted to include multiplying by larger numbers such as 3 or 4. The nature of this story facilitates the discussion of the inverse relationship between multiplication and division.

In later grades, this book may be used to help make fractions and ratios more meaningful. In Grade 5, for example, students are expected to "demonstrate an understanding of fractions by using concrete, pictorial and symbolic representations to: create sets of equivalent fractions; compare fractions with like and unlike denominators" (Alberta Education 2007, 18). The amount of cake left after each animal takes a piece can be represented as a fraction; the book even includes illustrations inside the front and back covers that show pieces of cake and their corresponding fraction. For example, after the elephant takes half, there is ½ of the cake left; after the gorilla takes his half, there is ¼ left; after the hippo takes her half, there is ⅛ left, and so on.

Lesson Plan

General Outcome: Develop Number Sense

Specific Outcome 6: Demonstrate an understanding of multiplication (two- or three-digit by one-digit) to solve problems by

- using personal strategies for multiplication with and without concrete materials,
- using arrays to represent multiplication,
- connecting concrete representations to symbolic representations,
- estimating products and
- applying the distributive property.
  [C, CN, ME, PS, R, V][1]

Achievement Indicators

- Create arrays using transparent mirrors and represent pictorially.
- Estimate the number of counters that will appear in the arrays for numerals 5 to 8.
- Use personal strategies when manipulating base-10 blocks to solve multiplication problems involving two or three digits multiplied by one digit.
- Display multiplication work symbolically using personal strategies, including distributive property.

MATERIALS	DISPLAYS
• All students can choose to have paper and pencils out during this lesson to support their problem-solving strategies and to keep track of the number of cakes being offered to the king. • Book—*The Lion's Share: A Tale of Halving Cake and Eating It, Too*. Prepare the book for the lesson by blocking out the numbers and pictures showing the number of cakes blocked out on page 16 and following. • Transparent mirrors—groups of two or three students can share one mirror. • One bag of counters per table • Arrays diagram sheet • Laminated animal faces from the story: ant, beetle, frog, macaw, warthog, tortoise, gorilla, hippo and elephant (one set per group) • Sets of base-10 blocks	• Have paper ready to write definitions and examples for the math word wall. Students will collaborate to generate definitions for the words *double* and *half/halving*.

TEACHER ACTIVITIES	STUDENT ACTIVITIES
Introduction: Begin by having students explore doubles using arrays.	
• Have students set up transparent mirrors at their table and share the counters between them. Two or three students can share one mirror. Allow them time to manipulate the counters, adding and taking them away from in front of the mirror.	• Discuss what they notice while exploring counters in front of the transparent mirrors.
• Have students place one counter in front of the mirror and draw what they see in the mirror, representing the arrays pictorially. Have students make additional arrays by adding one counter at a time, up to 4 counters.	• Create arrays; draw a picture of what they see using the arrays diagram sheet.
• Have students estimate what is going to happen when they put 2 more counters in front of the mirror. Ask for their reasoning. Students complete the diagrams estimating and illustrating what their arrays will look like for 6, 8, 9, and 10 counters within their groups. Discuss with students; use probing question 1 from the assessment items.	• Estimate and provide explanations for estimates; complete the diagrams with their group members.
• Have students discuss what the word *double* means and collaboratively create a definition to be placed on the math word wall.	• Discuss the word *double*.
• Show the students *The Lion's Share: A Tale of Halving Cake and Eating It, Too* and ask if any of them have read the book before.	
• If there is time, discuss the different meanings of the homonyms *having* and *halving*. Collaboratively create a definition for *halving* to be placed on the math word wall. This will be used in a later lesson. (Small 2009, 130) [12 min]	• Discuss the homonyms *having* and *halving*.

TEACHER ACTIVITIES (cont'd)	STUDENT ACTIVITIES (cont'd)
Development: • Provide groups with one set of animal faces from the story. The ant face goes in the middle of the table. Each student is responsible for one or two of the other characters (ideal group size would be four students with two animal faces each).	• Place the ant face in the middle of the table and take one or two character faces for themselves.
• Provide groups with one set of base-10 blocks.	
• Explain to students that you will be reading *The Lion's Share: A Tale of Halving Cake and Eating It, Too*. When the ant offers to bake the king a cake for the next day (p 15), ask the students to put one block on the ant's face in the middle of the table. Stress to the students that the rest of the animals are going to be *doubling* the number of cakes that they offer to bake for the king.	• Listen to the story. Place one block on the ant's face when he offers to bake a cake for the king.
• While reading, leave out the number of cakes each character is going to make for the king. When it is their character's turn to offer to bake cakes, the students are to solve how many cakes are going to be made. Discuss with students; use probing question 2 from the assessment items. • At the end of the story, have students tell the rest of the class the different strategies they used.	• Solve for how many cakes their character is going to offer to make for the king. Students can use paper and pencil to keep track of the other characters' numbers in the story, but they must use the base-10 blocks to arrive at their answer. Base-10 blocks may be used according to students' own personal strategies. Here are some possible strategies: ° Students place blocks representing the factor given on the table in front of them. To find out the number of cakes that their character has to make, students may place the same number of blocks on top of the original ones and count them. ° Students represent the known factor using base-10 blocks. To solve for how many cakes their character is going to offer to bake, they double each group of blocks (ones, tens, and hundreds) and simplify by regrouping the blocks that represent the product.
• Discuss the different ways that students could represent their work symbolically. Show the different strategies that students offer on the whiteboard and work through any misconceptions that arise. Encourage effective and appropriate strategies that represent multiplication. Discuss with students; use probing question 3 from the assessment items.	• Listen to peers talk about their explanations; offer individual strategies for showing work symbolically. A possible strategy offered may include the distributive property: ° $2 \times 128 = (2 \times 100) + (2 \times 20) + (2 \times 8)$ $= 200 + 40 + 16 = 256$
• Have students write their personal strategies in their math journals as a multiplication procedure.	• Write their strategies as written expressions in their math journals.
• Have students remove all the blocks from their characters and ask them to solve for what would have happened if the ant had offered to bake the king two or three cakes. [23 min]	• Solve the new problem. Students may use the same strategies as previously or may choose to use one of the strategies offered by their classmates.

TEACHER ACTIVITIES (cont'd)	STUDENT ACTIVITIES (cont'd)
Closure • Discuss with students what they noticed in the product similarities when the ant started with one cake and when he started with two cakes. • Discuss why the products were so different when the ant started with one or two cakes and when he started with three cakes. [5 min]	• Students may determine that all the products were the same because 2 was one of the products in the original problem. The difference will be seen with the elephant's character. Because the ant started with a higher multiple, it will result in a final product double that of the final product in the original problem. • Students may determine that because the ant started with three cakes and 3 was not a factor or a product in the original problem, all of the proceeding products will also be independent of the products from the original problem.

Differentiated Instruction

Differentiating for Above Readiness

- Using the original problem (the ant starts with 1 cake), have the students solve for how many guests were invited if the final guest offered to bake the king 512 cakes.
- Using the original problem, have students solve the problem if the animal guests wanted to triple the ant's original offer.

Differentiating for Below Readiness

- Ensure that students that need additional support are NOT solving for the hippo or elephant characters to begin with. Allow for practice with one- and two-digit numerals to begin with.
- Have students include another step during the development activity. The original activity has students transition from a concrete representation directly to a symbolic one. Have students who need additional support move from the concrete representation, using base-10 blocks, to a pictorial representation to further deepen their understanding. The picture will be a permanent resource for students to refer to when they encounter more complex problems.

Assessment

To assess this lesson, use the following checklist and accompanying probing questions. Behaviours that directly reflect the specific outcome are listed along the top row of the checklist, and students' names are in the left-hand column (Small 2009, 608). Circle the appropriate indicator for each student. The probing questions are meant to support students' learning and provide more information about students' understanding. It is important to let students answer fully and to "not interrupt during the explanation. Errors and possible reasons for misconceptions should be noted and used later to re-teach the student" (Bryant and Pedrotty 1997, 64).

The Lion's Share: A Tale of Halving Cake and Eating It, Too: Multiplication Lesson						
Student	Represents arrays pictorially	Makes an estimation to complete arrays	Uses base-10 blocks to support personal strategies to determine products	Makes a logical connection from strategies with base-10 blocks to a symbolic representation	Personal strategies include distributive property	Notes
	1 2 3 N/O	1 2 3 N/O	1 2 3 N/O	1 2 3 N/O	1 2 3 N/O	
	1 2 3 N/O	1 2 3 N/O	1 2 3 N/O	1 2 3 N/O	1 2 3 N/O	
	1 2 3 N/O	1 2 3 N/O	1 2 3 N/O	1 2 3 N/O	1 2 3 N/O	
	1 2 3 N/O	1 2 3 N/O	1 2 3 N/O	1 2 3 N/O	1 2 3 N/O	
	1 2 3 N/O	1 2 3 N/O	1 2 3 N/O	1 2 3 N/O	1 2 3 N/O	
	1 2 3 N/O	1 2 3 N/O	1 2 3 N/O	1 2 3 N/O	1 2 3 N/O	
	1 2 3 N/O	1 2 3 N/O	1 2 3 N/O	1 2 3 N/O	1 2 3 N/O	

1= never 2= sometimes 3=always N/O= not observed

Example Probing Questions

1. Can you tell me why you drew 12 counters in your next estimation for 6 counters in front of the mirror?
2. Ask about why students choose to use the base-10 blocks as they do.
   a. Can you explain to me why you put the same number of blocks on top of the original number of blocks to determine the product?
3. Can you explain your steps to me when you showed your work symbolically?
   b. Why did you convert 128 into 100, 20 and 8?
   c. Why did you add the products of the three equations?

Mathematical Processes

Communication—Students are given various opportunities to communicate with group members, the entire class and the teacher during this lesson. Students are encouraged to communicate with each other during the introductory activity, because they must share materials while exploring doubles in the transparent mirror. The teacher can listen for misconceptions as students talk in their groups about observations they make as they add counters or take them away from in front of the mirror. Students also engage in group discussion in the introductory activity when collaborating to develop mathematical definitions for the words *doubling* and *halving* for the math word wall.

As Small (2009) notes, it will be important to allow sufficient wait time after students contribute to allow other students to respond and add to the discussion (p 64).

During the development activity, students initially develop strategies on their own but tell the rest of the class about those strategies immediately after the story. To encourage listening, as suggested by Small (2009), once the strategies have been revealed there will be an opportunity for "... the listeners [to] describe what they learned from the explanations" (p 65). Students must also communicate information about symbolic representations that will be recorded in their math journals that can be used as a resource for future tasks. This written communication is important as it "... becomes available for later analysis and discussion ... and [can] help make mathematical relationships more obvious" (Greens 1999, 46).

Finally, students will communicate with the class during the closure about similarities and differences in the products of different multipliers.

Connections—This lesson gives students an opportunity to make connections between the introductory activity, in which they use transparent mirrors to explore doubles, and the development activity, in which they use base-10 blocks and symbolic representation to further explore the concept of multiplication.

The story also allows students to make connections to their personal lives. Some might have seen the animals from the book in a zoo or have attended a party where some of the guests took more than their fair share of dessert. Although these are not connections to mathematical concepts, Small (2009) stresses that "any and all connections are helpful for students to make sense of new ideas they encounter" (p 31).

Mental Mathematics and Estimation—During the introductory activity, students make estimations about what the arrays will look like in the transparent mirror for the numerals 5 to 8. This is done after they have practised using the counters to build arrays up to 4 × 4 and representing what they see pictorially.

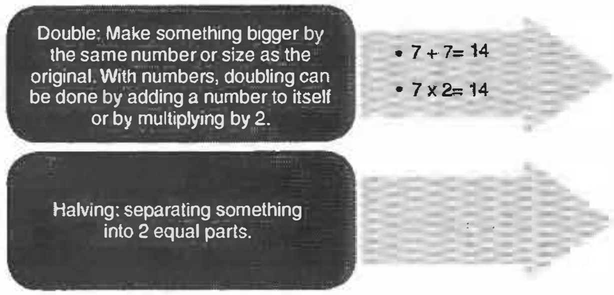
Problem Solving—As advised by Small (2009), students are first given a context—in this lesson, taken from the story—and are then required to develop mathematical procedures to problem solve (p 38). Students use the events in the story as their reason for using the base-10 blocks to solve for how many cakes their character will have to bake for the king. This lesson follows Small's (2009) three recommendations for teaching problem solving in a meaningful way. First, there is "an increased level of mathematical dialogue between students" (p 38). This is evident as students not only discuss the meanings of mathematical terms, but also talk about their personal strategies and respond to peers' comments in all three parts of the lesson. Second, there is an emphasis on "the teacher's role as a guide or coach more than as a presenter" (p 38). Students have time to explore manipulatives without being told how to use them. Students are also expected to develop and use personal strategies to problem solve—the teacher does not simply teach a strategy then expect students to use it. Rather, the teacher's role is to give students the materials needed for exploration and to provide the context for the lesson. Last, this lesson involves "the teacher's more judicious use of intervention" (p 38). While the teacher allows students to develop their own personal strategies, it is still the teacher's responsibility to encourage the use of effective and appropriate strategies and to help students realize and understand their misconceptions.

Reasoning—Students use reasoning skills throughout this lesson to generalize and verify conclusions (Small 2009, 30). The exploration of doubles in the introductory activity gives students an opportunity to begin to make inferences about the processes they are using. Those inferences can then be generalized during the development activity and verified through communication with peers and the teacher.

Visualization—Students can internalize images during the introductory activity. They are required to first represent their manipulatives as they see them in the transparent mirror. Then they must use visualization to estimate what the arrays will look like for the remaining numbers and represent those estimations using the arrays diagrams.

Notes

- The story can be used again for a division lesson on halving and for a fractions lesson.*
- Extension—What would happen if the animals wanted to triple the number of cakes they offered to bake?
- Math word wall additions—These will be organized with input from the class and placed on the math word wall for future references.



* Leave further explanation and examples for halving until later lesson on division.

Arrays Diagram Sheet

Estimation:

Estimation:

Note

1. Key:

C Communication
CN Connections
ME Mental Mathematics and Estimation
PS Problem Solving
R Reasoning
V Visualization

Source: Alberta Education 2007, 4.

References

Alberta Education. 2007. *Mathematics Kindergarten to Grade 9*. Edmonton, Alta: Alberta Education. Available at http://education.alberta.ca/media/645594/kto9math.pdf (accessed January 13, 2012).

Baroody, A J, and J L M Wilkins. 1999. "The Development of Informal Counting, Number, and Arithmetic Skills and Concepts." In *Mathematics in the Early Years*, ed J V Copely, 48–65. Reston, Va: National Council of Teachers of Mathematics.

Bryant, B R, and D Pedrotty Rivera. 1997. "Educational Assessment of Mathematics Skills and Abilities." *Journal of Learning Disabilities* 30, no 1: 57–68.

Greens, C. 1999. "Ready to Learn: Developing Young Children's Mathematical Powers." In *Mathematics in the Early Years*, ed J V Copely, 39–47. Reston, Va: National Council of Teachers of Mathematics.

Martinez, J R, and N C Martinez. 2000. "Teaching Math with Stories." *Teaching PreK-8* 30, no 4: 50–54.

McElligott, M. 2009. *The Lion's Share: A Tale of Halving Cake and Eating It, Too*. New York: Walker.

Small, M. 2009. *Making Math Meaningful to Canadian Students, K-8*. Toronto, Ont: Nelson Education.

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What a Lot of Trash!

Grade 3: Addition and Subtraction of Numbers to 1,000

Petra Nagtegaal

Introduction

While working at Vital Grandin Catholic Elementary School in St Albert, Alberta, I was privileged to see the success of the school's Wasteless Wednesday program. In this program, students' lunch waste was collected and weighed every Wednesday, with the goal of reducing the trash produced by the school.

As part of my preservice mathematics curriculum class I was required to produce an authentic math lesson plan, and I knew I wanted to create the beginnings of a unit on waste, using Wasteless Wednesday as the starting point.

Lesson Description

After weighing the garbage produced by all the classes in the school over a (minimum) two-week period, the students will graph the total amounts of trash from each classroom and note if there is a pattern—for example, if less trash is produced on Wasteless Wednesdays. In the lesson described in this paper, each trash item (eg, apple core, juice box) will be weighed separately (anticipatory set). Students will then use the data sheet provided (with daily trash weight) and subtract from their designated class total to reach a biweekly total of 725 grams,[1] or they may use actual numbers from the data they collected. Students will decide if they will subtract more lightweight items or fewer heavy items to reach the desired total. The teacher will use a checklist throughout the activity to monitor students' understanding and the rubric to evaluate the worksheet at the end of the lesson.

Specific Outcomes (Numbers)

Students will demonstrate an understanding of addition and subtraction of numbers with answers to 1,000 concretely, pictorially and symbolically by

- using personal strategies for adding and subtracting with and without the support of manipulatives, and
- creating and solving problems in contexts that involve addition and subtraction of numbers.

Mathematical Processes

Students will

1. demonstrate understanding of addition to 1,000 by adding daily totals of trash from data collected over a two-week period (lesson 1);
2. demonstrate understanding of subtraction (on the worksheet) by taking away from the total until close to 725 grams is reached (lesson 2);
3. relate information they are learning, through literature and scientific research, about the accumulation of trash in other classes (science, social studies, art, language arts) to make connections to the theme of trash and its reduction in their community;
4. use estimation and mental math to decide which trash items will work best to get closest to the goal;
5. use their knowledge of subtraction to make a plan of how much each class must reduce its waste to reach a goal of (close to) 725 grams of trash. Students will decide to reduce by fewer heavier items and/or more lightweight items;
6. prove that by reducing a number of specific trash items, the class can meet the goal of reducing trash to close to a biweekly total of 725 grams; and
7. weigh the actual trash items to see clearly what items take up the most/least amount of space in a landfill, then use that information to show how to reduce the mass of trash.

Achievement Indicators

- Model the subtraction of two numbers using visual representation and record the process symbolically.
- Determine the difference of two numbers using personal strategies in subtracting numbers to reach a given number; for example, for $972 - 55$ a student may record $972 - 50 - 5$ or $970 - 50 - 5 + 2$.

Materials

- Document camera or interactive whiteboard to demonstrate math procedures
- Data collected of weight of trash over two weeks for their designated class

- "How much does it weigh?" sheet (one per student—check the number of each level needed; see Appendix C)
- Activity sheet: What a lot of trash! (one per student; see Appendix D)
- Weigh scales
- Place-value mats (to 1,000)
- Base-10 blocks
- Trash (juice boxes, apple/banana peels, baggies, leftover sandwiches, and so forth) on tables around the classroom

Anticipatory Set

Students weigh trash items separately (no need to collect data, except for group 3). Discuss

- what weighed the most/least,
- whether the items weighed the previous weeks were similar,
- what items could be recycled/composted and
- the book *Dougal the Garbage Dump Bear* (Dray 2005) and the fact that it was not one bear in the landfill, but many bears.

Procedure

1. Discuss with students the totals of trash collected for each classroom and how that will affect the landfill (see Appendix E).
2. Tell students that they will work as environmentalists to figure out how each classroom can reduce its trash to reach a goal of 725 grams biweekly.
3. Model (using the document camera) how to determine what trash items (students must use a minimum of five different items) will reduce the amount of waste to reach the goal of 725 grams. For example, show students that the total (972 grams) is far from the goal of 725 grams, and therefore a heavy item will have to be removed. Choose the sandwich (42 grams) and demonstrate 972 – 42 = 930 using the place-value mat and manipulatives. Then draw the procedure on the activity sheet. Recognizing that 930 grams is still far from the goal and knowing that five different items must be used, the teacher will choose the next heaviest item (apple core, 35 grams) and demonstrate how to subtract 35 on the place-value mat, writing the numerical algorithm on the activity sheet. Next the teacher will use mental math and estimation to demonstrate how to subtract the next heaviest item (milk carton, 15 grams) and write the algorithm on the activity sheet. Then the teacher should do an example with the students by asking, "What should we take out of the total next?"
4. Distribute "How much does it weigh?" sheet (see Appendix C) to each student:
   - Sheet 1—at-grade-level activity sheet
   - Sheet 2—simplified activity sheet
   - Sheet 3—extended task sheet
5. Distribute one activity sheet, "What a lot of trash!" to each student. Students work in pairs (students should be paired with others working on the same level of activity sheet).
6. Closure: Ask each group how they chose the items that they subtracted from the total (volume and/or mass).

Adaptations

- Simplified task: Give the student the "How much does it weigh?" sheet 2 (see Appendix C), which uses benchmark numbers 5 and 10, but has two items to help students reach close to the goal of 725 grams.
- Extended task: Students will compute the number that is half their total and use this as a goal for their class (rather than the 725 grams). Students will also use the actual weight of each trash item.

Assessment for Learning

Using a checklist (see Appendix A) and moving between each group during the activity, ask questions to assess students' understanding or to scaffold learning. Questions may include the following:

- "Tell me how you got to that number." The student should be able to verbalize personal problem-solving technique/algorithm. Scaffold: Have students show their work using manipulatives or symbols.
- "How would it change if the number were (ten more/less)?" The student should describe the benchmark of 10 and that it is a simple algorithm of adding/subtracting from the 10 place-value. Scaffold: The student should use the place-value mat to see what occurs with the addition or subtraction of 10.
- "I see that you are close to 725 grams; how are you using mental math to decide what item weight to subtract next?" The student should be able to describe how he or she uses estimation and 1s, 5s and 10s (benchmarks) to choose the approximate weight necessary. Scaffold: Ask the student if he or she is more or less than 10 away from the goal of 725 grams, and from the answer, ask questions that will have student visualize the numbers.
- "Another student (use the student's name) showed a different way of calculating that. Can you tell me how he did it?" To show a deep understanding, students should be able to describe the algorithms used by other students in the group. Scaffolding:

If the student cannot explain another's algorithm, perhaps he or she does not understand it. In this case, have one student explain the algorithm to the other. Alternatively, the student can work through the problem with the teacher.

- "Do you prefer to use manipulatives like the place-value mat, draw out the problem or use numbers to solve the problem? Why?" Scaffold: Has the student tried three ways to demonstrate understanding?

Assessment of Learning

The teacher will evaluate the performance task (activity sheet; see Appendix D) using the rubric provided (see Appendix B).

Integration with Other Subjects

See Appendix F.

Note

1. The 725-gram figure was chosen because the number must be in the 0–1,000 range to meet the learning outcomes for this grade. Weights of specified objects have been estimated, not measured.

Reference

Dray, M. 2005. *Dougal the Garbage Dump Bear*. La Jolla, Calif: Kane/Miller.

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Appendix A

Assessment for Learning Checklist

Student	Models subtracting two numbers using visual representation	Records algorithms symbolically	Determines the difference of two numbers using a personal strategy	Notes

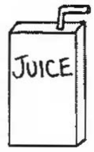
Appendix B
Assessment of Learning Rubric

Demonstrates an understanding of addition and subtraction of numbers with answers to 1,000 by	4	3	2	1
Using personal strategies for adding and subtracting with and without the support of manipulatives	Uses personal strategies for adding and subtracting with and without the support of manipulatives	Uses personal strategies for adding and subtracting with the support of manipulatives	Relies on one personal strategy for adding and one for subtracting	Is not able to use a personal strategy but can follow the steps of a procedure when given
Creating and solving problems in context that involve addition and subtraction of numbers	Rephrases the solution and there are few, if any, errors in procedure; student is flexible about the strategy and revises it as necessary	Solution is correct, thought there may be minor procedural errors; student revises the strategy as necessary	Solution is faulty due to several errors in procedure; student is hesitant to change strategies	Solution is seriously flawed due to major errors in procedure; student gives up if strategy does not work

Adapted from Small, M. 2009. *Making Math Meaningful to Canadian Students, K–8*. Toronto: Nelson, 613.

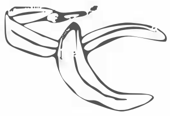
Appendix C
(1 of 3)

HOW MUCH DOES IT WEIGH? (#1)

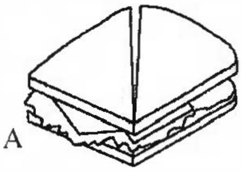


17 grams

35 grams



55 grams



42 grams

27 grams

15 grams

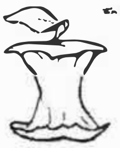
5 grams

11 grams

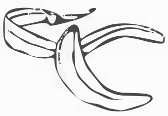
Appendix C
(2 of 3)

HOW MUCH DOES IT WEIGH? (#2)

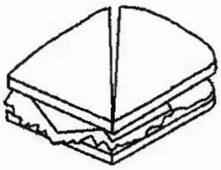
15 grams



35 grams



50 grams



100 grams

25 grams

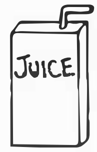
10 grams

3 grams

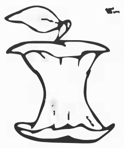
11 grams

Appendix C
(3 of 3)

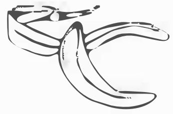
HOW MUCH DOES IT WEIGH? (#2)



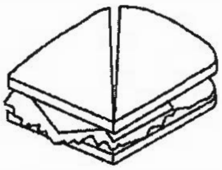
_____ grams



_____ grams



_____ grams



_____ grams

_____ grams

_____ grams

_____ grams

_____ grams

Appendix D

Name: ____

Activity sheet: What a lot of trash!

My group will study the trash weigh from Grade ________.

The total trash collected in the two weeks was ___________ grams. This is WAY too much trash!

Each class has a goal to reduce its trash to 725 grams every two weeks.

Use the base-10 blocks to find out how much LESS trash the grade you studied will have to produce. Show your algorithm pictorially, using base-10 blocks:

Explain, using numbers (symbolically), the algorithm you used:

You, as the environmentalist, will figure out how to help the class meet this goal by subtracting the weight of the trash items (see attached sheet) until you get as close as possible to 725 grams.

This will tell the class how much less of each item they need to throw out to get as close as possible to the goal of 725 grams.

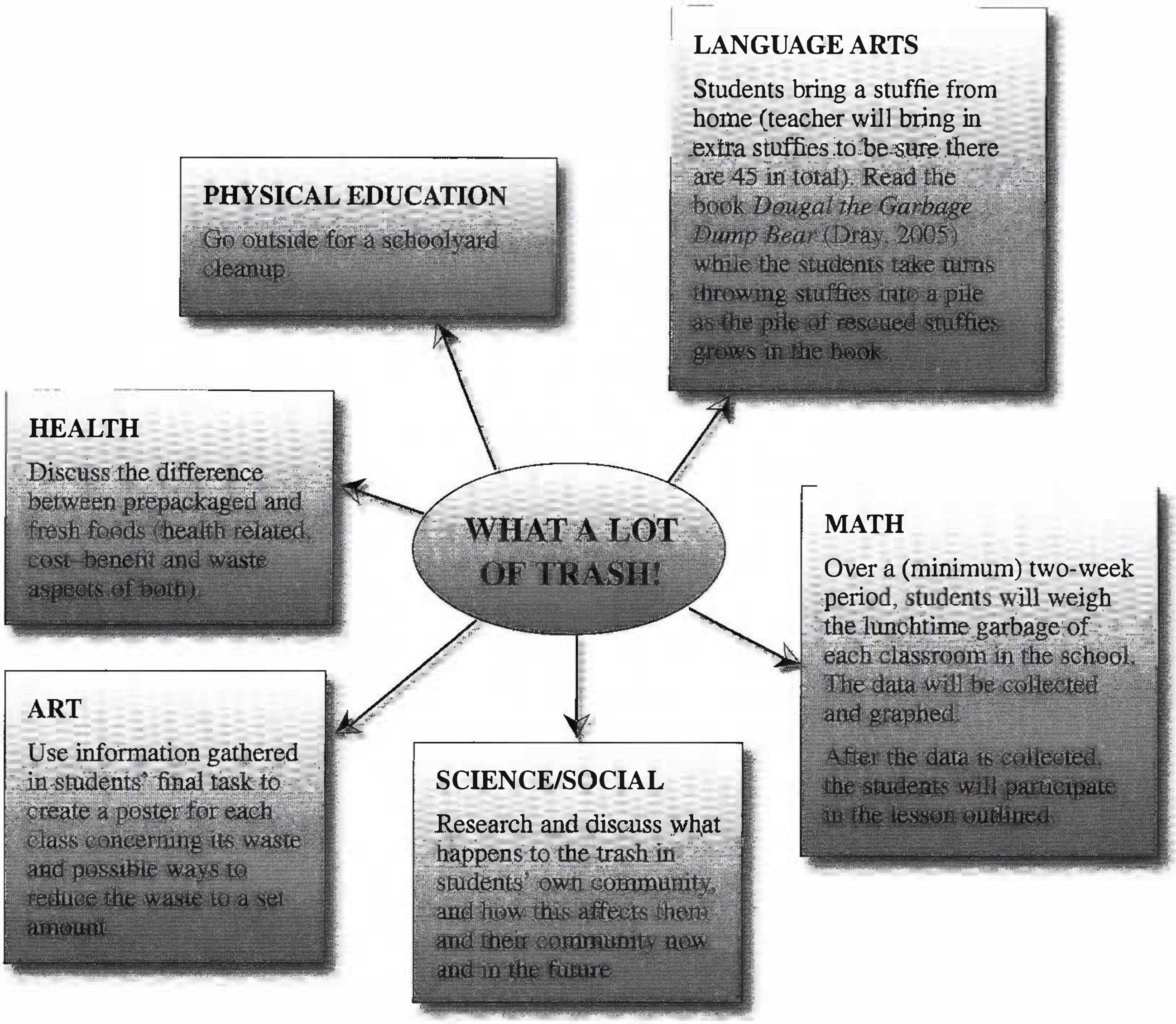
You must use at least five different items to reduce the number to 725 grams. Some items will have to be used more than once.

Show your work.

Appendix E

DATE	Grade 1	Grade 2	Grade 3	Grade 4	Grade 5	Grade 6
04-Apr	162	121	186	73	132	212
05-Apr	64	153	83	186	167	88
06-Apr	8	12	16	22	12	5
07-Apr	124	179	131	117	118	128
08-Apr	162	165	190	155	137	123
11-Apr	14	43	110	109	96	147
12-Apr	142	91	87	97	114	43
13-Apr	18	21	14	32	6	19
14-Apr	191	168	105	102	98	138
15-Apr	112	46	70	97	95	92

Appendix F


WHAT A LOT OF TRASH!
LANGUAGE ARTS
Students bring a stuffie from home (teacher will bring in extra stuffies to be sure there are 45 in total). Read the book *Dougal the Garbage Dump Bear* (Dray, 2005) while the students take turns throwing stuffies into a pile as the pile of rescued stuffies grows in the book.
PHYSICAL EDUCATION
Go outside for a schoolyard cleanup.
HEALTH
Discuss the difference between prepackaged and fresh foods (health related, cost–benefit and waste aspects of both).
MATH
Over a (minimum) two-week period, students will weigh the lunchtime garbage of each classroom in the school. The data will be collected and graphed.
After the data is collected, the students will participate in the lesson outlined.
ART
Use information gathered in students' final task to create a poster for each class concerning its waste and possible ways to reduce the waste to a set amount.
SCIENCE/SOCIAL
Research and discuss what happens to the trash in students' own community, and how this affects them and their community now and in the future.

Hungry Ants Explore Multiplication

Caitlin Dickinson

Looking back at my own experiences as an elementary mathematics student, I recall so many fellow students asking the inevitable question: "Why do we have to learn this?" Such a question was inevitably followed by the bold statement, "We'll never use this in real life!" The challenge that my own teachers faced was to convince my classmates and me that the concepts we studied in class were valuable and relevant; in particular, they searched for ways to engage us to prevent us from giving up before we'd begun. Literature can be an ideal medium through which students can personally connect to mathematics because it can provide interesting context for a problem and help students look at mathematics in a broader way.

This multiplication lesson uses the children's book *One Hundred Hungry Ants*, by Elinor J Pinczes. In *One Hundred Hungry Ants*, a colony of ants is on its way to procure food from a picnic site. The leader is trying to rush the colony along because he does not want to arrive when all of the food is gone. To speed their progress, he stops periodically to command the ants to regroup and form different arrays, inadvertently taking up *more* time and resulting in their being late to the picnic!

General Outcomes

- Develop number sense (Alberta Education 2007, 88)

Specific Outcomes

- Demonstrate an understanding of multiplication (two- or three-digit by one-digit) to solve problems by
  1. using personal strategies for multiplication with and without concrete materials,
  2. using arrays to represent multiplication,
  3. connecting concrete representations to symbolic representations,
  4. estimating products and/or
  5. applying the distributive property. [C, CN, ME, PS, R, V] [1] (Alberta Education 2007, 88)

Mathematical Processes

- **Communication**: Students will use base-10 blocks to demonstrate their understanding by showing the multiplication process and will participate in discussion and whole-class problems in which they will communicate their understanding verbally or by using the interactive whiteboard manipulatives. They will create their own problems and solve them using manipulatives, recording the processes they used pictorially and symbolically.
- **Connections**: Students will connect their prior experiences with arrays and single-digit by single-digit multiplication to single-digit by two- or three-digit multiplication. They will have the opportunity to connect this concept to their own lives by identifying a real-world situation in which such multiplication could occur when they create their own problems, and when discussing problems with their classmates.
- **Mental Mathematics and Estimation**: Students will break down larger multiplication problems into more manageable parts using the distributive property, and perform simpler mathematical problems mentally. They will estimate the product of two factors by simplifying the problem in a way they can understand and compensating afterwards, visualizing it in their minds, and so on.
- **Problem Solving**: Students will explore ways of solving these problems and experiment with different methods of finding an answer.
- **Reasoning**: Students will use their observations from using the distributive property to understand that the product of two factors does not change if the factors are split.
- **Visualization**: Students will use base-10 blocks to develop a mental image of single-digit by two- or three-digit multiplication. This helps them to visualize this process as a rectangle or an array.

Technology Tip

Using SMART Notebook, create a row at the bottom of a page that contains one of each of the virtual manipulatives (a 100s block, a 10s block, and a 1s block). Right-click on each of them and select Infinite Cloner. You can now drag an infinite number of copies of that shape away to use as examples and to help students show their work without needing to repeatedly create new ones.

Achievement Indicators

- Students can model a given multiplication problem using the distributive property.
- Students can use base-10 blocks to represent multiplication and record the process symbolically.
- Students can estimate the product of a one-digit and a two- or three-digit factor. (Adapted from Alberta Education 2007, 88)

Introduction

Review ways of using manipulatives to show multiplication:

- Have students use base-10 blocks to solve a problem (for example, I have three boxes, and each has eight marbles inside. How many marbles do I have?)
- As a class, discuss the different methods students used. Ask students to think about how they solved the problem and to tell a partner about it (National Council of Teachers of Mathematics 2011). Invite students to tell the entire class about this method. Did some students make groups and add them up? Did some students use an array? Ask students what the symbolic representation of this question would be, and invite one of them to write it on the board.
- Have students solve another problem using base-10 blocks, but have them use a different method to solve it (for example, I have 8 books, but Amelia has 9 times the number of books I have. How many books does Amelia have?)
- Ask students what the symbolic representation of this question would be and invite one of them to write it on the board. As a class, discuss the different methods students used. Which method did they find easiest to use? Was one of them better suited for the problem with larger factors?

Development

- Introduce the story *One Hundred Hungry Ants* (Pinczes 1993) by telling students the premise of the story and explaining that they must use the base-10 blocks to model the formations of the ants.
- Observe students as they perform this task:
  - Do they use various values of blocks appropriately? (For example, when creating a row of 100 ants, do they attempt to count out 100 1s blocks, or do they use 10 10s blocks?)
  - Can they successfully create the correct formation using the blocks?
- Go through some of the arrays of ants and ask students to build them again on their desks while one student builds it on the interactive whiteboard. Invite the students to figure out the symbolic representation for this array.
- Model the format for using base-10 blocks and the distributive property to solve a multiplication problem by adding on to this example—place the virtual base-10 blocks outside of the x and y axes and model the steps for solving it; involve the students by asking questions throughout the process (see Appendix A). Discuss how each section relates to the original equation (for example, "We really solved the problem as $(10 \times 5) + (10 \times 5)$").
- Task students with creating a multiplication problem (one-digit by two- or three-digit) and solving it, recording their process pictorially and symbolically. When they have finished, invite each group to solve the problems that the students in that group created.
  - The problem should be more than symbolic; students should create scenarios or identify real-life situations in which they can use multiplication to solve a problem.
- Model task: Create a problem (for example, on a very hot summer day, there were three lines at the ice cream stand. There were 212 kids in each line! How many kids were lined up in total?) Begin by estimating what the final product will be; discuss estimation techniques and methods. Ask students to compare solving methods with peers at their table. Did anyone solve the problem differently? Solve the problem together on the interactive whiteboard.
- Students will estimate the product and record what they think it might be and how they got that answer. They will then solve the problem using base-10 blocks and record their process pictorially and symbolically. They will then discuss their problems within their groups, and the group will work together to solve each one.
- Observe students as they perform this task:
  - Does their problem make sense as a multiplication problem? Are they able to represent it using concrete materials (base-10 blocks)?
  - Are they able to use an efficient method to solve the problem?
  - Can they record their multiplication process symbolically?

Closure

- Discuss
  - Did everyone in the group solve the problem exactly the same way? How were your methods the same? How were they different?
  - Could you solve all of the problems?
  - How close were your estimates?

Differentiated Instruction

- Support Activity
  - Perform the above activity using the base-10 blocks. This time, create an equation whose single-digit factor is 1 and whose second factor has either two or three digits (eg, 1 × 25 or 1 × 362). Invite the students to create such a problem, putting it in a real-world context, and solve it using the base-10 blocks. When they solve and record this problem, the students should rewrite their problem, replacing the factor 1 with 2 and performing their solution again. This simplifies the task and familiarizes students with the process.
  - If the students struggle with basic multiplication facts, they may benefit from using a grouping method in which they make groups of the factors (eg, in 6 × 124, make 6 groups each containing 1 100s block, 2 10s blocks, and 4 1s blocks) and, beginning with the blocks with the lowest values, trading in for blocks of higher values (for example, if the student has 24 1s blocks, she can trade 20 of them for 2 10s blocks). This method correlates to the traditional multiplication algorithm (Small 2009, 179).
- Extension Activity
  - Broken Calculator Problem (Adapted from Small 2009, 182): Give students an image of a calculator, with an X through the 8 button, and ask them to solve the problem 6 × 98 using the calculator without using the broken 8 button. They should use the base-10 blocks and their knowledge of the distributive property to determine the steps they would take to solve the problem with the calculator. There are many possible solutions to this problem. If (6 × A) + (6 × B) = 6 × 98, however, "A + B" must equal 98.

Assessment

- Observations Checklist (Small 2009, 608)
  - The teacher will use her observations of students' work to fill out an observations checklist (Appendix B) that lists the skills and concepts students are expected to utilize. The teacher will walk through the classroom and ask prompting questions if necessary to assess student understanding in these areas.
  - Prompts
    - How do your blocks show the groups?
    - How did you get to the product?
    - How would you write this down on paper?
    - How did you see these numbers when you estimated the product?
  - Criteria
    - Array
      - Students will use base-10 blocks during the story (and possibly during subsequent activities if they choose to use this method) to create arrays that represent multiplication.
    - Manipulatives
      - Students will use base-10 blocks during the story to solve given multiplication problems and to create and solve original multiplication problems.
    - Records process symbolically
      - Students will translate their concrete/pictorial representations of the problem symbolically, ensuring that they record the process that *they* used when solving the problem.
    - Estimates product
      - Students will estimate the product of two factors by rounding, using the distributive property, or by using other personal strategies.

Note

1. Key:

C	Communication
CN	Connections
ME	Mental Mathematics and Estimation
PS	Problem Solving
R	Reasoning
V	Visualization

Source: Alberta Education 2007, 4.

References

Alberta Education. 2007. *Alberta K–9 Mathematics Program of Studies with Achievement Indicators*. Edmonton, Alta: Alberta Education.

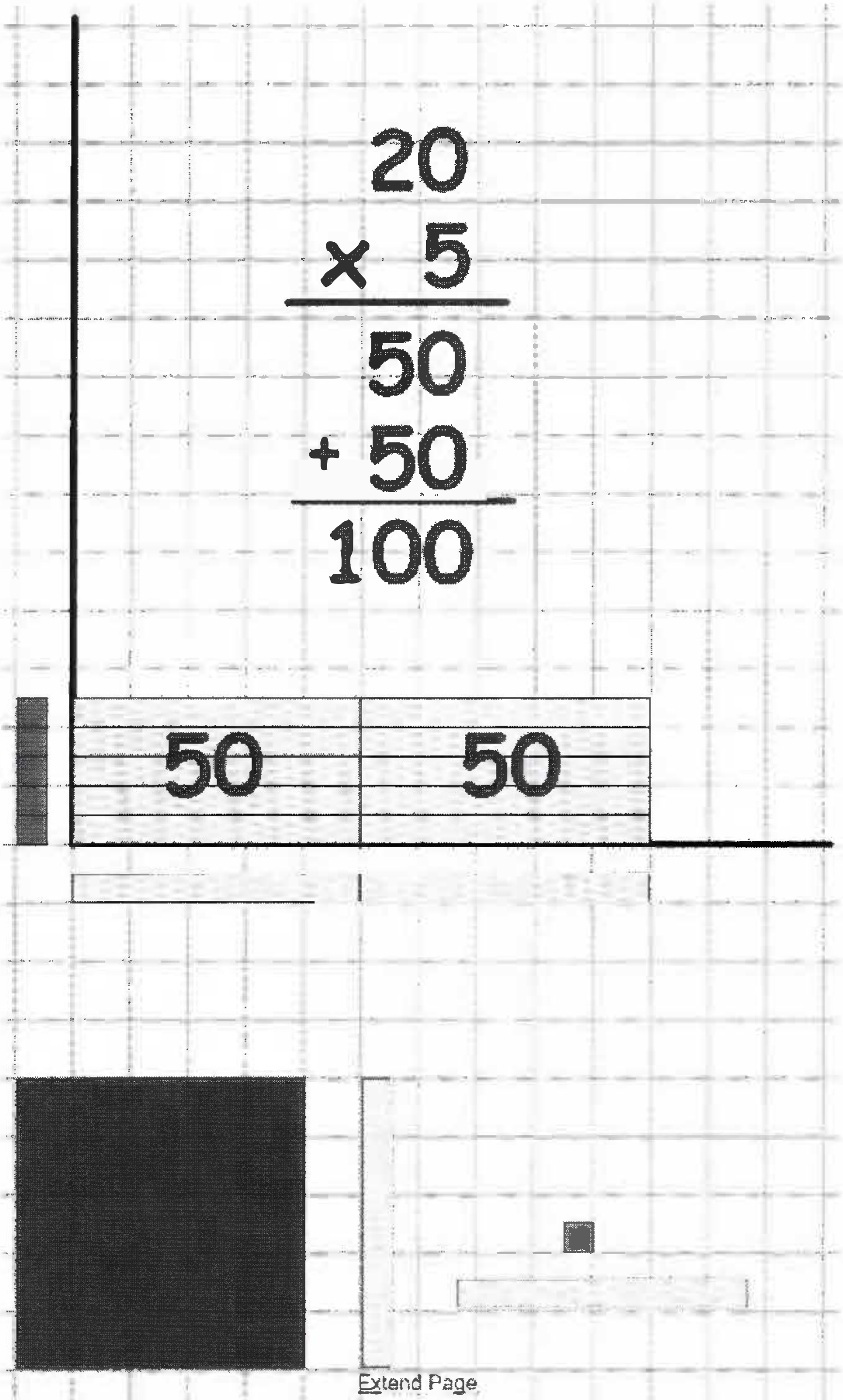
National Council of Teachers of Mathematics. 2011. "Asking Good Questions and Promoting Discourse (Part I)." Available at http://nctm.org/resources/content.aspx?id=25149 (accessed January 13, 2012).

Pinczes, E J. 1993. *One Hundred Hungry Ants*. New York: Houghton Mifflin.

Small, M. 2009. *Making Math Meaningful to Canadian Students, K–8*. Toronto, Ont: Nelson.

Caitlin Dickinson is completing her BEd at the University of Alberta. She has spent two years as a natural history interpreter, teaching children and adults about the world around them and the creatures in it. She has a passion for music as well as for learning and looks forward to learning alongside her students as she gains experience in the classroom.

Appendix A



Appendix B

Multiplication Assessment Checklist

√ = Student demonstrates clear understanding/performs this task accurately

• = Student needed assistance and/or prompting to demonstrate understanding/perform this task

/ = Student was unable to demonstrate clear understanding/perform this task without exceptional assistance

Name	Uses an array to represent multiplication	Uses manipulatives (or pictorial versions) to demonstrate multiplication process concretely/pictorially	Records personal process for multiplication symbolically	Estimates product using personal strategies

Digging Deeper into Fraction Addition

Jerry Ameis

Teaching mathematics methods courses to K–8 teacher candidates over a period of 20 years has led me to many observations. One of them concerns fractions. K–8 teacher candidates' conceptual understanding of fractions and their fraction algorithmic skills are weak. These observations are in line with formal research on the matter (Reeder and Utley 2007; Weller, Arnon and Dubinsky 2009; Ma, 1999; Ball 1990).

Reeder and Utley (2007) summarize the situation well.

> This study revealed that this group of prospective elementary teachers brought with them a limited understanding of fractions to their mathematics methods courses. Although these prospective teachers had many years of school mathematics … their reasoning about simple fraction concepts was often incorrect and based heavily on misconceptions they had previously developed or, at best, understandings of fractions as part of a whole. (p 249)

The research literature also indicates that middle-years students have similar difficulties understanding fractions and doing fraction arithmetic (Clarke, Roche and Mitchell 2007). One difficulty common to teacher candidates and middle-years students concerns fraction addition. There seems to be some tendency to add numerators to numerators and denominators to denominators. The edited version of a question submitted by a middle-years student to the website Ask Dr. Math illustrates this (the author of this article is a "math doctor" for that site).

The student's question:

> How do you solve: $2/5 + 3/8 + 1/4 + 140 = x$.
>
> My thoughts: $6/17 + 140 = x$.

Why might some middle-years students and teacher candidates add fractions in the way indicated in the question submission? There seem to be two overall explanations for this: (1) fractions are not understood and thus are treated as whole numbers and (2) fractions are partially understood and particular life situations seem to support an incorrect addition algorithm. This article concerns the second reason in the context of working with middle-years teacher candidates.

The article also concerns teaching for conceptual understanding. One of the goals of my middle-years mathematics methods course is to improve the teacher candidates' conceptual understanding (of fractions, in this case) while at the same time encouraging the development of question/problem posing as an important teaching strategy. Teaching for conceptual understanding relies on posing questions and problems, and using instructional materials (eg, diagrams and concrete materials). The National Council of Teachers of Mathematics (NCTM) (2000) notes that the mathematical discussion involved in question/problem posing is important for developing conceptual understanding because students become active participants in the development of mathematical ideas (Barlow and Cates 2006).

Fraction Addition with Middle-Years Teacher Candidates

I begin the session on fraction addition by telling the following anecdote from an age gone by (when there still was abundant hair on my head). While I was teaching fraction arithmetic to Grade 7 students, one of them challenged me with this situation.

> You taught us to add fractions by the common denominator method. But that doesn't work for my hockey team. We won 5 out of 6 home games and 1 out of 4 away games. Altogether we won 6 out of 10 games. Saying this with fractions, my team won 5/6 of our home games and 1/4 of our away games. Altogether, we won 6/10 of our games. So, $5/6 + 1/4$ is the total fraction of games won. The correct answer of 6/10 is figured out by doing 5 add 1 over 6 add 4. If you use the common denominator method to add, you don't get 6/10. The common denominator method is wrong.

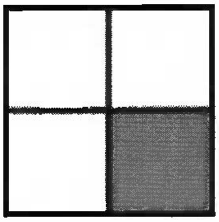
The incident points out to the teacher candidates why some Grades 7 and 8 students might be resistant to the common denominator method for adding fractions. My central purpose, though, is to probe deeply into fraction addition, an experience that inevitably generates controversy and thinking that goes beyond the conventional and relatively comforting world of using models (eg, a pie model) to develop fraction addition.

Developing the Part-of-a-Whole and Part-of-a-Set Meanings of Fraction

Before we consider the anecdote, two core meanings of fraction (part of a whole and part of a set) are developed in depth. That development proceeds by presenting the teacher candidates with a situation for each meaning (see Figures 1 and 2) and then, by question posing, extracting the critical features of each meaning.

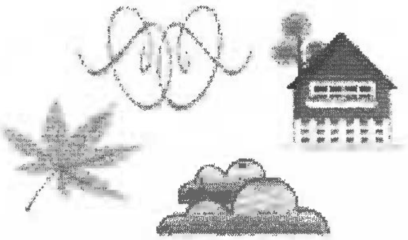
The questions concern wholes, cutting into parts, equality of parts, collections of things and attributes. My opening question for situation 1 is "Why can we say that the shaded rectangle is ¼ of the outside rectangle?"

Figure 1: Situation for part-of-a-whole meaning



The opening question for situation 2 is "There is no cutting into equal parts. Why can we say that ¼ of the pictures shows a house?"

Figure 2: Situation for part-of-a-set meaning

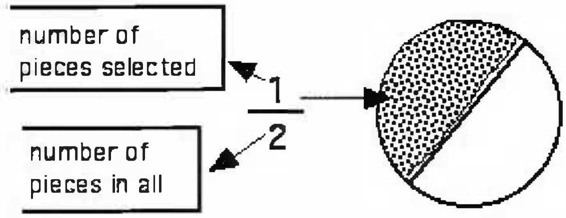


What follow are the critical features of the two fraction meanings that are uncovered by question posing and subsequent discussion.

Part-of-a-Whole Meaning of Fraction

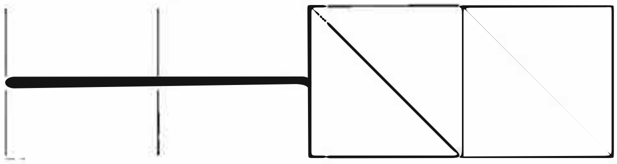
When a 5-year-old child says "I ate half the cookie," he/she is expressing a part–whole relationship. The child uses *half* not in the sense of a number but in the sense of an actual or imagined action that involves cutting a whole physical object in the middle. The imagined or actual action of cutting a whole object into n equal parts (according to a measurement concept such as length, area, volume and so on) underlies the part-of-a-whole meaning of fraction. We represent each part symbolically by the fraction notation $1/n$ (refer to Figure 3). The whole is typically a naturally existing thing. In other words, the whole is a conventional object such as an apple, rectangle, pie, loaf of bread and so on. The whole is not perceived as a collection of discrete objects.

Figure 3: Part of a whole (parts looking the same)



In Figure 3, the two pieces of the circle are equal in area and happen to look the same. This does not need to be the case. Consider a rectangle cut in the way shown in Figure 4. The eight pieces do not all look the same. Yet each piece is 1/8 of the rectangle in size because the pieces have the same area.

Figure 4: Part of a whole (parts not looking the same)



The two critical features of the part-of-a-whole meaning of fraction are that

- a natural whole exists, such as a piece of rope; and
- the whole is cut into equal parts according to size where *equal according to size* involves measurement (eg, length, area, volume, mass and so on). In the case of the piece of rope, it would be cut into sections of equal length.

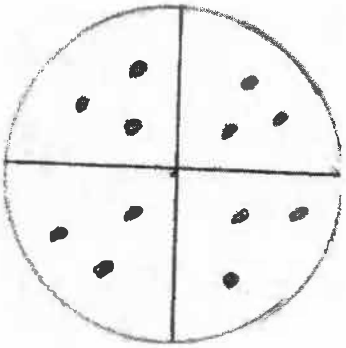
Part-of-a-Set Meaning of Fraction

The part-of-a-set meaning does not involve cutting a natural whole into equal parts—it involves selecting objects from a collection of discrete objects according to some attribute (eg, colour, being a student, made of glass). A set (or group) is not a naturally occurring whole as is a pie.

Suppose there are 23 books of varying size and content on a shelf and 14 of them are novels. We can represent this situation by the fraction 14/23. For this situation, we mean that 14 out of the 23 books are novels. The part-of-a-set meaning involves placing discrete things into categories for which the requirement is *belonging to*, not *equality of size*. This is a different enterprise than cutting up a whole object into parts of equal size based on a measurement concept.

Equal sharing is a special case of the part-of-a-set meaning. Suppose Mary receives a share of 12 candies shared equally among 4 people (including Mary). Mary's share is 3/12 (or 1/4). To some, equal sharing may seem like part of a whole (especially when a circle cut into 4 equal parts is used as the underlying prop for the sharing—refer to Figure 5), but it is not. The attribute is *equality of count* of discrete things (not equal parts of a whole). Counting is not equivalent to measuring. The things being counted do not have a stricture on them about being identical in length, mass and so on.

Figure 5: Part of a set (equal sharing using a fraction circle prop)

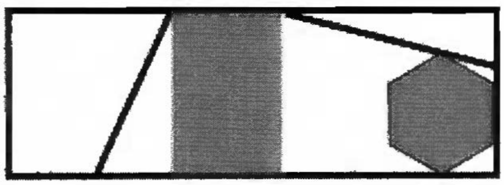


The three critical features of the part-of-a-set meaning of fraction are that

- there exists a collection of discrete things. This collection is seen as acting as a whole;
- equality of size is not required (although it could be present); and
- characteristics (attributes of interest) of the things in the collection are used to determine the fraction.

Following the development of the two core meanings of fraction, an example that compares and contrasts them further clarifies the distinction between them. Consider the large (outside) rectangle in Figure 6. The large rectangle (the whole) has been cut into eight parts, but the parts are not equal in area. We cannot use the part-of-a-whole meaning to say that the shaded area is 2/8 of the large rectangle.

Figure 6: A large rectangle cut into eight parts



However, from the perspective of the part-of-a-set meaning of fraction, equality of size is not required. What *is* required is a collection of things and an attribute of interest. There are eight shapes that comprise the large rectangle. They can be considered as the collection of things. If *shaded* is the attribute of interest, then, in that collection, two shapes are shaded. Thus, 2/8 of the collection is shaded.

When we use shading in a part-of-a-whole situation, we are not interested in shading as an attribute. Rather, shading is a way of identifying the number of equal parts to consider. Using shading for identifying the equal parts has become a tradition in educational circles. We could use other ways of identifying the equal parts—for example, we could put a checkmark in each of them.

The Anecdote Considered Through Question Posing and Discussion

Once the part-of-a-whole and the part-of-a-set meaning of fraction are understood, I retell the anecdote and shift to a question-posing and response session to delve into fraction addition. (For the sake of brevity, the teacher candidates' responses provided here are distillations of the actual ones. Also, not all of the questions and responses are included.)

Author: For the anecdote, what is the answer to the addition if you use the common denominator method that you were taught in junior high?

Teacher candidates: 5/6 + 1/4 is 10/12 + 3/12 = 13/12 or 1 1/12.

Author: Does this answer make sense?

Teacher candidates: No. It means that they won more games than they played. This is nonsense. Winning 6/10 of the games played is not nonsense.

Doubt about the correctness of the addition algorithm they were taught circulates around the room.

Author: Does the situation involve the part-of-a-whole or the part-of-a-set meaning of fraction?

There is no consensus. While the majority indicates part of a set, some see it as part of a whole. Further questioning about what is the whole and whether there is equality of size convinces a minority that the part-of-a-set meaning of fraction is involved. I return to the main thread.

Author: What is the set?

There is disagreement. Some say the number of home games is the set; others say away games. Some say home and away games. (To quote Sherlock Holmes, the game is now afoot.)

Author: If only home games or away games is the set, this creates a circumstance where one choice leaves out the other. Does it make sense to add a fraction that is not from the set? For example, suppose the set is home games. Five-sixths of the home games are won. Does it make sense to add ¼ to 5/6, if the away games are not included in the set?

There is reluctant agreement that it does not make sense. I follow up with another question.

Author: Suppose home games and away games together are the set. How many games belong to the set?

Teacher candidates: Ten games in all are in the set.

Author: What fraction of those ten games are home games won?

Teacher candidates: Five tenths.

Author: What fraction of those ten games are away games won?

Teacher candidates: One tenth.

Author: What fraction of the games played has the team won?

Teacher candidates: Six tenths.

Author: What fractions did you add to get 6/10 as the answer?

Teacher candidates: 1/10 and 5/10.

Author: Are you using the common denominator method you learned in junior high?

Teacher candidates: Yes. Both fractions already have the same denominator so we just added the numerators.

Author: What would you say to the Grade 7 student in response to his comment that the common denominator method is wrong?

Eventually, the conceptual quagmire pointed to by the question is sorted out through discussion. The teacher candidates come to realize that home and away games form a set of 10 games and that the attribute of interest is games won. While it may have appeared in the anecdote that the fraction of games won was obtained by adding numerators to numerators and denominators to denominators, what was actually happening was that the home and away games had to be combined into one set before fraction addition could take place. The combining gave the illusion of denominators being added to denominators. We could not have added the fractions had each one come from a different set.

We dig deeper into fraction addition. I pose the following problem.

There are two identical pies on the table. Joe eats 1/2 of pie #1. Hank eats ¼ of pie #2. How much pie is eaten in all?

Most of the teacher candidates add ½ and ¼, obtaining ¾ as the result. A few add ²⁄₈ and ⅛, obtaining ⅜ as the result. I ask what meaning of fraction is involved. There is consensus that it is part of a whole. I ask the obvious question: "What is the whole?"

Those who obtained ¾ as an answer respond that it is a pie, and that ¾ of one was eaten. Those who obtained ⅜ as a result respond that it is both pies combined and that there are 8 pieces of equal size on the table, 4 quarter-sections in pie #1 and 4 in pie #2—a total of 8 quarter-sections. We discuss the matter and conclude that the answer to the addition depends on what we view as the whole. The question "How much pie is eaten in all?" is ambiguous about what is the whole. A sharper question might have been "What fraction of the pie on the table was eaten?" This question suggests both pies together as the whole and thus the answer would be ⅜.

Conclusion

The teacher candidates have experienced the murky world of fraction addition. They are starting to realize that fraction addition depends on what we consider to be the whole or the set. Without clarity about that, the answers we obtain do not necessarily make sense. If we had just used models such as circles cut into equal parts, they would not have realized that. Question posing about fuzzy situations was needed.

The question that emerges from some at the end of the session is: Why did you make us go through all of this painful thinking?

After reminding them of the anecdote, I talk about how they may have a couple of students who see the adding-numerators-and-adding-denominators method of adding fractions as making sense. This is especially likely with students who play team sports. One reason, therefore, for putting them through the painful thinking was to prepare them for such a possibility and to help them be able to address the matter in an appropriate way.

I talk about another reason. I wanted to model a question-posing method of teaching, one that they will hopefully be comfortable in using. One of the current labels for this approach is *teacher as facilitator*. I point out that it is not really a new method of teaching. In the 1960s, the label was *Socratic teaching*—a variant of the Socratic debating method, which involves inquiry and debate between individuals with opposing points of view. In Socratic teaching, the focus is on posing questions to stimulate student thinking.

We also discuss their level of engagement during the question-posing experience. They invariably tell me that it was high. I ask them to imagine that I had used a show-and-tell approach instead and to compare their level of engagement with it to their level of engagement with the question-posing approach. Without fail, they inform me that they would not be as engaged with a show-and-tell approach to teaching. The point is made.

I conclude the methods course session by opening the door to another bout of painful thinking by presenting the following problem for them to think about for the next session.

> There are two pies on the table. Pie #1 is 30 cm across. Pie #2 is 15 cm across. Joe eats 1/2 of pie #1. Hank eats ¼ of pie #2. How much pie is eaten in all?

References

Ball, D L. 1990. "Prospective Elementary and Secondary Teachers' Understanding of Division." *Journal for Research in Mathematics Education* 21, no 2: 132–44.

Barlow, A T, and J M Cates. 2006. "The Impact of Problem Posing on Teachers' Beliefs About Mathematics and Mathematics Teaching." *School Science and Mathematics* 106, no 2: 64–73.

Clarke, D M, A Roche and A Mitchell. 2007. "Year Six Fraction Understanding: A Part of the Whole Story." In *Proceedings of the 30th Annual Conference of the Mathematics Education Research Group of Australasia* Vol 1, ed J Watson and K Beswick, 207–16. Hobart, Australia: MERGA.

Ma, L. 1999. *Knowing and Teaching Elementary Mathematics: Teachers' Understanding of Fundamental Mathematics in China and the United States*. Mahwah, NJ: Lawrence Erlbaum.

National Council of Teachers of Mathematics (NCTM). 2000. *Principles and Standards for School Mathematics*. Reston, Va: NCTM.

Reeder, S, and J Utley. 2007. "Developing Fraction Understanding in Prospective Elementary Teachers." In *Proceedings of the 29th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*, ed T Lamberg and L R Wiest. Reno, Nev: University of Nevada.

Weller, K, I Arnon and E Dubinsky. 2009. "Preservice Teachers' Understanding of the Relation Between a Fraction or Integer and Its Decimal Expansion." *Canadian Journal of Science, Mathematics and Technology Education* 9, no 1: 5–8.

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Specialist councils' role in promoting diversity, equity and human rights

Alberta's rapidly changing demographics are creating an exciting cultural diversity that is reflected in the province's urban and rural classrooms. The new landscape of the school provides an ideal context in which to teach students that strength lies in diversity. The challenge that teachers face is to capitalize on the energy of today's intercultural classroom mix to lay the groundwork for all students to succeed. To support teachers in their critical roles as leaders in inclusive education, in 2000 the Alberta Teachers' Association established the Diversity, Equity and Human Rights Committee (DEHRC).

DEHRC aims to assist educators in their legal, professional and ethical responsibilities to protect all students and to maintain safe, caring and inclusive learning environments. Topics of focus for DEHRC include intercultural education, inclusive learning communities, gender equity, UNESCO Associated Schools Project Network, sexual orientation and gender variance.

Here are some activities the DEHR committee undertakes:

- Studying, advising and making recommendations on policies that reflect respect for diversity, equity and human rights
- Offering annual Inclusive Learning Communities Grants (up to $2,000) to support activities that support inclusion
- Producing *Just in Time*, an electronic newsletter that can be found at www.teachers.ab.ca; Teaching in Alberta; Diversity, Equity and Human Rights.
- Providing and creating print and web-based teacher resources
- Creating a list of presenters on DEHR topics
- Supporting the Association instructor workshops on diversity

Specialist councils are uniquely situated to learn about diversity issues directly from teachers in the field who see how diversity issues play out in subject areas. Specialist council members are encouraged to share the challenges they may be facing in terms of diversity in their own classrooms and to incorporate these discussions into specialist council activities, publications and conferences.

Diversity, equity and human rights affect the work of all members. What are you doing to make a difference?

Further information about the work of the DEHR committee can be found on the Association's website at www.teachers.ab.ca under Teaching in Alberta, Diversity, Equity and Human Rights.

Alternatively, contact Andrea Berg, executive staff officer, Professional Development, at andrea.berg@ata.ab.ca for more information.