THE CALGARY MATHEMATICAL ASSOCIATION

35^{th} JUNIOR HIGH SCHOOL MATHEMATICS CONTEST

May 4, 2011

NAME: SOLUTIONS GENDER: ☐ M ☐ F

PLEASE PRINT (First name Last name)

SCHOOL: ____________________ GRADE: ____ (7,8,9)

- You have 90 minutes for the examination. The test has two parts: PART A — short answer; and PART B — long answer. The exam has 9 pages including this one.
- Each correct answer to PART A will score 5 points. You must put the answer in the space provided. No part marks are given.
- Each problem in PART B carries 9 points. You should show all your work. Some credit for each problem is based on the clarity and completeness of your answer. You should make it clear why the answer is correct. PART A has a total possible score of 45 points. PART B has a total possible score of 54 points.
- You are permitted the use of rough paper. Geometry instruments are not necessary. References including mathematical tables and formula sheets are **not** permitted. Simple calculators without programming or graphic capabilities **are** allowed. Diagrams are not drawn to scale. They are intended as visual hints only.
- When the teacher tells you to start work you should read all the problems and select those you have the best chance to do first. You should answer as many problems as possible, but you may not have time to answer all the problems.

MARKERS' USE ONLY	
PART A ×5	
B1	
B2	
B3	
B4	
B5	
B6	
TOTAL (max: 99)	

BE SURE TO MARK YOUR NAME AND SCHOOL AT THE TOP OF THIS PAGE.
THE EXAM HAS 9 PAGES INCLUDING THIS COVER PAGE.

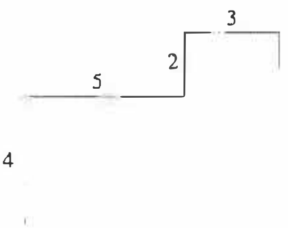
Please return the entire exam to your supervising teacher at the end of 90 minutes.

PART A: SHORT ANSWER QUESTIONS

A1 A store sells pies. Each pie costs the same price and two pies cost \$8. How much do three pies cost?

\$12

A2 Nahlah's living room is as shown in the diagram, with all distances in metres and with all angles 90°. What is the area (in square metres) of her living room?



38

A3 Doan mixes together 1 litre of 1% butterfat milk, 2 litres of 2% butterfat milk and 4 litres of 4% butterfat milk. What percentage of the resulting seven litres of milk is butterfat?

3%

A4 Nine people, all with different heights, are sitting around a circular table. What is the greatest possible number of people that could be taller than both persons sitting next to him/her?

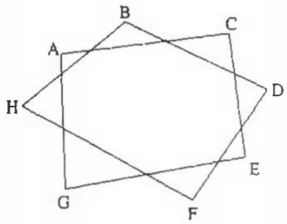
4

A5 The number $111\ldots1$ has 102 ones, and the number $222\ldots2$ has 101 twos. Suppose you do the subtraction $111\ldots1 - 222\ldots2$ to get a whole number. What is the sum of the digits of this whole number?

809

A6 In the following 8-pointed star, what is the sum of the angles A, B, C, D, E, F, G, H?

720 degrees

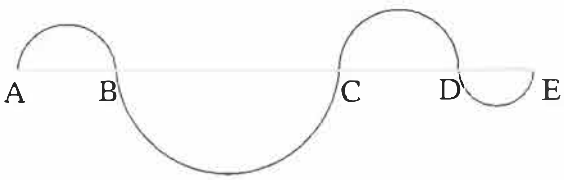


A7 Sixteen coins, numbered 1 to 16, are each red on one side and blue on the other side. Initially, all coins have their red sides facing up. The coins that are multiples of 2 are turned over. Then the coins that are multiples of 4 are turned over. Then the coins that are multiples of 8 are turned over. Then the coins that are multiples of 16 are turned over. Afterwards, how many of the coins have the red side facing up?

11

A8 Five points A, B, C, D, E lie on a line segment in order, as shown. The segment AE has length 10cm. Semi-circles with diameters AB, BC, CD, DE are drawn, as shown. The sum of the lengths of the semicircles $\widehat{AB}, \widehat{BC}, \widehat{CD}, \widehat{DE}$ can be written in the form $k\pi$ for some number k. What is k?

5



A9 Suppose that a and b are positive integers, and the four numbers

$$a+b, \quad a-b, \quad a \times b, \quad a \div b$$

are all different and are all positive integers. What is the smallest possible value of $a+b$?

8

PART B: LONG ANSWER QUESTIONS

B1 Ariel purchased a certain amount of apricots. 90% of the apricot weight was water. She dried the apricots until just 60% of the apricot weight was water. 15 kg of water was lost in the process. What was the original weight of the apricots (in kg)?

Solution 1: Let x be the original weight of the apricots (in kg). Then in the original apricots, $\frac{9x}{10}$ kg of the apricots is water. Since 15 kg of the water is lost during the drying process, $\frac{9x}{10} - 15$ kg of the water remains and the apricots weigh $x - 15$ kg. Since 60% of the dried apricot weight is water, we have

$$\frac{\frac{9x}{10} - 15}{x - 15} = \frac{60}{100} = \frac{3}{5}.$$

Cross multiplying yields $5(\frac{9x}{10} - 15) = 3(x - 15)$, or equivalently, $\frac{9x}{2} - 75 = 3x - 45$. Therefore, $\frac{9x}{2} - 3x = 30$, which simplifies to $\frac{3x}{2} = 30$. Hence, $3x = 60$. Solving for x yields $x = 20$.

Therefore, the original weight of the apricots is 20 kg.

Solution 2: In the dried apricots, the water is $\frac{60}{100} = \frac{3}{5}$ of the apricots. Therefore, the ratio of the water to the non-water part of the dried apricots is $3 : 2$. The ratio of the water to the non-water part of the original apricots is $9 : 1 = 18 : 2$. The non-water weight remains the same throughout the drying process. Hence, we can let $2x$ be the weight of the non-water part of the dried apricots. Then $18x$ is the weight of the water before drying and $3x$ is the weight of the water after drying. Since 15 kg of the water is lost, $18x - 3x = 15$, i.e. $15x = 15$, which yields $x = 1$. The original weight of the apricots is $18x + 2x = 20x = 20$.

Therefore, the original weight of the apricots is 20 kg.

B2 A group of ten friends all went to a movie together. Another group of nine friends also went to the same movie together. Fourteen of these 19 people each bought a regular bag of popcorn as well. It turned out that the total cost of the movie plus popcorn for one of the two groups was the same as for the other group. A movie ticket costs \$6. Find all possible costs of a regular bag of popcorn.

Solution 1: Since more people are in the first group than in the second group and the costs of the two groups are equal. there must be more people in the second group that ordered popcorn than in the first group. Since a total of 14 people ordered popcorn. but only 9 people are in the second group. there are two possibilities as to how many people from each group ordered popcorn.

Case 1: Five people from the first group ordered popcorn and all nine people from the second group ordered popcorn.

Case 2: Six people from the first group ordered popcorn and eight of the nine people from the second group ordered popcorn.

In the first case. the four extra popcorns for the second group must be equal in cost to the one extra movie ticket for the first group. Therefore. four popcorns must cost \$6. so one regular popcorn must cost \$1.50.

In the second case. the two extra popcorns for the second group must be equal in cost to the one extra movie ticket for the first group. Therefore two popcorns must cost \$6. so one regular popcorn must cost \$3.

Therefore. the possible costs of a regular bag of popcorn are \$1.50 and \$3.

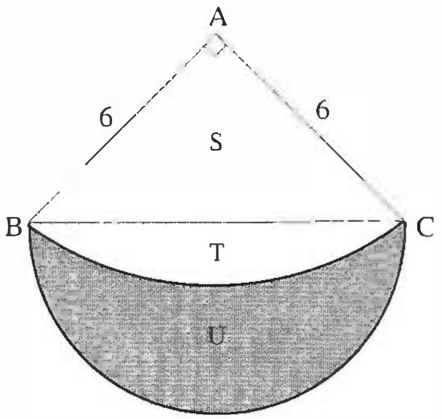
Solution 2: We proceed up to the two cases of Solution 1. Let x be the cost of one regular popcorn.

In the first case. the first group paid $10 \times 6 + 5x$ dollars and the second group paid $9 \times 6 + 9x$ dollars. Therefore. $10 \times 6 + 5x = 9 \times 6 + 9x$. Hence. $60 + 5x = 54 + 9x$. Therefore. $6 = 4x$. which yields $x = \frac{6}{4} = \frac{3}{2}$. Therefore. a regular popcorn costs \$1.50.

In the second case. the first group paid $10 \times 6 + 6x$ dollars and the second group paid $9 \times 6 + 8x$ dollars. Therefore. $10 \times 6 + 6x = 9 \times 6 + 8x$. Hence. $60 + 6x = 54 + 8x$. Therefore. $6 = 2x$. which yields $x = 3$. Therefore. a regular popcorn costs \$3.

Therefore. the possible costs of a regular bag of popcorn are \$1.50 and \$3.

B3 In the diagram, $AB = 6$ cm, $AC = 6$ cm and $\angle BAC$ is a right angle. Two arcs are drawn: a circular arc with centre A and passing through B and C, and a semi-circle with diameter BC, as shown.



(a) (1 mark) What is the area of ΔABC?

The area of the triangle is $\frac{1}{2} \times AB \times AC = \frac{1}{2} \times 6 \times 6 = 18$ cm^2.

(b) (2 marks) What is the length of BC?

By Pythagorean Theorem, the length of BC is $\sqrt{AB^2 + AC^2} = \sqrt{6^2 + 6^2} = \sqrt{72} = 6\sqrt{2}$. Therefore, the length of BC is $\sqrt{72}$ cm, or equivalently, $6\sqrt{2}$ cm.

(c) (6 marks) Find the area between the two arcs, i.e. find the area of the shaded figure in the diagram.

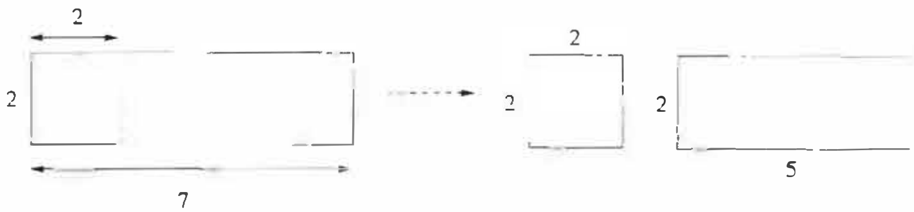
Solution: Let S, T, U be regions labeled in the above diagram, i.e. S the area of the triangle, T the area between side BC and the arc with centre A passing through BC and U the area between the two arcs.

We first find the area of the semi-circle with diameter BC. This is the area of T and U. The radius of the circle is half of BC, which is $3\sqrt{2}$. Hence, the area of the semi-circle is $\frac{1}{2}(\pi(3\sqrt{2})^2) = \frac{1}{2} \times 18\pi = 9\pi$ cm^2. Therefore, the area of T and U is 9π cm^2.

We now find the area of the quarter-circle with centre A passing through B and C, i.e. the area of S and T. The area of S and T is $\frac{1}{4} \times \pi \times 6^2 = 9\pi$ cm^2. Therefore, the area of S and T is 9π cm^2. Since the area of T and U is also 9π cm^2, S and U have the same area. The area of S is 18 cm^2 by (a), the area of U is also 18 cm^2.

Hence, the area of the shaded figure is 18 cm^2.

B4 Given a non-square rectangle, a *square-cut* is a cutting-up of the rectangle into two pieces, a square and a rectangle (which may or may not be a square). For example, performing a square-cut on a 2×7 rectangle yields a 2×2 square and a 2×5 rectangle, as shown.



You are initially given a 40×2011 rectangle. At each stage, you make a square-cut on the non-square piece. You repeat this until all pieces are squares. How many square pieces are there at the end?

Solution: We first cut as many squares with side 40 as we can. Upon division of 2011 by 40, the quotient is 50 and the remainder is 11. Therefore, we will have 50 squares with side 40 and what remains is a 11×40 rectangle.

With the resulting 11×40 rectangle, we use the similar idea as in the previous paragraph. The quotient of $40 \div 11$ is 3 and the remainder is 7. Therefore, we will have 3 squares with side 11 and what remains is a 7×11 rectangle.

The quotient of $11 \div 7$ is 1 and the remainder is 4. Therefore, we will have one square with side 7 and what remains is a 7×4 rectangle.

The quotient of $7 \div 4$ is 1 and the remainder is 3. Therefore, we will have one square with side 4 and what remains is a 4×3 rectangle.

The quotient of $4 \div 3$ is 1 and the remainder is 1. Therefore, we will have one square with side 3 and what remains is a 3×1 rectangle.

The quotient of $3 \div 1$ is 3 and the remainder is 0. Therefore, we will have three squares with side 1. Now, every piece is a square.

Counting all of the squares we have cut, the number of squares is $50+3+1+1+1+3 = 59$ squares. Therefore, there are 59 square pieces at the end.

B5 Five teams A, B, C, D, E participate in a hockey tournament where each team plays against each other team exactly once. Each game either ends in a win for one team and a loss for the other, or ends in a tie for both teams. The following table originally showed all of the results of the tournament, but some of the entries in the table have been erased.

Team	Wins	Losses	Ties
A	3		
B	1		1
C	1		
D	0		
E			4

The result of each game played can be uniquely determined. For each game in the table below, if the game ended in a win for one team, write down the winner of the game. If the game ended in a tie, write the word "Tie".

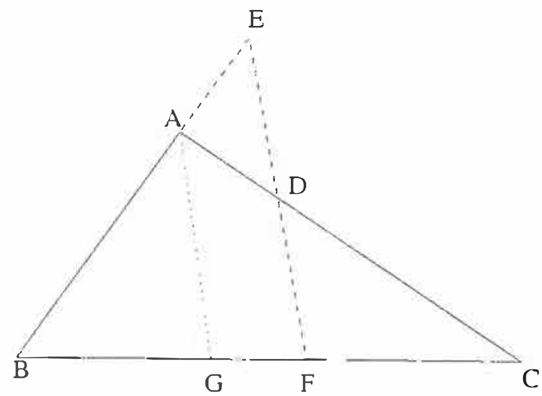
Solution:

Team A vs Team B	Team A
Team A vs Team C	Team A
Team A vs Team D	Team A
Team A vs Team E	Tie
Team B vs Team C	Team C
Team B vs Team D	Team B
Team B vs Team E	Tie
Team C vs Team D	Tie
Team C vs Team E	Tie
Team D vs Team E	Tie

Each team played four games. Therefore, team B lost 2 games. Furthermore, team E did not win or lose any game and team E tied with every team. Since team A won 3 games, and tied one game (with team E) and won every other game. Team A won against Teams B, C and D.

We now figure out the result amongst the games between teams B,C,D. Note that the total number of wins in the tournament is 5. Therefore, five games ended in a win/loss, which implies that the other five games ended in a tie. Therefore, the sum of the number of ties for the five teams is $5 \times 2 = 10$. Since all four games involving team E ended in a tie, there is another tied game in a game played amongst A,B,C,D. Since each of teams A,B has only one tie (with team E), then teams C and D tied in their game. Hence, the number of ties teams C and D each have is 2. Therefore, team C lost 1 game and team D lost 2 games. We know team C lost to team A and has one win unaccounted for. Therefore, team C won against team B. Finally, team B has one win unaccounted for. Therefore, team B won against team D. This completes the table.

B6 A triangle ABC has sides $AB = 5, AC = 7, BC = 8$. Point D is on side AC such that $AB = CD$. We extend the side BA past A to a point E such that $AC = BE$. Let the line ED intersect side BC at a point F.



(a) (2 marks) Find the lengths of AD and AE.

Since $DC = 5$, $AD = AC - DC = 7 - 5 = 2$. Since $BE = AC = 7$, $AE = BE - AB = 7 - 5 = 2$.

Therefore, AD has length 2 and AE has length 2.

(b) (7 marks) Find the lengths of BF and FC.

We draw a line passing through A parallel to the line EF, as shown. Let this line intersect the side BC at a point G. Then by similar triangles and parallel lines, we have

$$\frac{BG}{GF} = \frac{BA}{AE} = \frac{5}{2}, \text{ and } \frac{GF}{FC} = \frac{AD}{DC} = \frac{2}{5}.$$

Hence, $BG : GF : FC = 5 : 2 : 5$. Therefore,

$$\frac{BF}{FC} = \frac{7}{5}.$$

Since $BC = 8$, $BF = \frac{7}{12} \times BC = \frac{7}{12} \times 8 = \frac{14}{3}$. Hence, $FC = 8 - BF = 8 - \frac{14}{3} = \frac{10}{3}$.

Therefore, the length of BF is $\frac{14}{3}$ and the length of FC is $\frac{10}{3}$.