The Alberta High School Mathematics Competition
Part I, November 16, 2010

1. The number of positive integers n such that the number $4n$ has exactly two digits is

   (a) 21 (b) 22 (c) 23 (d) 24 (e) 25

2. A 4×6 plot of land is divided into 1×1 lots by fences parallel to the edges of the plot, with fences along the edges as well. The total length of fences is:

   (a) 58 (b) 62 (c) 68 (d) 72 (e) 96

3. The greatest common divisor and least common multiple of two positive integers are 1 and 10 respectively. If neither of them is equal to 10, their sum is equal to

   (a) 3 (b) 6 (c) 7 (d) 11 (e) none of these

4. The number of pairs (x, y) of non-negative integers such that $3x + 2y = 27$ is

   (a) 4 (b) 5 (c) 8 (d) 9 (e) 10

5. In the sequence 1, 2, 3, 4, 6, 7, 8, 9, ..., obtained by deleting the multiples of 5 from the sequence of the positive integers, the 2010th term is

   (a) 2511 (b) 2512 (c) 2513 (d) 2514 (e) none of these

6. Alice, Brian, Colin, Debra and Ethel are in a hotel. Their rooms are on floors 1, 2, 3, 21 and 40 respectively. In order to minimize the total number of floors they have to cover to get together, the floor on which the get-together should be is

   (a) 18 (b) 19 (c) 20 (d) 21 (e) none of these

7. A square pigeon coop is divided by interior walls into 9 square pigeonholes in a 3×3 configuration. Each of two pigeons chooses a pigeonhole at random, possibly the same one. The probability that they choose two holes on the opposite sides of an interior wall is

   (a) $\frac{1}{18}$ (b) $\frac{1}{9}$ (c) $\frac{4}{27}$ (d) $\frac{8}{27}$ (e) $\frac{1}{3}$

8. The set of all values of the real number x such that $\frac{1}{x} \leq -3 \leq x$ is

   (a) $\{x \leq -1/3\}$ (b) $\{-3 \leq x \leq -1/3\}$ (c) $\{-3 \leq x < 0\}$

   (d) $\{-1/3 \leq x < 0\}$ (e) none of these

9. In the quadrilateral $ABCD$. AB is parallel to DC, $DC=2AB$. $\angle ADC=30^\circ$ and $\angle BCD=50^\circ$. Let M be the midpoint of CD. The measure of $\angle AMB$ is

(a) 80° (b) 90° (c) 100° (d) 110° (e) 120°

10. We are constructing isosceles but non-equilateral triangles with positive areas and integral side lengths between 1 and 9 inclusive. The number of such triangles which are non-congruent is

(a) 16 (b) 36 (c) 52 (d) 61 (e) none of these

11. In each of the following numbers, the exponents are to be evaluated from top down. For instance, $a^{b^c} = a^{(b^c)}$. The largest one of these five numbers is

(a) $2^{2^{2^{2^3}}}$ (b) $2^{2^{2^{3^2}}}$ (c) $2^{2^{3^{2^2}}}$ (d) $2^{3^{2^{2^2}}}$ (e) $3^{2^{2^{2^2}}}$

12. A gold number is a positive integer which can be expressed in the form $ab + a + b$, where a and b are positive integers. The number of gold numbers between 1 and 20 inclusive is

(a) 8 (b) 9 (c) 10 (d) 11 (e) 12

13. The edges DA, DB and DC of a tetrahedron $ABCD$ are perpendicular to one another. If the length of DA is 1 cm and the length of each of DB and DC is 2 cm, the radius, in cm, of the sphere passing through A, B, C and D is

(a) $\frac{3}{2}$ (b) $\frac{\sqrt{5}+1}{2}$ (c) $\sqrt{3}$ (d) $\sqrt{2}+\frac{1}{2}$ (e) none of these

14. Let $f(x) = x^2$ and $g(x) = x^4$. We apply f and g alternatively to form

$$f(x) = x^2,\ g(f(x)) = g(x^2) = (x^2)^4 = x^8,\ f(g(f(x)) = f(x^8) = (x^8)^2 = x^{16},$$

and so on. After we have applied f 50 times and g 49 times, the answer is x^n where n is

(a) 148 (b) 296 (c) 2^{148} (d) 2^{296} (e) none of these

15. Triangle ABC has area 1. X, Y are points on the side AB and Z a point on the side AC such that $XY = 2AX$, XZ is parallel to YC and YZ is parallel to BC. The area of XYZ is

(a) $\frac{1}{27}$ (b) $\frac{2}{27}$ (c) $\frac{1}{9}$ (d) $\frac{2}{9}$ (e) $\frac{1}{3}$

16. The number of integers n for which $n^3 - 3n + 2$ is divisible by $2n + 1$ is

(a) 3 (b) 4 (c) 5 (d) 6 (e) 8

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1. Since $10 \le 4n \le 99$, $3 \le n \le 24$. Hence there are $24 - 3 + 1 = 22$ such values. The answer is **(b)**.

2. There are 7 fences of length 4 and 5 fences of length 6. The total length is $7 \times 4 + 5 \times 6 = 58$. The answer is **(a)**.

3. Since the least common multiple is even, at least one number is even. Since the greatest common divisor is odd, exactly one number is even. We can show in a similar manner that exactly one of the two numbers is divisible by 5. Since neither is 10, one of them is 2 and the other is 5, yielding a sum of 7. The answer is **(c)**.

4. Note that x must be odd, and $x = 9 - \frac{2y}{3}$. Since $y \ge 0$, $x \le 9$. Thus there are 5 triples $(x, y) = (9, 0)$, $(7, 3)$, $(5, 6)$, $(3, 9)$ and $(1, 12)$. The answer is **(b)**.

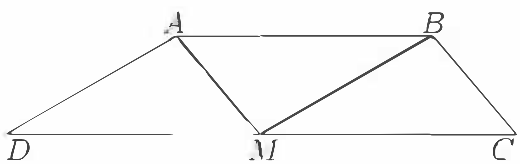
5. Note that $2010 \times \frac{5}{4} = 2512.5$. There are 502 multiples of 5 from 5 to 2510 inclusive. Hence 2512 is the $(2512 - 502)$-th or 2010-th number in the punctured sequence. The answer is **(b)**.

6. Alice and Ethel are 39 floors apart, and as long as the get-together floor is in between, the total number of floors they cover is 39. Similarly, the total number of floors Brian and Debra cover is 19. The minimum number of floors Colin covers is 0, when they get together on floor 3. The answer is **(e)**.

7. The pigeons can choose the pigeonholes in $9 \times 9 = 81$ ways. There are 12 pairs of rooms separated by an interior wall. Since the pigeons can choose these two rooms in 2 ways, the desired probability is $\frac{2\times12}{81} = \frac{8}{27}$. The answer is **(d)**.

8. Since $\frac{1}{x} \le -3$, $x \le 0$. Multiplying by $-\frac{x}{3}$, we have $-\frac{1}{3} \le x$. Hence the set of all values of x is $\{-\frac{1}{3} \le x < 0\}$. The answer is **(d)**.

9. Since $AB = DM$ and AB is parallel to DM, $ABMD$ is a parallelogram. Similarly, $ABCM$ is a parallelogram. Therefore, $\angle AMD = \angle BCD = 50°$ and $\angle BMC = \angle ADC = 30°$. Therefore, $\angle AMB = 180° - \angle AMD - \angle BMC = 180° - 50° - 30° = 100°$. The answer is **(c)**.



10. We first count the triangles in which the equal sides of length k are longer than the third side, which can be of length from 1 to $k-1$. Summing from $k = 1$ to 9, we have $0+1+2+\cdots+8 = 36$ such triangles. We now count the triangles in which the equal sides of length k are shorter than the third side, which can be of length from $k+1$ to $2k-1$. Summing from $k = 1$ to 9, we have $0 + 1 + 2 + 3 + 4 + 3 + 2 + 1 + 0 = 16$. The total is $36 + 16 = 52$. The answer is **(c)**.

11. The first three choices are equal respectively to $2^{2^{256}}$, $2^{2^{512}}$ and $2^{2^{81}}$. Clearly, the second one is the largest among them. The fourth number is equal to $2^{3^{16}}$. Since $2^{512} = 4^{256} > 3^{256} > 3^{16}$, the second number is larger than the fourth one. The fifth number is equal to $3^{2^{16}}$. Clearly, $2^{2^{256}} = 4^{2^{255}} > 3^{2^{255}} > 3^{2^{16}}$. Hence the second number is the largest among the five choices. The answer is **(b)**.

12. Note that $ab + a + b + 1 = (a+1)(b+1)$. Every composite number can be written in this form and no prime number can be written in this form. Therefore, the positive integers that are not gold numbers are those that are one less than a prime. By simple counting, we see there are 8 primes from 2 to 21. Therefore, the number of gold numbers between 1 and 20 inclusive is $20 - 8 = 12$. The answer is **(e)**.

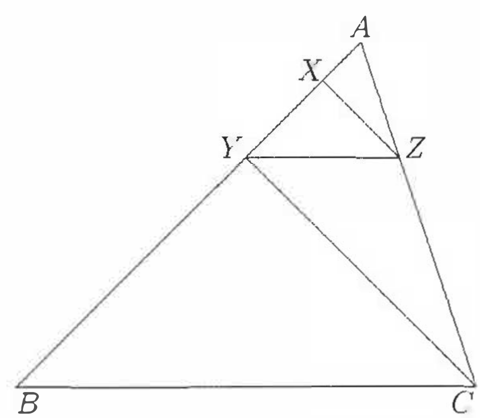
13. The sphere which passes through A, B, C and D also passes through the other four vertices of a $1 \times 2 \times 2$ block having A, B, C and D as four of its vertices. Since the space diagonal of this block is of length $\sqrt{2^2 + 2^2 + 1^2} = 3$, the radius of the sphere is $\frac{3}{2}$. The answer is **(a)**.

14. Each application of f doubles the exponent while each application of g quadruples the exponent. After 50 applications of f and 49 applications of g, the exponent has been doubled $50 + 2 \times 49 = 148$ times so that $n = 2^{148}$. The answer is **(c)**.

15. Denote the area of triangle T by $[T]$. Since triangles AXZ and AYC are similar, $ZC = 2AZ$. Since triangles AYZ and ABC are similar, $YB = 2AY$. It follows that

$$[XYZ] = \frac{2}{3}[AYZ] = \frac{2}{9}[AYC] = \frac{2}{27}[ABC] = \frac{2}{27}.$$

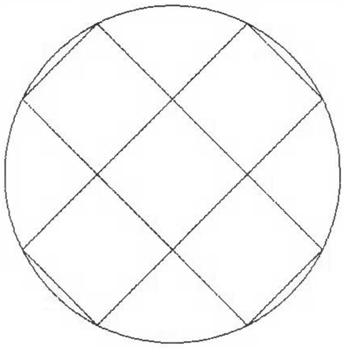
The answer is **(b)**.

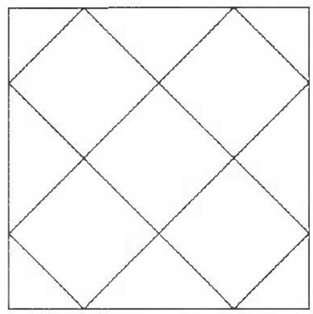


16. When $n^3 - 3n + 2$ is divided by $2n+1$, the quotient is $\frac{n^2}{2} - \frac{n}{4} - \frac{11}{8}$ and the remainder is $\frac{27}{8}$. Hence $8(n^3 - 3n + 2) = (2n+1)(4n^2 - 2n - 11) + 27$, so that $2n + 1$ divides $n^3 - 3n + 2$ if and only if it divides 27. The set of all values of n for which $2n+1$ divides 27 is $\{-14, -5, -2, -1, 0, 1, 4, 13\}$, and there are 8 such values. The answer is **(e)**.

The Alberta High School Mathematics Competition
Part II, February 2, 2011.

Problem 1.
A cross-shaped figure is made up of five unit squares. Determine which has the larger area, the circle touching all eight outside corners of this figure, as shown in the diagram below on the left, or the square touching the same eight corners, as shown in the diagram below on the right.





Problem 2.
There is exactly one triple (x, y, z) of real numbers such that $x^2 + y^2 = 2z$ and $x + y + z = t$. Determine the value of t.

Problem 3.
On the side BC of triangle ABC are points P and Q such that P is closer to B than Q and $\angle PAQ = \frac{1}{2}\angle BAC$. X and Y are points on lines AB and AC, respectively, such that $\angle XPA = \angle APQ$ and $\angle YQA = \angle AQP$. Prove that $PQ = PX + QY$.

Problem 4.
Determine all the functions f from the set of integers to the set of positive integers such that $f(n-1) + f(n+1) \leq 2f(n)$ for all integers n.

Problem 5.
Seven teams gather and each pair of teams play one of three sports, such that no set of three teams all play the same sport among themselves. A triplet of teams is said to be *diverse* if all three sports are played among themselves. What is the maximum possible number of diverse triplets among the seven teams?

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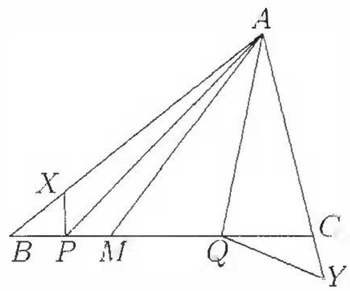
Problem 1.
The diameter of the circle, being the diagonal of a 1×3 rectangle, is $\sqrt{10}$, so the area of the circle is $\pi(\sqrt{10}/2)^2 = \frac{5\pi}{2}$. The diagonal of the square is 4, so the side of the square is $\frac{4^2}{2} = 8$. Since $\pi < 3.2 = \frac{16}{5}$, we have $\frac{5\pi}{2} < 8$. Thus the square has greater area than the circle.

Problem 2.
Eliminating z, we have $x^2 + y^2 = 2(t - x - y)$ so that $(x+1)^2 + (y+1)^2 = 2(t+1)$. In order to have a unique solution for x and y, we must have $2(t+1) = 0$ or $t = -1$.

Problem 3.
Let M be the point on PQ such that $\angle MAP = \angle BAP$. Then

$$\begin{aligned}
\angle MAQ &= \angle PAQ - \angle MAP \\
&= \frac{1}{2}(\angle ABC - \angle MAB) \\
&= \frac{1}{2}\angle MAC \\
&= \angle CAQ.
\end{aligned}$$

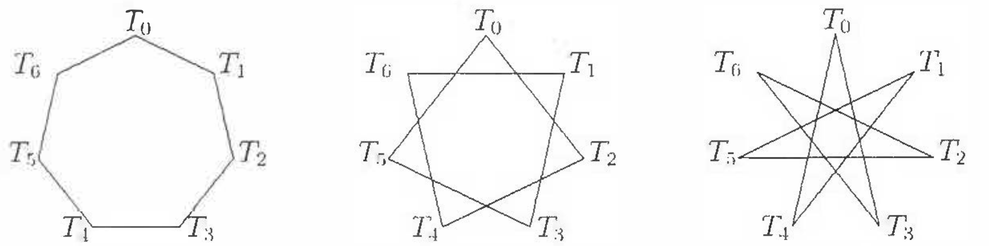
Since $\angle XPA = \angle MPA$, triangles XAP and MAP are congruent by the ASA Postulate, so that $PX = PM$. Similarly, we can prove that $QY = QM$, so that $PX + QY = PM + QM = PQ$.



Problem 4.
Since $f(n)$ takes on only positive integral values, it has a minimum value m. Let n be such that $f(n) = m$. Then $2m \leq f(n-1)+f(n+1) \leq 2f(n) = 2m$, which implies that $f(n-1) = f(n+1) = m$ also. It follows easily that $f(n) = m$ for all integers n.

Problem 5.

We first show that the conditions of the problem can be satisfied. Construct a graph where the teams are represented by vertices T_i, $0 \le i \le 6$. In the diagram below, we partition the graph into three subgraphs. Two teams play each other in the first sport if and only if the vertices representing them are joined by an edge in the first subgraph, the second sport in the second subgraph and the third sport in the third subgraph. None of the subgraphs contains a triangle.



The edges in the same subgraph have the same length, and those in different subgraphs have different lengths. In geometric terms, a diverse triple is a scalene triangle. There is basically one such triangle, namely $T_0T_1T_3$. Six others can be obtained from it by rotation, and seven more by reflection. Thus we may have as many as 14 scalene triangles.

We now prove that there are at most 14 diverse triples. Construct a complete graph on 7 vertices which represent the 7 teams. Paint an edge in the i-th colour if the teams represented by its endpoints play each other in the i-th sport, $1 \le i \le 3$. A triangle is diverse if all three sides are of different colours, and non-diverse otherwise. Since there are no monochromatic triangles, a non-diverse triangle has two sides of the same colour. Call the vertex at the junction of the two sides of the same colour its *pivot*. The number of pivots is equal to the number of non-diverse triangles. There are six edges incident with each vertex. If at least 3 of them are of the same colour, then this vertex is the pivot of at least 3 non-diverse triangles. If not, then exactly 2 edges are of each colour, so that the vertex is the pivot of exactly 3 isosceles triangles. Hence each vertex is the pivot of at least 3 non-diverse triangles. Since there are 7 vertices, this brings the total to at least 21, so that the maximum number of diverse triangles or diverse triples is 14.