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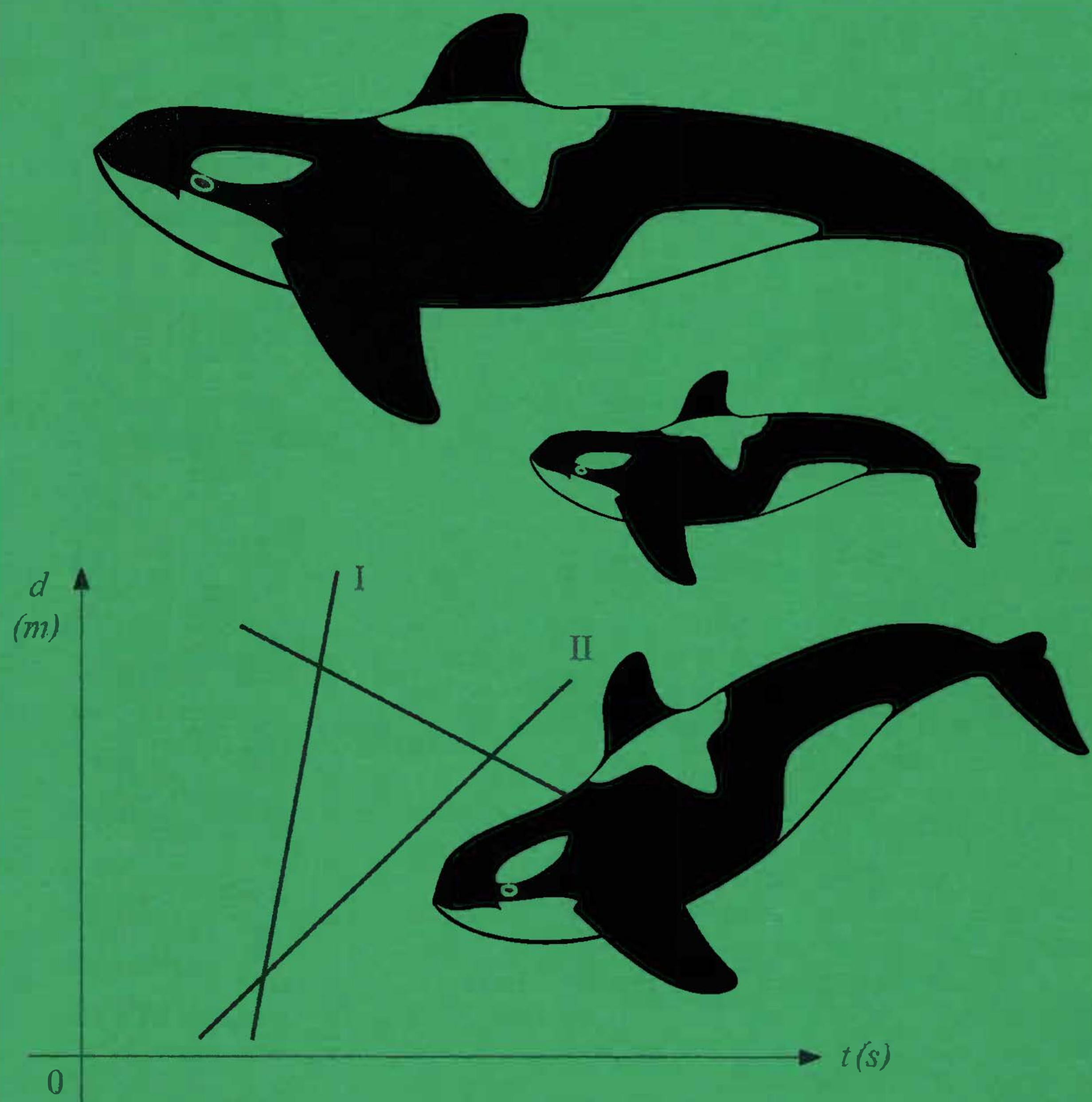

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Guidelines for Manuscripts

delta-K is a professional journal for mathematics teachers in Alberta. It is published twice a year to

- promote the professional development of mathematics educators, and
- stimulate thinking, explore new ideas, and offer various viewpoints.

Submissions are requested that have a classroom as well as a scholarly focus. They may include

- personal explorations of significant classroom experiences;
- descriptions of innovative classroom and school practices;
- reviews or evaluations of instructional and curricular methods, programs or materials;
- discussions of trends, issues or policies;
- a specific focus on technology in the classroom; or
- a focus on the curriculum, professional and assessment standards of the NCTM.

Suggestions for Writers

1. *delta-K* is a refereed journal. Manuscripts submitted to *delta-K* should be original material. Articles currently under consideration by other journals will not be reviewed.
2. If a manuscript is accepted for publication, its author(s) will agree to transfer copyright to the Mathematics Council of the Alberta Teachers' Association for the republication, representation and distribution of the original and derivative material.
3. All manuscripts should be typewritten, double-spaced and properly referenced. All pages should be numbered.
4. The author's name and full address should be provided on a separate page. If an article has more than one author, the contact author must be clearly identified. Authors should avoid all other references that may reveal their identities to the reviewers.
5. All manuscripts should be submitted electronically, using Microsoft Word format.
6. Pictures or illustrations should be clearly labelled and placed where you want them to appear in the article. A caption and photo credit should accompany each photograph.
7. References should be formatted consistently using *The Chicago Manual of Style*'s author–date system or *The American Psychological Association (APA)* style manual.
8. If any student sample work is included, please provide a release letter from the student's parent/guardian allowing publication in the journal.
9. Articles are normally 8–10 pages in length.
10. Letters to the editor or reviews of curriculum materials are welcome.
11. Send manuscripts and inquiries to the editor: Gladys Sterenberg, 195 Sheep River Cove, Okotoks, AB T1S 2L4; e-mail gladyss@ualberta.ca.

MCATA Mission Statement

Providing leadership to encourage the continuing enhancement of teaching, learning and understanding mathematics.

delta-K

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From the Editor's Desk

Gladys Sterenberg

It's hard to believe that another semester is complete. Many teachers provincewide are engaged with the implementation of the new mathematics program of studies. The feature articles of this issue focus on making mathematics meaningful and teaching for deep understanding. Drawing on his vast experience in mathematics education, Werner Liedtke considers the impact of the curriculum changes. Jérôme Proulx provides a perspective of how mathematics teaching and learning can be revitalized. Both articles prompt us to think about the effectiveness of our instruction.

As we move toward a complete program implementation, I am reminded of some of the essential components that have made teaching and learning mathematics in Alberta so effective. Problem solving continues to be a focus of our instruction, and the articles on teaching ideas are consistent with this emphasis. Craig Loewen, a regular contributor and former editor of *delta-K*, presents an example of how a probability problem can be explored using technology. Lesley Ross and Brenda Wells explore how a rich task can be assessed in their classrooms. Three Alberta mathematics contests are provided as a source of classroom problems for junior and senior high school students.

Concluding this issue is a problem page. When this problem was submitted, one of the reviewers, Darryl Smith, suggested that we have our own math contest. The MCATA executive thought this was a great idea. The contest will be held for two groups: students and teachers. I invite you to submit your solutions and/or those solutions of your students to me at gladyss@ualberta.ca by March 30, 2011. All entries will be evaluated by selected reviewers. Solutions that demonstrate creative problem-solving strategies and correct answers will be published in upcoming issues of *delta-K*. It is my hope that this will provide an interesting way of engaging in the process of problem solving.

As always, I want to encourage you to consider publishing your teaching and scholarly ideas in *delta-K*. The guidelines are listed on the inside of the front cover. I would be more than willing to assist you with this process.

Happy reading!

Piano Keys

Kevin Wang

On a full-sized piano, there are 88 keys. Many different combinations of keys can be pressed, and hence many sounds can be made. For example, a one-keyed piano can make one sound. A piano with two keys can make three possible sounds from pressing the keys: two at the same time, key number one and key number two. A piano with three keys can make seven sounds. Thus, on a full-sized piano with 88 keys, how many different sounds can be made?

Answer: 309 485 009 821 345 068 724 781 055 combinations/sounds, or 3.0948501×10^{26}.

The least mind-blowing way to solve it would be to form a table of values:

Keys pressed	Combinations
1	1
2	3
3	7
4	15
5	31

Here we see a simple chart. After some careful looking, we can see that when a value of one is added to each combination, it forms:

Keys pressed	Combinations
1	2
2	4
3	8
4	16
5	32

Now it gets clearer. The combinations seem to be just one more than 2 to the exponent of the value of the keys. That would result in a formula $2^x - 1$, where x is the number of keys.

This solution gives the answer of 309 485 009 821 345 068 724 781 055 combinations on a full-sized piano.

Kevin Wang is a Grade 8 student at Grandview Heights School, in Edmonton. He has always loved to do math and has witnessed the wonders it can do. One of his recent inspirations for math was Mr Walsh, who taught him to learn math and not just do it. Kevin is extremely interested in computers as the world seems to revolve around them, along with math, of course.

The New Curriculum: Will It Make Mathematics Meaningful for Students?

Werner Liedtke

Looking Back

The framework for the previous British Columbia mathematics curriculum (Ministry of Education 1995) included general goals related to use of the imagination, tolerance for ambiguity, positive attitude, the ability to communicate, sense making, enjoyment and effective problem solving. Developing or demonstrating number sense was listed at the top of every grade level. Similar goals were part of the previous curricula in western and northern Canada.

Does any concrete evidence exist that is indicative of

- positive behavioural changes regarding the major goals of the mathematics curriculum?
- changes in use of the imagination, willingness to take risks, growth in tolerance for ambiguity and/or achievement of high levels of development in mathematical thinking?
- changes in or improvement of development of number sense?

During a conversation, a mathematics educator who is interested in key aspects of the elementary mathematics curriculum responded to the question "Do you have any data that indicate that we reached these goals?" with, "Perhaps the goals were too high."

Hume (2009) suggests that "it's time to take a new approach to math." In part, his conclusion is based on the fact that about 30 per cent of Canadian parents hire tutors to assist their children with mathematics. This suggests that for many students the goals of the curriculum were not reached.

Hume also reports that "kids like and enjoy mathematics in the elementary grades," but then things worsen for many students. Is liking and enjoying mathematics sufficient? One Grade 7 BC teacher said that according to a survey in his school all Grade 7 students enjoyed problem solving, but test results showed that they could not solve problems. Enjoyment does not imply that curriculum goals are reached.

My data collection and years of teaching span several mathematics curriculum changes. The observations originate from data about conceptual understanding and number sense collected annually from interacting with K–7 students in many classrooms; conducting thorough diagnostic interviews with Grades 1–7 students; talking to teachers-to-be, students and parents in and around schools; and talking to adults who went through the school system.

Each group of students enrolled in courses about mathematics teaching, learning and assessment (including distance education courses with teachers from different areas of BC as well as from other provinces) completed at least one assignment that involved a conversation about mathematics learning and/or assessment with a student and an adult. Over the years many hundreds of transcripts of conversations with subjects from urban and rural areas, even with teachers on picket lines, were read and analyzed.

My data led me to conclude that many students did not achieve some of the key goals of the previous curriculum and that mathematics is not meaningful for them despite the time, energy and money spent on several revisions and additions to the curriculum. Could the goals be too high?

Looking Ahead

Is it possible that the Western and Northern Canadian Protocol (Alberta Education 2006), which provides the framework, topics and general goals for the new elementary mathematics (K–9) and was adopted, or adopted with minor revisions, by jurisdictions in western and northern Canada, can provide the new approach that Hume is looking for? This new curriculum includes the following list of critical components that students must encounter in a mathematics program and goals that can positively affect students' learning about mathematics. Ensuring that students encounter these components and reach the goals requires paying special attention to important ideas related to teaching and assessment.

The Critical Components—Selected Issues Related to Teaching

Communicating Mathematically

Students are expected to communicate in order to learn and express their understanding. The ability to talk and write in one's own words about mathematics and to connect familiar language to mathematical language is essential to the development of conceptual understanding, an important goal of the new curriculum. Writing tasks allow students to think about their thinking. As recordings are shared, ideas may be reinforced or modified. Discussions in cooperative settings and comparisons of strategies and reports by students can contribute to flexible as well as advanced thinking.

Writing was mentioned as part of communicating mathematically in the previous curriculum, but neither specific outcomes nor ideas were suggested for making it part of teaching. This curriculum included other oversights and errors, which indicate a need for final editing. As a result, it is not surprising that surveyed students did not believe that writing was an important part of learning about mathematics (Liedtke and Sales 2008). When asked to write, some students stated, "Writing is not what we do in math." The "must" condition in the new curriculum implies that talking and writing are an essential part of mathematics learning.

Connecting

Students are expected to connect mathematical ideas to other concepts in mathematics, to everyday experiences and to other disciplines. Connecting can be thought of as a problem-solving strategy. Students should be able to connect everything they learn or have learned to previous learning and to experiences outside the classroom. It is discouraging when students are unable to respond to, "Who would want to multiply two two-digit numerals (for example, 32×25) or two decimals (for example, 4.2×3.6), and why would people want to do that?" During interviews, two students answered with, "A teacher" and "The smartest person in the classroom."

My data support the results collected from American teachers (Howe 1999). Graduates of our school system are unable to make up a meaningful word problem for the equation $1\frac{3}{4} \div \frac{1}{2} = [\ ]$. Many confuse it with $1\frac{3}{4} \div 2 = [\ ]$. (Satisfy your curiosity. Ask students from middle or high school to make up word problems for $8 \div \frac{1}{2} = [\ ]$ and $8 \div 2 = [\ ]$. The results may be surprising.)

The statement in the new curriculum about connecting can be used to illustrate an important issue: the need for specificity of language. Lack of specificity can be detrimental to reaching goals. Specific language or examples or even both are required to make it possible to translate statements like "connecting ideas to other concepts" into appropriate action.

Mental Mathematics and Estimation

Students are expected to demonstrate fluency with mental mathematics and estimation. How is it possible to foster the development of fluency and assess it without knowing the meaning of the term? Examples, criteria and a definition are required to make implementation possible.

Why is it that estimation has been and is very problematic? Students' responses to requests that involve estimating can be fascinating. Whenever the opportunity arises, I ask questions like:

- How many students do you think are in that part of the playground?
- How many bicycles do you think are in the bicycle stand?

Many students will give answers with a digit in the ones place or to the nearest five. Rarely is the word *about* used as part of a response nor do explanations of estimation strategies mention the use of a referent.

Many people are confused about *guessing* and *estimating* and the language that is used with these terms. Some authors of crossword puzzles give the same meaning to guess and estimate or they equate estimation with making a rough guess. Some teachers use descriptors like *good, logical* and *rough* when students are requested to make a guess or when guesses are categorized; for example, "That's a good guess." I have heard "guesstimate" used as part of a request (how confusing is that?), and student teachers ask young students to estimate without telling them what that means.

An article about lions in the May 15, 2006, *Vancouver Sun* stated, "Estimated number of those attacks that are fatal: 66." Air Canada's website (www.aircanada.com/en/about/fleet/a319-100xm.html) states that the range of one Airbus is 4,442 km. How many test pilots were lost to determine this result? How was the distance that smokers must keep between themselves and a gas pump determined and at what cost? The sign informs us that it is 7.5 m. Reporting results in this fashion is inappropriate, and the statements indicate a lack of number sense.

The learning outcomes related to estimation in the new curriculum can resolve the existing issues related to estimation and making statements about numbers.

Problem Solving

Students are expected to develop and apply new mathematical knowledge through problem solving. The previous curriculum (Ministry of Education 1995, 2) states: "Students must learn the skills of effective problem solving, including the ability to communicate solutions, so that they will become reasoning, thinking individuals able to contribute to society." An impressive sounding statement indeed, but what are the skills of effective problem solving? How is effective defined? How are solutions to be communicated and why? How does sharing solutions contribute to becoming reasoning, thinking people who are able to contribute to society? Does any data exist to indicate that the students who went through the system reached any of the ambitious goals included in the statement? This example is included to reiterate the need for specificity of language.

Teaching through problem solving represents a major change from the previous curriculum and requires special strategies, questioning techniques and assessment procedures. Open-ended questions and high-order thinking questions are essential. All the different types of student responses need to be accommodated (Liedtke 2010).

Mathematical Reasoning

Students are expected to develop mathematical reasoning. The goals included under Reasoning Mathematically in the previous curriculum (p 3) include "Mathematics instruction should help students develop confidence in their ability to reason and to justify their thinking" and "Mathematics should make sense, be logical, and be enjoyable." Without elaboration it is impossible to interpret logical and enjoyable as part of a goal.

At one time student teachers were asked to write specific learning outcomes for the cognitive and the affective domain. Some students tried to transfer this to my courses and on their lesson plans recorded as a goal, "The students will enjoy the lesson." After a smile, I completed these outcomes with "or else" (with an apology to the student and an explanation). The previous WNCP included the specific outcome, "Students will enjoy optical illusions." It is impossible to be serious about this outcome in attempting to reach it with a group of students and then assessing it!

The criteria for mathematical reasoning I have adopted (Greenwood 1993, 144) include "everything should make sense," "getting unstuck on one's own," "ability to identify errors in answers, use of materials and thinking," "using a minimum of counting, and rote pencil and paper computations," and "being willing to try another strategy." My data led me to conclude that students have not reached these goals. The strategies and questioning techniques that are required for teaching through problem solving will contribute to the development of mathematical reasoning.

Technology

Students are expected to select and use technology as tools for learning and solving problems. The new curriculum (Alberta Education 2006, 9) includes a list of possible uses for calculators and computers. The following are key entries of this list for the elementary grades:

- Develop number sense: Any activities that can contribute to the development of number sense will be of benefit to students.
- Develop personal strategies for mathematical operations: Questions can be posed that lead students to develop their own strategies and contribute to other aspects of learning as well; that is, number sense, flexible thinking.
- Explore and demonstrate mathematical relationships: Students can use the calculator to test and describe results of tests.

Visualization

Students are expected to develop visualization skills to help them process information, make connections and solve problems. This critical component requires elaboration. What are specific examples of visualization skills? How do these skills help students process information and what kinds of information? What are examples of the connections that are facilitated? How do the skills assist in problem solving? The discussion under Visualization in the WNCP (Alberta Education 2006) does not provide any answers. In fact, it includes other statements that require definitions, specific examples or both. For example, how many readers would visualize the following statement in the same way? "The use of visualization provides students with opportunities to understand concepts and make connections among them" (p 9).

Interactions with groups of students and with individuals during interviews or discussions have provided me with data that support the conclusion that many students are unable to visualize. This is not surprising, because visualization is not mentioned or explicitly dealt with in the previous curriculum.

However, what is surprising is that reference to developing or demonstrating number sense is part of every grade level though there are no references to ability to visualize number, a key aspect of number sense. Visualization is essential for making mathematics meaningful. Fostering students' ability to visualize requires special strategies.

The New Curriculum—Possible Impact on Students

Despite entries that lack the necessary specificity, the inclusion of undefined terms, and subjective statements, translating the new curriculum into action can make mathematics meaningful for students. However, this positive effect depends on several factors related to teaching and assessment.

Reaching the goals that are part of any major curriculum change depends on appropriate professional development for those whose task it is to translate the curriculum into action. Professional development is required that shows how to make the critical components a must part of a mathematics program.

Professional Development

The essential components of this development must deal with strategies that are new or may differ from those employed to translate the previous curriculum into action. That means the focus should be on "Developing ability to visualize, learning through problem solving, and 'fluency' with estimation and mental mathematics" (Alberta Education 2006, 6), on "Attaching meaning to what is learned" (p 2) that implies the acquisition of conceptual knowledge and on the "opportunity to develop personal strategies" (p 6). Since number sense is the key foundation of numeracy, the accommodation of key aspects of number sense as part of ongoing teaching must be isolated and addressed. The critical components and the goals of the new curriculum require that the appropriate assessment strategies be discussed.

Development of Visualization

Visualization is an important component of all aspects of sense making, especially number sense (the key foundation for numeracy), spatial sense (a requisite for the ability to solve problems) and problem-solving ability in general. The following examples illustrate the importance of the ability to visualize. As far as number sense is concerned, when students hear the names for numbers (whole numbers, fractions, decimals, integers) or see numerals, they should be able to "see" the numbers behind these names. This ability is illustrated by the Grade 5 student, who as part of her explanation that involved relating fractions included the phrase, "I see it in my brain." Without the ability to visualize numbers or "see the numbers in one's brain," and without being able to think flexibly about numbers and numerals, it is very difficult, if not impossible, to reach some of the major learning outcomes in the new curriculum.

The focus of a lesson with a group of Grade 1 students last year was on recognizing, visualizing and relating numbers. For one type of task, one student showed the age of a younger sibling (two) and another his age (seven). Sample questions that were posed to the group included: "Show and quietly name a number close to two," "Show and name a different number that is just as close," "Show and name a number between two and seven that is close to seven," "Show and name a different number between two and seven," "Show and name a number greater than seven that is not close to seven," and "Show an even number between two and seven" (I had watched the students talk about and illustrate even numbers as sharing numbers).

Another task involved an attempt to identify a mystery number. After each hint was presented—for example, "The number is between two and nine"—the students were requested to show what they thought the number or numbers could and could not be. During the procedure a boy in the front row looked up and asked, "How old are you anyway?" This was a legitimate question considering the age of his teacher and the age of the even younger student teacher. I asked him to guess. "81," he said. "How about 181," I answered to which he uttered a drawn out, "Oh." I am still curious about the discussion that might have taken place at the dinner table about the old teacher who taught about mathematics.

The ability to visualize has to be a focus for whole numbers, decimals, fractions and integers. The role of visualization as part of spatial sense, number sense and problem-solving ability is illustrated by such abilities as translating a problem or an equation into a diagram that indicates the meaning of the numbers, the intended order and/or the intended action or actions. A lack of ability to visualize the appropriate type of division accounts for the inability to create a meaningful word problem for equations like $1\frac{3}{4} \div \frac{1}{2} = [\ ]$.

Why do so many students lack the ability to visualize? Possible reasons include lack of specific goals for lessons, inappropriate practice and assessment, ineffective use and/or lack of use of materials, use of nonspecific and incorrect language, and insufficient introductions to operations and procedures (Leidtke 2007). The majority of these shortcomings are addressed in the new curriculum.

Opportunities to Learn Through Problem Solving

One assumption for learning through problem solving is that it is better to solve a problem in many ways than to solve many problems in the same way. Teaching through problem solving presents challenges because it requires special teaching strategies and questioning techniques, the ability to accommodate different types of responses from students and special assessment techniques. Questions must be asked that allow students to use what they know to calculate an answer in a new way. Learning new ideas through problem solving ensures that these ideas are meaningful.

Number Sense the Key for Numeracy

Why do so many responses from students and adults indicate a lack of number sense? An examination of entries related to number sense in the previous curriculum indicates the following problems: key aspects are not identified, possible strategies for fostering the development of number sense are not suggested, activity settings for the key aspects are not identified, practice appropriate for the key aspects is not illustrated, assessment strategies and suggestions for reporting to parents are missing, and settings that could be detrimental to a development of number sense are not identified. The main aspects of number sense include visualizing number, recognizing number without having to count, flexible thinking about number and numerals, estimating number, relating numbers or numerals, and connecting numbers or numerals. The accommodation of these aspects requires special classroom settings and appropriate practice and needs to become part of ongoing teaching.

Ability to visualize, learning through problem solving and number sense are interrelated and are essential for the acquisition and development of conceptual understanding, a major goal of mathematics teaching and learning. Conceptual understanding facilitates new learning. This understanding is fostered in settings where students are allowed to communicate their reasoning orally as well as in writing. Communication in the mathematics classroom is essential (Elliott and Garnett 2008). Research results (Hiebert 2000, 437) indicate "that instruction can emphasize conceptual understanding without sacrificing skill proficiency."

Number sense is essential because it is a requisite for major goals of the new curriculum that include mental mathematics strategies for the basic facts, estimation and mental mathematics strategies for the operations, personal algorithms or strategies for computational procedures, and "fluency" with estimation and mental mathematics.

Professional development needs to focus on strategies and settings that are conducive to the development of number sense and conceptual understanding since this development is vital to making mathematics meaningful.

Selected Issues Related to Assessment

A new curriculum framework that includes components that students must encounter requires discussions and reflections about specific learning outcomes, teaching strategies and assessment techniques as well as possible reporting procedures about the key goals and components. During a discussion among teachers, school boards and the ministry about the assessment data that is collected annually from Grades 4 and 7 BC students, one board member observed that present test results "are too narrow an assessment to measure the overall state of student achievement" (Sherlock 2010).

Assessment instruments are required that yield information about students' ability to think and to think mathematically. Data about how students think is essential not only for reports to caretakers but also for meaningful intervention. Some assessment instruments do not yield any meaningful data about the critical components and key goals for students; that is, multiple-choice tests.

Any attempt to collect assessment data about the critical components of the curriculum requires appropriate and fair assessment items. Many assessment items students face lack these characteristics. This is true for all types of tests (Liedtke 2005). Simple questions that identify one answer as the correct response or ask students to select a response from several choices do not yield any meaningful data. For example, Which does not belong? Which one is different? Which one comes next? Which one is incorrect (correct)? Which statement is true (false)? These simple questions can be very unfair because students must guess what an author was thinking. For example, consider any item that shows part of a repeating or growing pattern and the question, "What comes next?" Since a repeating pattern can be extended in many ways and can easily be changed to increasing patterns, identifying one correct response can punish many students who are flexible in their thinking. If it is an author's intent to look for one correct response, specific instructions are required.

At the end of presentations to parents about such ideas as confidence building, risk taking and flexible thinking, some will share concerns about their children. Anxiety that is caused by timed basic fact tests (for example, mad minute) is a frequent concern. Last year one mother told a story about her Grade 5 son, whose response to the request to extend a pattern was marked incorrect. After he explained his thinking to his mother, he shared the conclusion, "I think my teacher has lost her imagination."

The following further illustrates examples of inappropriate types of requests that identify one response as correct. Several friends forwarded an e-mail of a four-item one-minute test. They had failed the test and wondered whether or not I could pass it. Not one sender commented about the format of the items, nor did others who looked at them. People automatically assume that if items appear on a test they must be appropriate. Three of the items are mathematical:

Continue this sequence in a logical way: M T W T _ _ _

Correct this formula with a single stroke: 5 + 5 + 5 = 550

Draw a rectangle with three lines:

Some authors of test items assume that the one extension for sequences they have in mind is logical and others are not. Could it be, to quote the Grade 5 student, that these authors have lost their imagination? Students who are taught through problem solving, who are confident risk takers, and who persevere and exhibit curiosity (all goals of the curriculum) will be able to present different logical ways of extending this sequence other than thinking of what an author has in mind, in this case, days of the week. According to the author, the correct response to the second item involves drawing a segment to change the first plus sign to 4. It is just as easy to change the "equal" to a "is not equal" sign.

These examples illustrate a few concerns about some types of assessment. If only one response is identified as correct and is expected, appropriate detailed instructions are required.

The third example illustrates the use of incorrect terminology or language in the request as well as in the answer. Three segments are shown in the interior of a rectangular region. Is the use of inappropriate or incorrect language an unusual occurrence? The answer to the question might surprise a few people (Liedtke 2005).

A columnist of the local newspaper wrote the Foundations Skills Assessment (FSA) test for Grade 4 and included the following item in his report: "The students will take a bus 62 km to a nature park. The bus travels at 40 km/h. About how long will the trip take?" For many ESL students the answer would be 60 km or 62 km or none of the above, depending on the choices that are provided.

The framework of the new curriculum requires a careful look at the types of questions that appear on assessment instruments. Perhaps the critical components of the new curriculum require a focus on Foundations Ideas Assessment rather than on Foundations Skills. If such a test is administered at the beginning of the year, teachers could use the results to make instructional adjustments and plan effective IEPs for students who require them. Everyone could come out a winner!

Looking Back from the Future

Here are some questions that should be part of a backward glance five or ten years after the implementation of the new curriculum:

- Does observable and measurable data exist that show that students encountered the critical components of the curriculum?
- Is there evidence that the goals for students were or are reached?
- Did the new curriculum provide the "new approach to math" that columnist Hume suggested?
- Did the new curriculum reduce the need for tutors as was suggested by a former BC minister of education?
- Were there noticeable changes in assessment procedures and reporting to parents?
- Did commercial tutoring companies focus on the critical components of the curriculum or did they continue to present and assess procedural learning?
- Are there some who claim that perhaps the goals in the new curriculum are too high?

Speculations About Answers to the Questions

The new curriculum presents an opportunity to have a positive and lasting impact and to make mathematics meaningful for students. For many teachers the ability to make mathematics meaningful for students will depend on the professional development that is provided. This development must focus on characteristics of classroom settings, teaching strategies and assessment techniques that centre on development of visualization, learning through problem solving, and developing number sense and conceptual understanding.

It took a lot of time, effort and money to produce the new curriculum. However, it has been introduced without setting aside the monies required for support services during implementation and for final editing of the answers that are predictable for most students. As a result of a period of cutbacks, future researchers and columnists will likely conclude that nothing has changed and it's time to take a new approach to math.

Is there a possible solution? Perhaps monies need to be set aside at the outset of a revision for professional inservice. As well, textbook publishers must explain and illustrate the teaching and assessment strategies that are required for accommodating the critical components and general goals of the program.

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Research in and Development of School Mathematics: Delving Deeper into Concepts to Revitalize the Mathematics Taught and Learned in Schools

Jérôme Proulx

Contextualizing the Research

Mathematics teaching and learning happen every school day. Albeit sometimes celebrated and advocated for, most mathematics teaching and learning are often questioned, criticized or negatively judged by people involved in mathematics education. These criticisms have often led these same people to call for changes to how mathematics is taught and learned in classrooms. It has always seemed odd to me that these criticisms are addressed mainly to teachers and their teaching practice. For me, as a mathematics education researcher, the problem has always been something else. To contextualize this better, I offer you this quote from French didactician of mathematics, Guy Brousseau:

> I am never critical toward teaching as it is practiced. If you see 200 000 teachers doing the same thing and it looks stupid to you, it is not because there are 200 000 stupid people. It is because there is a *phenomenon* that orients this same type of reaction in these people. And it is this *phenomenon* that we need to understand. ... We won't improve it with an ideology, nor by moralizing to teachers. (Brousseau 1988; my translation)

It is this "phenomenon," this "something else," that the research project I report on in this paper attempts to address. In short, the underlying belief of this project is that it is not the people who are problematic in mathematics teaching and learning, but the mathematics itself: the problem resides in the mathematics being taught and learned and not how it is taught or learned. My entry into the problem is, therefore, mathematical.

This paper outlines the significance of this research for the continuing improvement of mathematics teaching and learning. I explain the various phases of the research to give the reader an idea of the approach. Then, through this, I report on the specific project conducted, focused on the notion of area of planar figures, and discuss the results and products of this research orientation and their outcomes for the teaching practice.

The Research Approach: Delving Deeper into the Mathematics

The intention to address the mathematics being taught and learned in schools came as a result of diverse reform movements in mathematics that advocated for "less mathematics, but deeper," to promote more conceptual forms of understandings in students. But what does it mean to go deeper into the mathematics? It is this question that this research project explored, taking the notion of area of planar figures as a specific example.

In this project, the choice of area of planar figures as an example to work on was based on personal interest and on my teaching experience in secondary schools, where I have often been unsatisfied with the treatment of this topic. Area is often seen as an easy topic simply consisting of helping students memorize a number of diverse formulas. Thus, learning area is pinned down to knowing and memorizing formulas, recognizing which ones to use and applying them in a problem. Consider this typical quote taken from the Purplemath website (www.purplemath.com/modules/geoform.htm) addressed to learners:

> Some instructors like to give all needed geometric formulas, so your test will have a listing of anything you might need. But not all instructors are this way, and you can't just expect a new instructor ... to give you all this information. Ask your instructors for their policies, but remember

that there does come a point (high school? SAT? ACT? College? "real life"?) at which you will be expected to have learned at least some of these basic formulas. Start memorizing now!

Thus, the central question of this research project became, what would it mean to work deeper in area of planar figures? To address this question, a number of phases were designed. I report on them below.

Phase 1: What Is Out There?

Interesting as it may seem as a research orientation, wanting to delve deeper in mathematics supposes that one has some awareness of the mathematics usually worked on in schools. This led to the consideration of what is out there in the area of planar figures. Thus, the first phase of the research reviewed various forms of textbooks and curricular materials to gather a sense of what is out there concerning the teaching and learning of area of planar figures.

This review of Grades 4 to 8 resources revealed a somewhat poor treatment of the notion of area of planar figures. From most of the resources reviewed, three important tendencies could be highlighted:

1. An explicit and important focus given to area formulas of planar figures, which were either given, constructed or explained
2. A large number of isolated formulas, one for each planar figure; for example, rectangle [$L \times l$], parallelogram [$B \times h$], square [s^2], rhombus [$\frac{(D \times d)}{2}$], trapezoid [$\frac{(B+b) \times h}{2}$]
3. A focus placed on numerical calculations of areas for the planar figures, often triggered by a command, such as "Calculate the area of the following rectangle."

It was therefore felt that area received a rather poor treatment through these various resources. Thus, the project's central question was again brought to the fore: what would it mean to work deeper in area of planar figures than what these resources already offer? The challenge was to delve into the topic and draw out more of it than this review outlined to enrich mathematically the concept of area of planar figures. This paved the way for the second phase of the research concerning the development of a deeper approach to the concept of area of planar figures.

Phase 2: Digging into and Developing the Mathematics

As mentioned, the challenge was to probe into the concept of area to draw out an approach that enriched its treatment. Therefore, the work centred on exploring, making sense and delving into the mathematical concept of area of planar figures to unpack some of its underlying meanings, relations and subtleties often hidden within it.

However, it is important to note that this work was explicitly seen as being able to produce/develop only one of the many possible approaches to the concept of area of planar figures, as many other ideas could be explored. The intention was not to develop all possible treatments for area of planar figures but to offer one rich approach to illustrate what it could mean to delve deeper into a topic of study (here, with area of planar figures taken as an example).

Two important issues emerge from the overabundance of formulas, numbers and calculations in the study of area: (1) the absence of geometry and (2) the enormous number of isolated and disconnected formulas to memorize. Richard Skemp discusses this second issue:

> There is a seeming paradox here, in that it is certainly harder to learn. It is certainly easier for pupils to learn that "area of a triangle = ½ base × height" than to learn why this is so. But they then have to learn separate rules for triangles, rectangles, parallelograms, trapeziums; whereas relational understanding consists partly in seeing all of these in relation to the area of a rectangle. It is still desirable to know the separate rules; one does not want to have to derive them afresh everytime [sic]. But knowing also how they are inter-related enables one to remember them as parts of a connected whole, which is easier. (Skemp 1978, 12–13)[1]

These issues triggered specific guidelines in the research inquiry for developing the approach that would delve deeper into area of planar figures: (1) attempting to go back to and work in geometry in area of planar figures and (2) finding a way to draw out the links existing between the usual area formulas. To illustrate where these guidelines led the work, I offer a glimpse at ideas that were brought together for delving into area of planar figures. (However, because of space constraints I cannot go into great length about these ideas, so I refer the reader to three other papers that were produced on these ideas: Proulx 2007; Proulx 2008; Proulx and Pimm 2008).

A Glimpse at the Ideas Developed for Area of Planar Figures

Let's begin by introducing a mathematical principle that influenced the inquiry: the Cavalieri principle.

This principle, for planar figures and solids, asserts that:

> If between the same parallels any two plane figures are constructed, and if in them, any straight lines being drawn equidistant from the parallels, the included portions of any one of these lines are equal, the plane figures are also equal to one another; and if between the same parallel planes any solid figures are constructed, and if in them, any planes being drawn equidistant from the parallel planes, the included plane figures out of any one of the planes so drawn are equal, the solid figures are likewise equal to one another. ... The figures so compared let us call analogues, the solid as well as the plane, ... (Cavalieri 1653)

Gray (1987, 13) expresses the two-dimensional aspects of the Cavalieri principle as follows:

> The principle asserts that two plane figures have the same area if they are between the same parallels, and any line drawn parallel to the two given lines cuts off equal chords in each figure.

Figure 1 offers an illustration of the Cavalieri principle with two plane figures:

Informally, the principle asserts that if you cut each polygon horizontally at the same height and each chord obtained is of equal length, then the two polygons are of the same area. To make the comparison, both polygons need to be the same height; if not, the comparison appears not possible because one polygon would be cut where there would not be any of the other polygon left to cut.

When used for the study of area, the Cavalieri principle allows for insightful comparisons between figures, and in that sense appears helpful in establishing geometric links and relating planar figures. For example, one can establish links between rectangles and parallelograms (of same base and same height), where both figures lie between the same parallels (see Figure 2).

The 2-D version of the Cavalieri principle helps us also to see "families of planar figures;" for example, a family of rectangle and parallelograms of same area. To see this, let the parallelogram be as slanted as you want: any parallelogram with the same height and same base would have the same area, because each cross-section is always of the same length, creating the equivalent family of rectangles and parallelograms (Figure 3).

Figure 1. An Illustration of the Cavalieri Principle in Two Dimensions

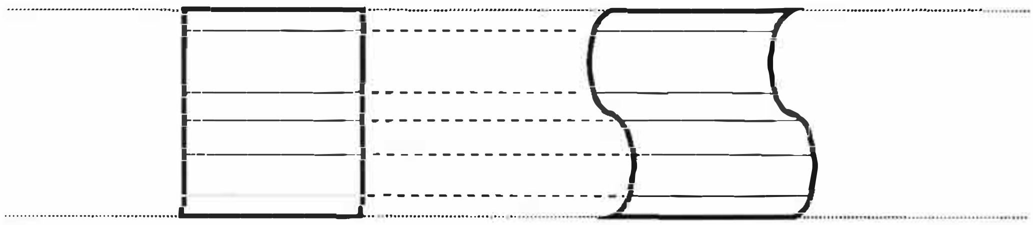


Figure 2. Using the Cavalieri Principle with a Rectangle and a Parallelogram

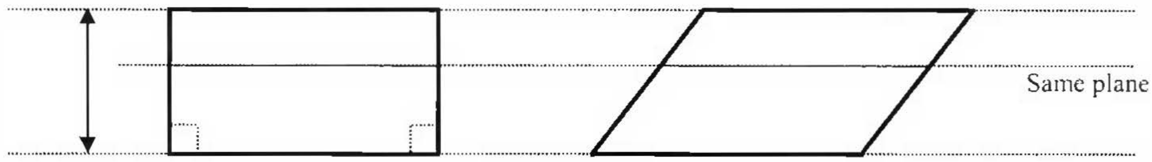
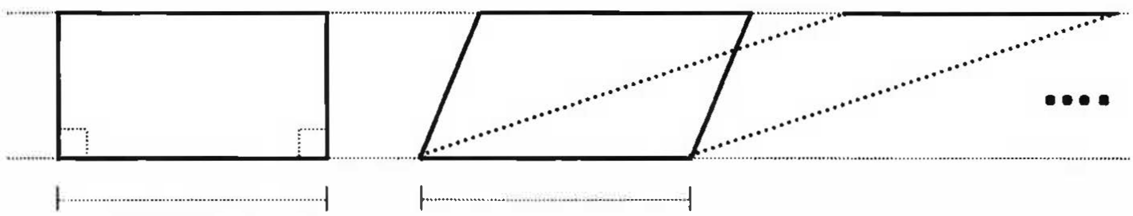


Figure 3. Family of Rectangle and Parallelograms



A parallelogram thus possesses an associated rectangle, which has the same base and same height, implying that the area of a parallelogram can be obtained through the same formula as for rectangles:

Area of the parallelogram	=	length of the base	×	height

The Cavalieri principle, and families of parallelograms, offers an interesting entry into rhombuses and squares. Obviously the square, frequently defined as a rectangle with four equal sides, is directly related to the preceding formula of $Area = base \times height$. Thus, this prevents the need for inventing a specific formula for the square (often given with s^2), a prevention that in fact strengthens the link between the square and the rectangle.

The rhombus can be defined as a parallelogram that possesses all identical sides and therefore can be seen as part of a family of parallelogram and directly associated with a specific rectangle (having the same base and height as the rhombus). However, defining a rhombus as a parallelogram raises some questions concerning the usual "data" given to calculate its area. Diagonals are usually provided (or to be found) for finding the area of the rhombus; its formula being in fact directly related to it [$\frac{(D \times d)}{2}$]. Hence, defining the rhombus as a parallelogram leads one to aim for the "base" and "height" of the rhombus, something quite unusual for rhombuses.[2] However, one has to acknowledge that this strengthens the link between rhombuses, parallelograms, squares and rectangles; a link that, in addition, simplifies the various formulas for these four planar figures by providing a single and general formula for them: $Area = base \times height$.

Concerning families, the same thing can be said for triangles, where all triangles with the same base and height are part of the same family, be they as slanted as one wants (Figure 4 or Figure 5).

It is also possible to establish a family of trapezoids, where the small base slides on the same plane, producing a family of trapezoids with the same base and the same height (Figure 6).

Figure 4. A Family of Triangles with Same Height and Same Base

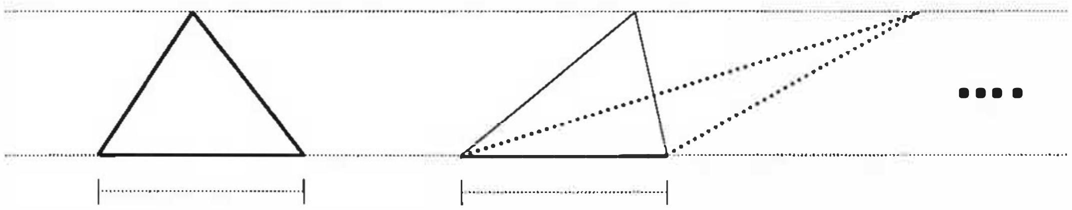


Figure 5. Another Representation of a Family of Triangles

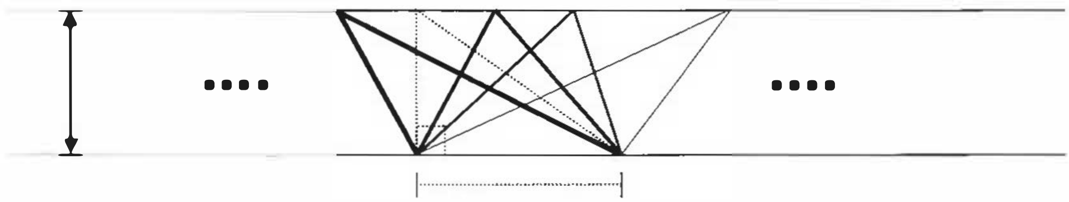
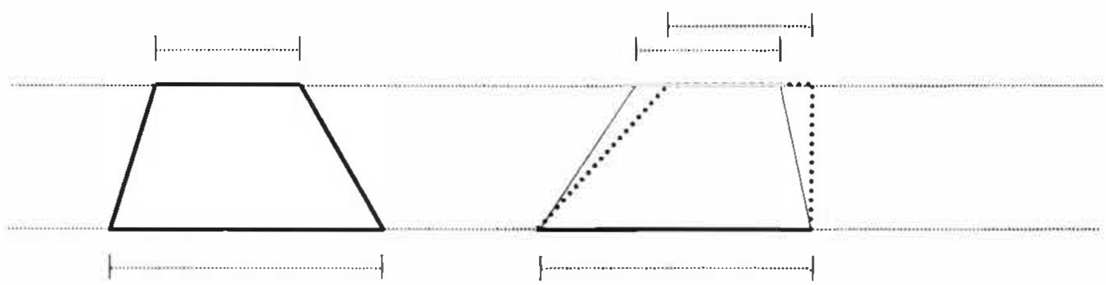


Figure 6. A Family of Trapezoids with Same Height and Same Base



Moreover, as the family of trapezoids gets established, it is interesting to note that a trapezoid is indeed defined as "a quadrilateral with two sides parallel" (see, for example, Wolfram MathWorld website http://mathworld.wolfram.com), which takes into account not only standard trapezoids that we are often used to seeing (Figure 7) but also any quadrilateral that has a pair of opposite sides that are parallel, which are indeed trapezoids, without being parallelograms (Figure 8)[3].

In that sense, the family is composed of any quadrilateral that has the same height and a pair of opposite and parallel sides, making the family of trapezoids look like the following (Figure 9).

One notices, however, that no family of equivalent trapezoids contains a rectangle. However, there is a fixed relationship between a trapezoid and its associated rectangle. By rotating the rectangular trapezoid (that is part of any trapezoid family) about the midpoint of the remaining slant side, a rectangle that is the double of the trapezoid is produced (Figure 10), establishing a significant relationship between trapezoids and rectangles. Trapezoids become perceived as the half of a related rectangle that has the same

Figure 7. Some Examples of Standard Trapezoids

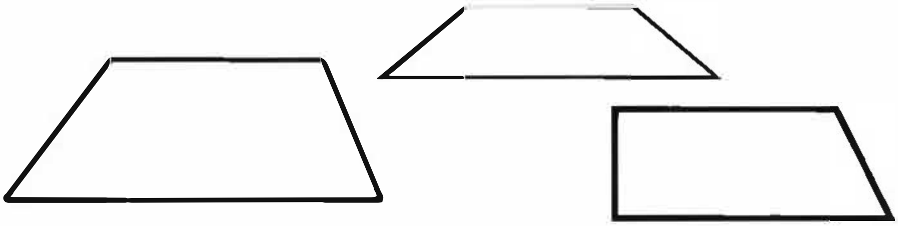


Figure 8. Another Type of Trapezoid

Figure 9. A Family of Trapezoids with Nonstandard Ones

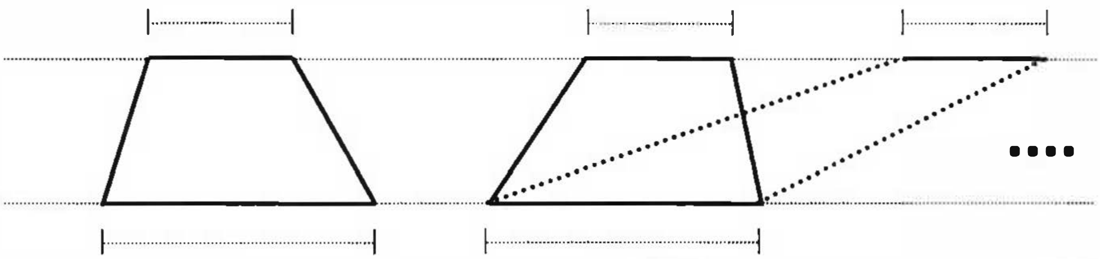
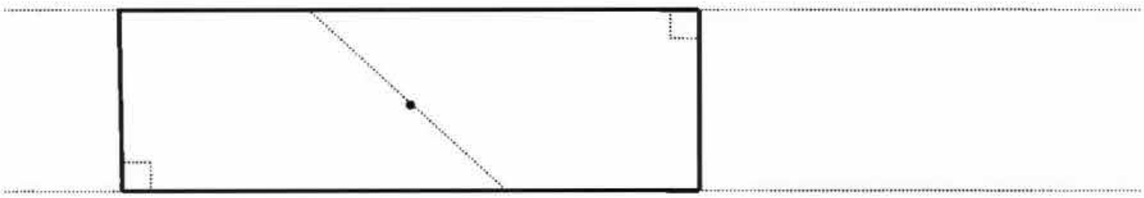


Figure 10. Two Copies of a Right Trapezoid Creating a Rectangle



height and a base equal to the sum of both the trapezoid's bases.

This helps establish a relationship between trapezoids and rectangles, where the trapezoid is seen as a half of its associated rectangle (with same height and a base being the sum of both trapezoid's bases).

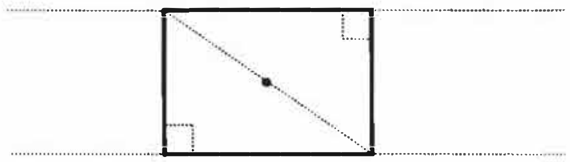
Area of trapezoid = ½ × **area of the *associated* rectangle (same height and base is the sum of the two bases of the trapezoid)**

↓

Area of trapezoid = ½ × (**Length of both bases** × **height**)

Like trapezoids, there are no rectangles in the triangle's family either (see Figures 4 and 5). But, again, it is possible to establish a relationship between the right-angled triangle and its associated rectangle, which is once more the double of it obtained as well by rotation about the midpoint of the hypotenuse (see Figure 11). As well, this rotation or double of the triangle can lead triangles to be called "half rectangles," a connotation that emphasizes the relationship between a triangle and its associated rectangle (see for example, Jamski 1978).

Figure 11. Two Copies of a Right-Angled Triangle Forming a Rectangle



At the level of formulas, this offers a specific conceptualization because the triangle is not defined anymore in regard to its own formula [$\frac{(B \times h)}{2}$] but mainly in its relationship with its associated rectangle [½ of rectangle]. Subtle as it may seem, it provides the occasion to draw out a strong link between the rectangle and all other planar figures mentioned above. The triangle is therefore defined in relation to its relationship with its associated rectangle; that is, as being half of the area of a rectangle that has the same height and the same base.

Area of triangle = ½ × **area of *associated* rectangle**

↓

Area of triangle = ½ × (**Length of base** × **height**)

The glimpse at some of the ideas presented above illustrates how the approach developed for conceptualizing more deeply the area of planar figures offers a different view. It can transform one's view of the area of planar figures far away from a calculational view (Thompson et al 1994) and from a collection of disconnected formulas. In addition to attempting to strengthen the existing link between figures and their formulas, this approach grounded these links in geometrical aspects, an approach different from a view focused on numbers and calculations for area.

However, this conceptual work was still incomplete. The project intended to work on a more concrete phase, structured around the construction of physical devices that would support the approach aimed at delving into the concept of area of planar figures. Thus, a third phase was designed and consisted of building devices and materials that would embody and support the mathematical ideas and issues developed in phase 2 of the project. I present a number of the designed devices below.

Phase 3: Building the Supporting Devices[4]

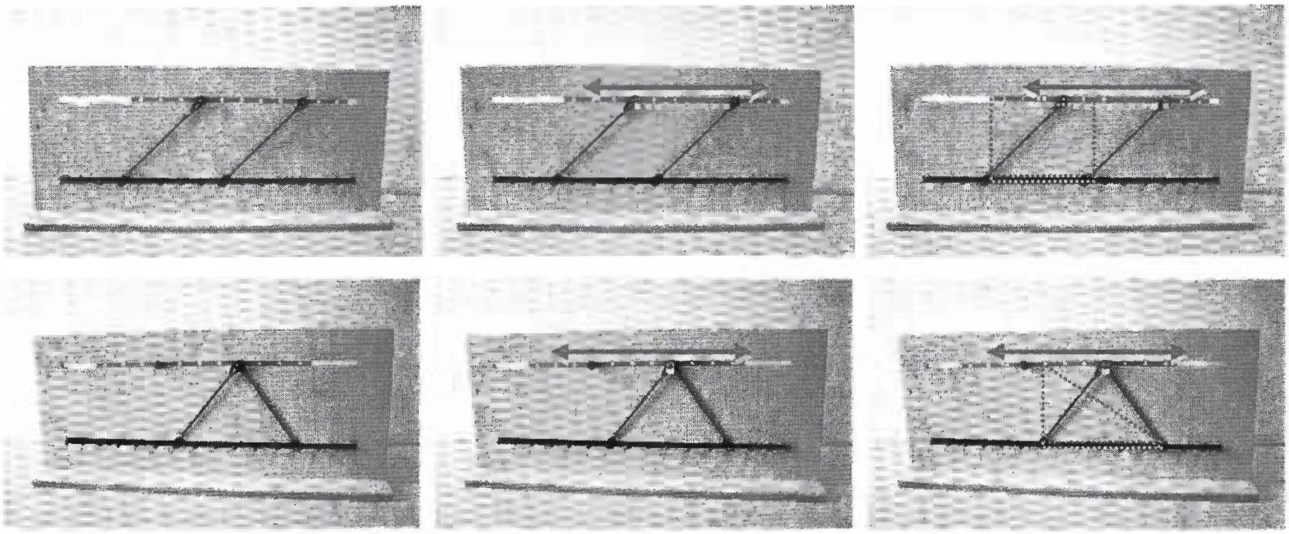
Device 1: Variations Keeping the Area Constant

This device was designed to show, dynamically, that the area stays constant as one moves from left to right or right to left the upper base of the figure (for a quadrilateral) or the vertex (for a triangle). One important mathematical property that can be drawn out of this device is that for quadrilaterals, for example, any parallelogram with equal bases and equal height is of the same area independently of how slanted it is (in relation to Figure 3). It can also help to show that this family of equivalent-area parallelograms can be related to the rectangle with same base and same height. For the triangle, it also can be related to a right-angled triangle of the same base and same height (in relation to Figures 4 and 5). As well, the same could be represented for trapezoids in relation to Figures 6 and 9.

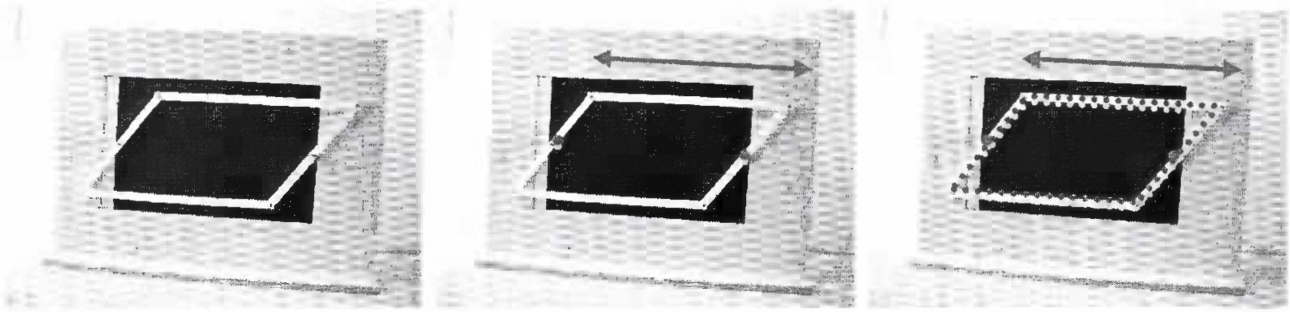
Device 2: Variations Keeping the Perimeter, But Not the Area, Constant

This device was built to contrast with Device 1, where the variation changes the area while keeping the perimeter constant. Albeit similar work can be done with geostrips, the decrease in area can be shown in relation to the black rectangle drawn on the back of the device offering something to compare the new area with. Thus, as one varies the angles, the area of the rectangle decreases in comparison with the black-initial rectangle that had the same area as the white rectangle when all angles were at 90 degrees.

Device 1: Variations Keeping the Area Constant



Device 2: Variations Keeping the Perimeter, But Not the Area, Constant



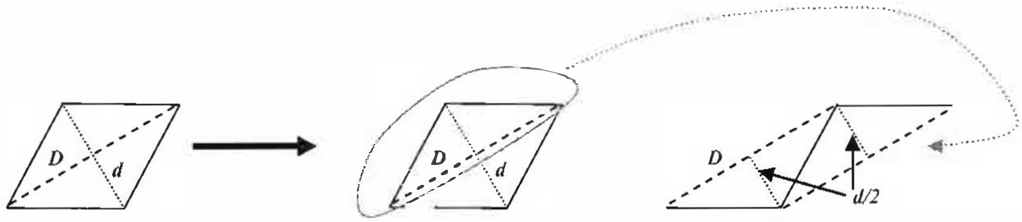
Device 3: An Accumulation of Cross-Sections

Device 3 shows aspects of the Cavalieri principle in regard to cross-sections of equal length that can be taken from a figure (between a rectangle and a parallelogram) in relation to Figures 1 and 2. The stripes or cross-sections can be tossed aside to create a parallelogram (or another figure) that keeps the same area.

Device 4: The Area of the Rhombus Calculated as the Area of a Parallelogram

Device 4 illustrates how to overcome the difficulty of considering the rhombus as a parallelogram in regard to the data given to calculate its area (normally its diagonals). It shows that a rhombus, for which one does not know the length of its side and thus cannot compute its height using the parallelogram/rectangle formula of base × height, can be reorganized to create a parallelogram for which the long diagonal is the base and half of the small diagonal is the height. This device intends to help link the rhombus with parallelograms and rectangles to continue contrasting with the need to opt for having a different and disconnected formula for the rhombus. See Figure 12.

Figure 12. Reorganizing a Rhombus into Another Parallelogram, Using the Device Developed



Device 5: The Relation Between Trapezoids and Rectangles

This device illustrates the idea of Figure 10 in order to see, through doubling the figure, that twice a trapezoid constitutes a rectangle (or a parallelogram if the trapezoid is not a right-angled one). Hence, this uses the ratio of 1:2 between the trapezoid and the rectangle that has the same height and a base constituted of the sum of the trapezoid's bases, supporting the idea that the area of a trapezoid can be calculated in relation to the rectangle (again setting aside the need for another specific formula for the trapezoid).

Device 6: The Relation Between Triangles and Rectangles

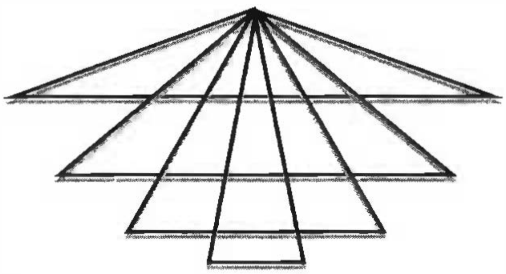
In the same vein as for the trapezoid, this device supports the idea of the triangle as half of the rectangle with the same base and same height (illustrated in Figure 11). This device, however, embodies a very specific case, the one with a right-angled isosceles triangle leading to a square (one could think of a variation in the triangle used, leading to various rectangles and parallelograms that would be the double of area).

Device 7: Variations on the Triangle

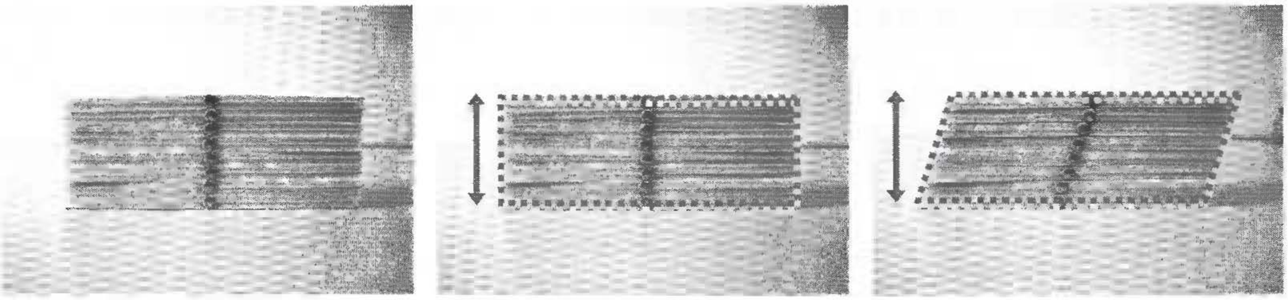
This device was built in the same spirit as Device 2, but for triangles to contrast with Device 1 that keeps the area constant in the case of the triangle. Here, it shows a different sort of family of triangles: isosceles triangles with the same equal sides and a third one that varies. This leads us to ask probing questions; for example, which triangle in the family has the bigger area? (see Figure 13 inspired by Avital and Barbeau 1991). What happens to a triangle in terms of area when only one side varies?

Thus, those devices were built to support the ideas put forth in Phase 2 of the project. Specifically, they were designed to draw out the geometric properties in regard to area through a dynamic interplay with the devices (changes, variations, constants, relationships

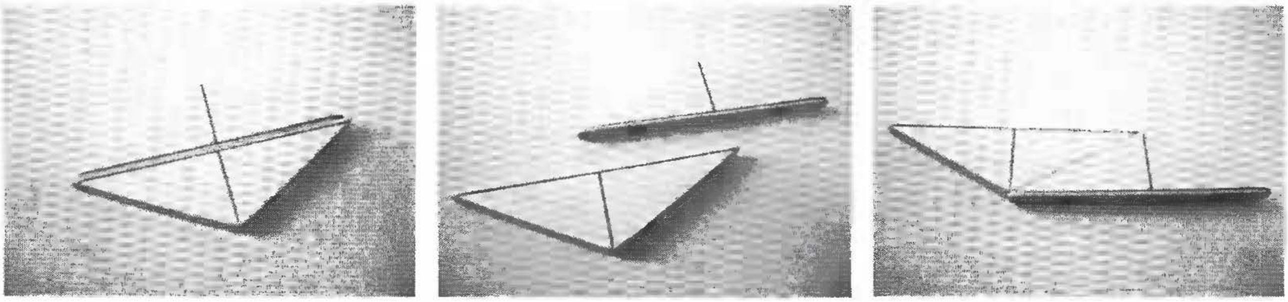
Figure 13. Another Sort of Family of Triangles: The Isosceles Triangles Obtained Through a Variation of the Angle Joining Both Equal Sides



Device 3: An Accumulation of Cross-Sections



Device 4: The Area of the Rhombus Calculated as the Area of a Parallelogram



between figures, families and so on), and supporting the "delving deeper approach" to area of planar figures developed.

Final Remarks

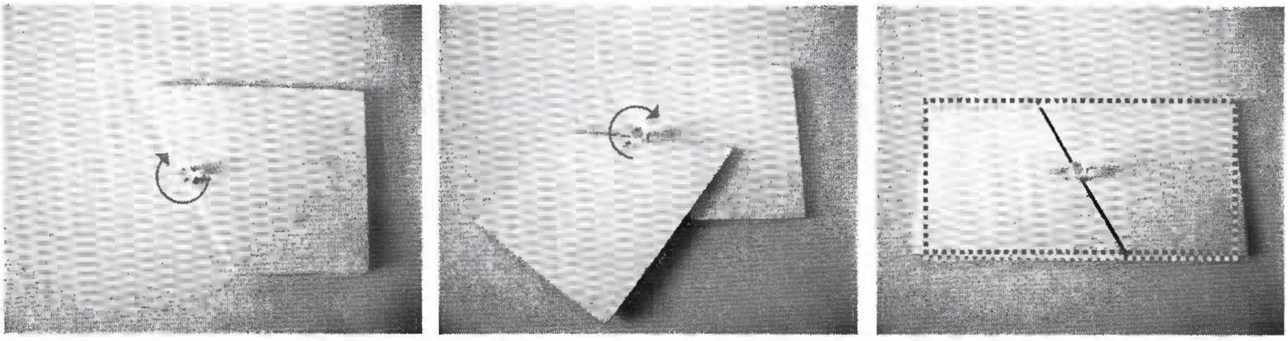
The development reported on about area of planar figures, specifically using the Cavalieri principle, brings forth the interrelations between the usually studied planar figures at a geometrical level. It also links these figures through their formulas, instead of having numerous area formulas to make sense of or memorize. Whereas the study of the many different formulas tends to isolate figures from each other, this geometrical approach helps to reunite them and stimulate significant reasoning for the concept of area.

This sort of work on delving deeper in mathematical concepts, where only the example of area of planar figures was offered, appears important to stimulate the development of rich mathematical reasoning that enables a deeper understanding of mathematical concepts. This said, continuous work must be conducted along those lines as area of planar figures appear only as one of many topics in school mathematics that can be delved into and from which richer mathematics can be brought forth. For example, similar work has been conducted in a project led by Elaine Simmt around systems of equations (for details on the work done and its outcomes, see Proulx et al 2008) as well as a project I have led on the study of trigonometry (reported in Proulx 2003).

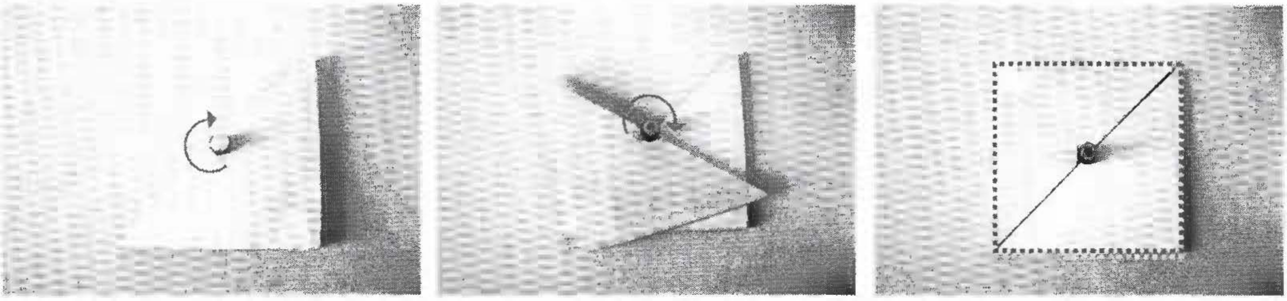
In each of these projects, the outcomes led to questioning the study of mathematical topics and the orientation they received in the curriculum. As mentioned, it is not the people who teach or learn mathematics who are problematic, it is mathematics itself. Thus, how can school mathematics topics be enriched and dug into deeper to revitalize the teaching and learning of mathematics in schools? Paying attention to the development of school mathematical concepts appears to be a fruitful line of inquiry in mathematics education in relation to the goal for continuing enhancement and revitalization of the teaching and learning of mathematics in classrooms.

Mathematics needs mathematicians to continue to evolve as a field of study. Perhaps what schools need are not mathematicians but school mathematicians: people who probe into the mathematics of the curriculum. Simply put, there is a lot of mathematics to delve into and develop within school mathematics. We need people to work intensely on these issues to enhance and revitalize the teaching and learning of mathematics in schools. This project has attempted to move toward that goal.

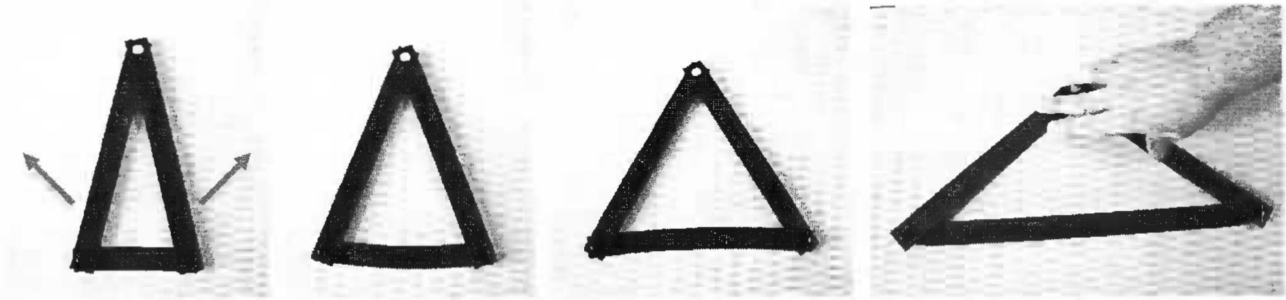
Device 5: The Relation Between Trapezoids and Rectangles



Device 6: The Relation Between Triangles and Rectangles



Device 7: Variations on the Triangle



Notes

1. A rapid Internet search concerning area formulas yields a horrifying list of numerous and different formulas for the area of planar figures, giving the impression that planar figures like rectangles, parallelograms, rhombuses and squares are very different figures requiring very different treatments concerning their area.

2. In this case, it is possible to transform the rhombus with long diagonal D and short diagonal d in a parallelogram of base D and of height d/2; a parallelogram then associated with a rectangle of base D and of height d/2.

3. I say "used to seeing" mostly because from looking at various textbooks and websites, rarely does one of them offer a picture different from the ones offered in Figure 7. It seems indeed that this type of trapezoid is not a current form that is often studied.

4. I need to acknowledge here the tremendous work done by one of my research assistants, Tom Hillman, in physically constructing these devices.

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The Unusual Die: Exploring a ProblemThrough Technology

A Craig Loewen

I recently had a wonderful problem-solving experience that taught me a great deal about ways to explore problems, the role technology can play in exploring problems and the qualities of a truly intriguing problem. It started with this seemingly simple game that I introduced to a group of beginning teachers:

Kara is playing a dice game. In this game each player selects a die, rolls it five times and adds the values rolled. The player with the greatest sum wins. Kara may choose between two dice. One die is a regular six-sided die; the other die has three ones and three sixes on its six sides. Kara knows that the selected die must be used for the entire game, and she knows that her opponent must use the other die. Which die should Kara choose?

It is amazing how this problem tends to divide people into two groups: optimists and pessimists. The optimists will select the unusual die knowing it has three sixes and therefore the greatest chance of rolling a six. The pessimist will select the regular die given that the unusual die has three ones and the greatest chance of rolling a one.

This simple game is a wonderful problem to introduce to groups of people as it sparks a lot of good discussion. Of all the information given, what is the most important information, and what is it that makes the problem so difficult? To many solvers it is disconcerting that the player must stick with whatever die is chosen, and it is a bit confusing in that the player will make only five rolls but the die has six sides. In other words, no matter which die is selected only a maximum of five of the different faces on each die can possibly appear. But does this matter?

The number of rolls is really not a significant variable in this problem. After all, if the die is fair (unusual, but fair), each face has exactly the same probability of turning up on each roll (that is, 1/6). We know it is quite possible for the same face to turn up five times in a row, but we also know it is more likely that more than one face will appear within the five rolls.

How do we tie all this information together, or more to the point, how can we reasonably compare the dice?

A Solution

What we need to calculate is the average possible roll on each die, and then we can simply multiply that value by five to find the average score that die would likely produce.

The average roll is calculated as the sum of the values on the faces divided by the number of faces. For the regular die:

$$\frac{1+2+3+4+5+6}{6} = \frac{21}{6}$$

The average roll is 3.5 and the average score across many games with this die would be 17.5 (= 5 × 3.5). Likewise, the average score obtained with the unusual die is:

$$\frac{1+1+1+6+6+6}{6} = \frac{21}{6}$$

The unusual die also has an average roll of 3.5, so the average score over many games with this die will likewise be 17.5 (= 5 × 3.5).

In other words, it theoretically doesn't matter which die we pick. Ultimately, these two dice give the same results at least while playing this game.

Exploring the Problem

It was at this point that the solvers decided that they would like to try it for themselves; they were hesitant to give up their pessimistic or optimistic view of the unusual die. To conduct the experiment the teachers took two dice and rolled them, treating one die as the regular die and the other die as the unusual die (counting twos and threes as ones, and fours and fives as sixes). They kept track of which die seemed to win most often and after several games they began to share and compile their results. However, even with a large group there was still doubt whether enough examples had been generated to clearly prove which was the better die.

A Monte Carlo Simulation

I came prepared to show my students a Monte Carlo simulation, a way to help generate more examples quickly using a computer. In the case of this problem, I had simply programmed a computer to play this game repeatedly, keeping a tally of the winning die. As long as we can assume that the computer generates values in a manner comparable to that of rolling a die, the simulation should give similar results to playing the game many hundreds, even thousands of times.

The following program was entered into my TI-83+ calculator to simulate the game.

```
PROGRAM:DICE
:ClrHome
:Prompt M
:0→R:0→S:0→T
:Output(3,1,1234
56)
:Output(4,1,1116
66)
:Output(5,1,"TIE
S")
:For(A,1,M)
:0→D:0→U
:For(B,1,5)
:D+randInt(1,6)→
D
:randInt(1,2)→X
:If X=2:6→X
:U+X→U
:End
:If D>U:R+1→R
:If U>D:S+1→S
:If D=U:T+1→T
:Output(3,9,R)
:Output(4,9,S)
:Output(5,9,T)
:End
:█
```

To run the program all that needs to be entered is the number of times you wish the calculator to play the game, and the screen will display the results of the simulation, showing the number of times each die wins and the number of tie games. The following screen resulted from having the calculator play the game 10,000 times.

```
M=?10000

123456   4649
111666   4751
TIES     600
```

From these results it is easy to calculate that in this Monte Carlo simulation the regular die won approximately 46.5 per cent of the games, the unusual die won approximately 47.5 per cent of the games, and 6 per cent of the games resulted in ties. But, would we get the same results if we ran the program again, or if we had the computer play the game 100,000 times? To do that, we would need to use a much faster computer!

Going Deeper

After showing these teachers this simulation, one of them asked me this question: "Why is it that 6 per cent of the games are ties?" After a careful explanation of how it is possible for games to result in ties, the student re-asked his question: "Yes, I can see that it is possible for there to be ties, but what I don't understand is why it is 6 per cent of the games that would result in ties. Why not 5 per cent or 8 per cent or any other number?" This is a marvellous question! And frankly, I didn't know the answer! So, I began to play with the problem again, this time with the help of a spreadsheet.

To construct the spreadsheet I had to calculate all of the possible outcomes (final scores) for each die and the probabilities of obtaining those scores. For example, with the regular die the smallest possible game score is five, and that can be achieved only one way: rolling a one every time the die is tossed. It is also possible to get a score of six, seven, eight and so on all the way up to thirty. Just like there is only one way to roll a game score of five, there is only one way to roll a game score of thirty; there is a certain symmetry to the table of values. Similar computations were necessary for the unusual die. The table below shows all of the possible game scores and their probabilities for the regular die along the left edge, and all of the possible game scores and probabilities for the unusual die along the top.

Score	# Ways	Probability	5	10	15	20	25	30	Sums:
		Score:	5	10	15	20	25	30	Sums:
		# Ways:	243	1215	2430	2430	1215	243	7776
		Probability:	0.031250	0.156250	0.312500	0.312500	0.156250	0.031250	1.000000
Score	# Ways	Probability							
5	1	0.000129	0.000004	0.000020	0.000040	0.000040	0.000020	0.000004	
6	5	0.000643	0.000020	0.000100	0.000201	0.000201	0.000100	0.000020	
7	15	0.001929	0.000060	0.000301	0.000603	0.000603	0.000301	0.000060	
8	35	0.004501	0.000141	0.000703	0.001407	0.001407	0.000703	0.000141	
9	70	0.009002	0.000281	0.001407	0.002813	0.002813	0.001407	0.000281	
10	126	0.016204	0.000506	0.002532	0.005064	0.005064	0.002532	0.000506	
11	205	0.026363	0.000824	0.004119	0.008238	0.008238	0.004119	0.000824	
12	305	0.039223	0.001226	0.006129	0.012257	0.012257	0.006129	0.001226	
13	420	0.054012	0.001688	0.008439	0.016879	0.016879	0.008439	0.001688	
14	540	0.069444	0.002170	0.010851	0.021701	0.021701	0.010851	0.002170	
15	651	0.083719	0.002616	0.013081	0.026162	0.026162	0.013081	0.002616	
16	735	0.094522	0.002954	0.014769	0.029538	0.029538	0.014769	0.002954	
17	780	0.100309	0.003135	0.015673	0.031346	0.031346	0.015673	0.003135	
18	780	0.100309	0.003135	0.015673	0.031346	0.031346	0.015673	0.003135	
19	735	0.094522	0.002954	0.014769	0.029538	0.029538	0.014769	0.002954	
20	651	0.083719	0.002616	0.013081	0.026162	0.026162	0.013081	0.002616	
21	540	0.069444	0.002170	0.010851	0.021701	0.021701	0.010851	0.002170	
22	420	0.054012	0.001688	0.008439	0.016879	0.016879	0.008439	0.001688	
23	305	0.039223	0.001226	0.006129	0.012257	0.012257	0.006129	0.001226	
24	205	0.026363	0.000824	0.004119	0.008238	0.008238	0.004119	0.000824	
25	126	0.016204	0.000506	0.002532	0.005064	0.005064	0.002532	0.000506	
26	70	0.009002	0.000281	0.001407	0.002813	0.002813	0.001407	0.000281	
27	35	0.004501	0.000141	0.000703	0.001407	0.001407	0.000703	0.000141	
28	15	0.001929	0.000060	0.000301	0.000603	0.000603	0.000301	0.000060	
29	5	0.000643	0.000020	0.000100	0.000201	0.000201	0.000100	0.000020	
30	1	0.000129	0.000004	0.000020	0.000040	0.000040	0.000020	0.000004	
Sums:	7776	1.000000							

Probability of Normal Die Winning (Sum of shaded boxes):	0.47130
Probability of Unusual Die Winning (Sum of non-shaded boxes):	0.47130
Probability of Ties (Sum of blackened boxes):	0.05740

The values in the middle of the table show all of the probabilities of the various game outcomes; that is, the product of the probabilities of the given final scores. For example, there is a probability of 0.3125 that the final score with the unusual die will be 15, and there is a probability of 0.100309 that the final score with the regular die will be 17. Therefore, the probability of that particular result (a score of 15 and 17 on the respective dice) is 0.031346.

The values in the boxes that are shaded show all of the probabilities related to game outcomes where the normal die wins. Likewise the values in the nonshaded boxes show all of the probabilities of game outcomes where the unusual die wins. The values in the blackened boxes show the probabilities of tied game outcomes.

There are lots of patterns of symmetry in this table, but a quick inspection shows that there exists the same number of ways for the unusual die and the regular die to win, and that there are six ways for the game to result in a tie. Further, by separately summing the values in the shaded boxes and nonshaded boxes (I had the spreadsheet do this), we see that the probability of each die winning is about 47 per cent of the time. More importantly, by adding the values in the blackened boxes, I could show that the probability of tie games is about 5.7 per cent!

Creating a table like this would be almost impossible without the aid of a spreadsheet. There are just too many values to tabulate and compute, and it is likely that too many errors would be made. The spreadsheet however does this accurately and quickly, and provides a credible answer to the question that started the exploration.

Looking Back

Solving this problem with these teachers helped to remind me of some of the most important qualities of a good problem-solving experience.

First, a good problem-solving experience should be well rounded; it should involve exploration (preferably a hands-on component where possible), the asking of why, and the opportunity to explore the problem on many levels. This problem provided each of these when the teachers first picked their favourite die, conducted an experiment, compiled and compared their results, and then began to challenge the conclusions.

This particular problem also provided an opportunity to integrate technology into the search for a solution, including both a Monte Carlo simulation and the creation of a complex spreadsheet. The simulation provided a way to test our hypotheses and the spreadsheet aided in making very complicated calculations that would otherwise be too tedious or difficult.

The most important outcome though was this: it reminded me yet again that in most problem-solving experiences the answer is much less important than the process of solving the problem. The process we engaged in brought enjoyment, debate and a realization of the power of technology, and above all else, stimulated a question arising from genuine curiosity. These are important hallmarks of a successful problem-solving experience.

Extension Problems

1. If you need to roll a value of 20 or greater in five rolls, with which die will you have the best chance?
2. How do you calculate the number of ways each game outcome can be reached? In other words, how do we know there are only 15 ways of getting a score of 7 with a regular die in this game?
3. Design a third die and compare it to the two dice used in this game. Does your die improve your odds of winning? How do you know? Design a different unusual die that has the same chance of winning as a regular die.
4. Playing the same game, assume you may pick between a regular four-sided die, and an unusual four-sided die that has two ones and two fours on it. Which die would you pick? Why?

A Craig Loewen is a professor of mathematics education at the University of Lethbridge, where he has been teaching for 23 years. He is particularly interested in acts of problem solving, building problem-solving competence and the role that technology can play in the solution of many different kinds of problems. His other areas of interest include the use of manipulatives and manipulative-based games in enhancing both the understanding and enjoyment of mathematics. In his spare time, Loewen enjoys music and woodworking in his small home shop.

Home Ice Advantage: Representing Numbers to 20, Grade 1

Lesley Ross and Brenda Wells

This lesson is part of a teachers' resource, Home Ice Advantage: Classroom Assessment of Mathematics. *It will be available in 2011 from the Centre for Mathematics, Science and Technology Education (www.uofaweb.ualberta.ca/cmaste/).*

This lesson is part of a professional development project on assessment in mathematics initiated by Joel Canete, Connie Farrell and Lori Weinberger, the mathematics consultants of the Fort McMurray public and separate school divisions, and facilitated by Gladys Sterenberg, a faculty member of the University of Alberta. As part of our work on assessing children's mathematical ways of knowing, we decided to create assessment tools and to implement them in the teachers' classrooms. We were inspired by the book *The Hockey Sweater*, by Roch Carrier, and created literature-based lessons that included assessment strategies, descriptions of the teaching and learning context, student artifacts and interpretations of the assessment results. We decided that the best way to increase our own understanding of classroom assessment was to visit our colleagues' classrooms during the implementation of the lessons. Both the public and separate boards of education sponsored these visits by providing teacher release time. This was a definite strength of the project as it enabled teachers to collect assessment data in their colleagues' classrooms. What follows is the lesson we designed and implemented as part of this project.

Lesson Description

Using numbers to 20, students will identify and label two complete sets of uniforms. They will determine how many uniforms will be needed in each group, offer possible solutions for identifying the uniforms and provide explanations for the identifications. This is part of the number strand. The assessment is intended to be used midway or at the end of the unit, Representing Numbers to 20. It may also be used later in the year as a means of review.

Specific Outcomes (Numbers)

1. Say the number sequence 0–100 (at this time to 20) by
   • ones forward and backward between any two given numbers.
2. Demonstrate an understanding of counting by
   • indicating that the last number said identifies how many,
   • showing that any set has only one count,
   • using the counting on strategy and
   • using parts or equal groups to count sets.
3. Represent and describe numbers to 20, concretely, pictorially and symbolically.
4. Compare sets containing up to 20 elements to solve problems, using
   • referents and
   • one-to-one correspondence.

Specific Outcomes (Patterns and Relations)

5. Describe equality as a balance and inequality as an imbalance, concretely and pictorially.

Essential Questions (Mathematical Processes)

1. How will the students communicate their understanding of counting?
2. How will the students develop connections within their understanding of number sense?
3. How will the students develop mathematical reasoning when counting and comparing numbers?
4. How will the students develop visualization skills to represent, describe and compare numbers from 0 to 20?
5. How will the students develop number sense through problem solving?
6. How will the students use mental estimation skills to compare numbers from 0 to 20?

Achievement Indicators

1. Recite forward by ones the number sequence between two given numbers (0 to 100).
2. Recite backward by ones the number sequence between two given numbers (20 to 0).
3. Answer the question, "How many are in the set?" using the last number counted in a given set.
4. Identify and correct counting errors in a given counting sequence.
5. Determine the total number of objects in a given set, starting from a known quantity and counting on.
6. Partition any given quantity up to 20 into two parts, and identify the number of objects in each part.
7. Compare two given sets, using one-to-one correspondence, and describe the sets, using comparative words, such as more, fewer or as many.
8. Determine if two given concrete sets are equal or unequal, and explain the process used.

School Context

St Martha is an early-entry elementary Catholic school with a grade range from prekindergarten children of 3½ years old (or 2½ years old with severe language delays or disabilities) to Grade 8 students. St Martha School is currently in its fourth year of operation in the newly developed area of Timberlea in Fort McMurray. There are approximately 650 students in the school, of which roughly 70 are in Grade 1. There is a mixture of low-, middle- and high-income families. There are many ESL students from around the world including countries such as South Africa, Zimbabwe, India, Venezuela and the Philippines.

Dr K A Clark School is a K–8 public school, serving students in the downtown core of Fort McMurray. This school offers many programs to help children succeed, such as junior skills and intermediate skills district programs, Native education and mentorship programs.

The Learning Plan

Task

Maurice wants to buy 20 new hockey jerseys to make two teams for the children. He doesn't know how many to buy for each team. What can he do to find out how many sweaters each team should have? How will the fans know who is playing? How will he be able to tell the teams apart?

You and your partner need to find out how many sweaters each team needs. Decide how to tell the teams apart and how the audience will know which player is wearing what jersey. Design two different team sweaters and label them accordingly.

Plan

1. Discuss background knowledge about the game of hockey. Ask students: How do you know who is playing? How many players are on each team? Do you have a favourite hockey team? Do you or anyone you know play hockey?
2. Read *The Hockey Sweater*, by Roch Carrier.
3. Ask students: What did you notice about all the players in the pictures? If you were the referee, how would you be able to tell the teams apart? What is the problem with everyone wearing the same sweater?
4. Introduce the math task for children to solve. Ask students: What do you know from the problem and what do you need to find out? Discuss with your partner what you will need.
5. Provide the children with sweater cut-outs. Each pair is to count out as many as they need (20) and take them back to their desks to decide how to divide them into two teams.
6. Once the sweaters have been divided into two teams, ask students to label each sweater with a corresponding numeral and colour code for each team's sweater. Students may choose any numeral from 0 to 20 or beyond to put on the sweater.

Assessment Strategies

(See assessment tools at the end of the lesson.)

- Rubric for direct observation
- Checklist for finished product
- Video recording of student conversations

Assessment Questions and Observations

The students were assessed while they were doing their work and after they finished. Using the rubric designed specifically for the task, teachers tracked outcome objectives and recorded general observations that related to the following questions:

a. Can the student justify his or her choice for the number of sweaters on each team?
b. Can the student explain what might happen if there is a player on one team who has the same sweater number as a player on the other team (for example, both players are a number 6)?
c. How is the child counting the sweaters (for example, one-to-one correspondence, pairs, five-frame or ten-frame arrangements)?
d. Is the child counting accurately and consistently?

e. Is the child communicating math language, demonstrating cooperation and sharing ideas? Math language examples include number words zero to twenty, count, number, equal groups, singles, estimate, more fewer, same as, equal, as many as.

Samples of Student Responses

Brenda Wells's Grade 1 classroom has 18 students (7 boys and 11 girls) and a regular part-time educational assistant. Many of the math practice and enquiry sessions that the students routinely participate in have the students working with partners or in small groups. This rich task project used a partner format with partners preassigned by the teacher. While the students were completing this assignment, three guest teachers observed the lesson. They interacted with the students by asking questions or clarifying student actions, in addition to responding to student enquires, recording observations on rubrics and videotaping the class. One student who completed the rich task assignment was Adam. At the time of the assignment, Adam was 6 years 7 months old and working at grade level in both math and language arts. Adam has two younger siblings, and both of Adam's parents work outside the home.

On this day Adam worked with his partner, counting out the sweaters and dividing them into two equal groups. When asked to count his set and describe the solution that he and his partner came up with, Adam found that he had 12 team sweaters. His partner said she tried to tell him he had too many, but he said he had counted 10 sweaters. Adam was asked to then count both his and his partner's, and he counted to 22. After the recount, Adam used mental math strategies to decide that he had two extra team jerseys. He did not hesitate to have them removed from his set. Adam was able to articulate $10 + 10 = 20$ and both partners should have 10. The numerals Adam chose to write on the jerseys were very interesting when compared to those of the other students in the class. All other students either chose the digits 1–10 or randomly selected any numbers between 1 and 100. Adam used odd numbers to successfully skip count to 15. He made an error with the last two team jerseys by using 18 and 20.

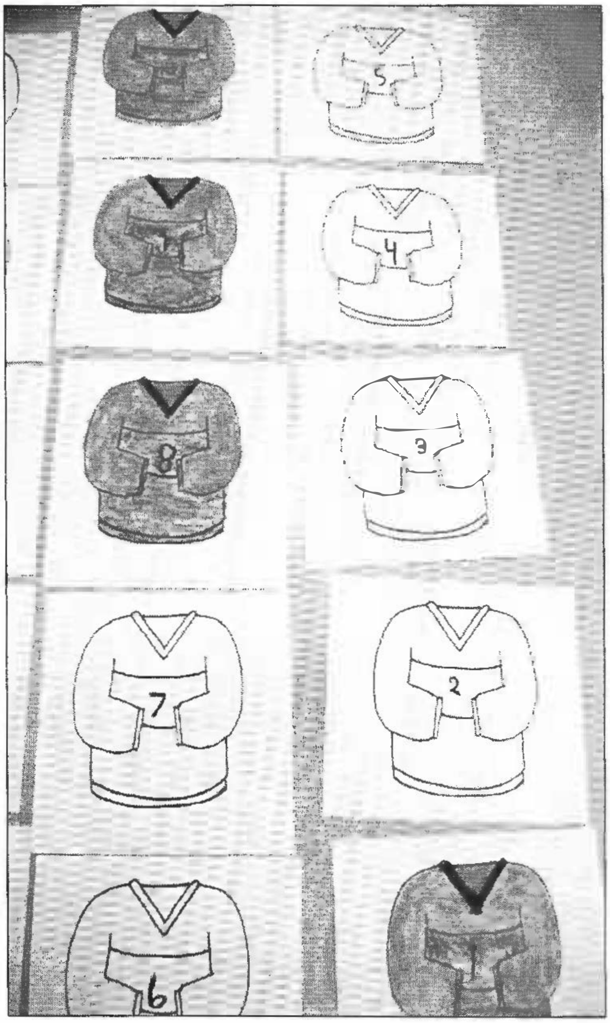
Adam was not asked afterward to explain why this error occurred. Possible reasons could be that he lost his train of thought because of interruptions, or he got mixed up with odd and even numbers. It was interesting to observe that if he had been counting by twos, he would have used the last two numbers correctly. Adam had difficulty printing the number 7 with a reversal, which he self-corrected. Adam demonstrated his awareness of sports jerseys by printing the numerals on the back of the cards and making logos on the front. He also exemplified his awareness of patterns and the fraction of ½ when colouring the uniforms. Specific examples of mathematical language were not recorded; however, when discussing the project, Adam did use math language independently. Overall Adam clearly demonstrated his proficiency when using numbers to 20, solving problems and showing his work.

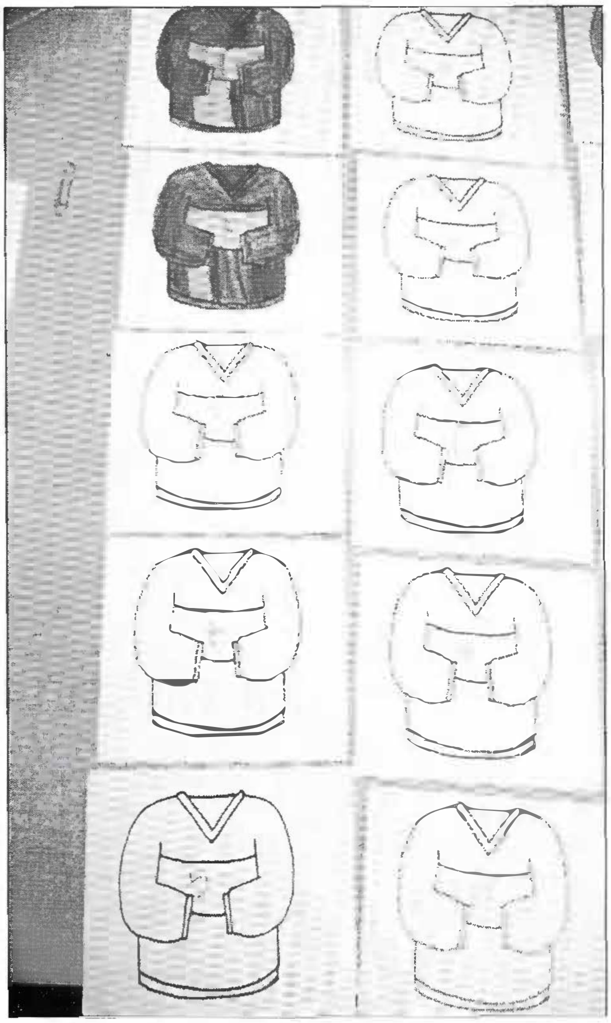
Rachel worked on the task, demonstrating that her counting and problem-solving skills were excellent at this time. She effectively counted out 20 objects and worked with her partner to produce a solution to the problem. When asked, Rachel correctly identified that the total set had 20 objects and now each smaller group had 10. She had 10 sweaters for her team and correctly labelled each with its own identifying numeral, which were clearly printed and had no reversals. Rachel selected a wide range of digits (1, 6, 9, 11, 12, 22, 36, 50, 80 and 90) showing her understanding of numbers to 100. When asked, Rachel explained her solution to the observers. Some math language was used independently but mostly this language was used only with prompting. Rachel successfully completed the finished product by producing 10 team sweaters with number labels and colouring the jerseys. The neatly completed task also indicated her familiarity with patterns, and her understanding of what team jerseys look like.

Chad and Jeff, working partners, counted 20 hockey sweaters. From the onset, this pair identified that each team would need 10 sweaters, demonstrating their ability to produce a solution to the problem using mental math. They thought they were successful at counting out the correct number of objects and dividing them into two equal groups until they were asked to show their sweaters to the teacher observing them. Chad accurately identified that his set had 10 objects and used the last number (10) as his starting place to label each sweater. He counted backward (10–1) as he labelled the sweaters. While this skill was not indicated in the assignment rubric, it is one of the outcomes prescribed in working with numbers to 20. Chad accurately printed the numerals legibly, with no reversals. When the boys were asked to count their team sweaters out loud, Jeff realized he somehow had 12 sweaters and this was more than he needed. He then discarded the two extra sweaters. In contrast to the numerals Chad selected to label his sweater, Jeff used various numbers to 100. His printing was legible and there were no reversals. With assistance, this partnership accurately produced two equal teams, and they were able to explain their solution using some math language. Other examples of mathematical procedures and knowledge were also observed within this partnership. When asked how many sweaters he had left to colour after completing two, without going back and recounting Chad replied there were eight, thus indicating mental math procedures. Jeff counted the team sweaters by twos a few times and arrived at different answers. He then rechecked by counting by ones. He selected a counting strategy that was efficient, but when the results were different from his first attempt, he selected a different strategy to verify his answer.

Lesley Ross's class is made up of 17 Grade 1 students. Just as Brenda had prearranged partners, Lesley put much thought into matching up students into groups of two to guarantee that students would feed off one another and learn the most from this rich task.

Kadar and Bhadra were partners in completing the hockey sweater task. They seemed to work very well together. As Kadar counted out the 20 sweater cards one by one, he passed them to Bhadra, who recounted and organized the cards. When asked to explain the numbers that these partners had chosen to put on the hockey sweaters, Bhadra responded that she would label all of the sweaters with the number one. Instantly, Kadar knew that this was incorrect because, as he said, "You won't know who that is," or as Bhadra rephrased, "If the referee blows his whistle and says, 'Number one just scored a goal,' would we know which number it would be?" Without much prompting, Kadar explained to Bhadra, "You need to do different numbers than one. You did one the first time and then you can change to nine, ten, eleven or twelve." Bhadra still did not completely understand, and Kadar went on to explain, "Because when number one scores, they won't know which player scored." With a nod of her head, Bhadra seemed to understand and got back to work on the paper jerseys. The following are photographs of the jerseys that each student completed, and the rubric used to assess their work.





Representing Numbers to 20 Rubric

Partners Kadar Bhadra

	Excellent	Proficient	Adequate	Not yet adequate
Count a set with one to one correspondence	Counts to 20 with no errors K ✓	Counts to 20 with 1-2 errors B ✓	Counts to 20 with 3-5 errors	Counts to 20 with greater than 5 errors
Shows that the set has only one count – the last number is how many.	Accurately identifies each set without assistance K ✓ B ✓	Accurately identifies each set with limited assistance	Requires several attempts to identify each set with assistance	Unable to demonstrate that each set has only one count
Printing the numerals – legibility & reversals	All numerals are legible with no reversals K ✓ B ✓	1 reversal and legible	2 reversals and legible	3 or more reversals and/or illegible
Able to produce a potential solution to the problem	Identifies a plausible solution independently K ✓	Identifies a plausible solution with limited assistance	Identifies a plausible solution with direct guidance B ✓	Unable to identify a plausible solution
Demonstrate how 20 items would be Divided into 2 teams.	Accurately produces 2 equal groups K ✓	Accurately produces 2 equal groups with assistance B ✓	Accurately produces 2 equal groups with direct guidance	Unable to produce 2 equal groups
Describe how the teams are balanced Or unbalanced	Accurately explains their solution to others independently K ✓	Accurately explains their solution to others with limited assistance B ✓	Accurately explains their solution to others with direct guidance	Unable to provide an explanation
Using Math Language	Uses math language independently K ✓ B ✓	Uses math language with limited prompting	Uses math language with direct prompting	Unable to use math language

10 + 10 = 20

Kadar - chose number 1-10. Counted backward as he printed.
- Explained to Bhadra, why all her players couldn't be #1.
She accepted his explanation.

Kadar can explain both teams are different colour so that it doesn't matter that both sets are 1-10

Bhadra - started to label all of her sweaters with #1

4

Through this observation of Kadar and Bhadra, we learned that Kadar had an excellent understanding of what was expected of him in this rich task, and he had the appropriate mathematical knowledge of numbers to 20 to facilitate his success in this project. He was also able to assist his partner in realizing what was needed to complete the task. Although Bhadra required guidance, we learned that with assistance, she was able to reach a possible solution to the hockey sweater project.

Self-Assessments of the Lessons

Lesley felt that one of the strengths of the lesson was the opportunity to be in Brenda's class the previous week and to see some of the avenues the students took and where they got off-track. This helped her anticipate some of the difficulties her students might have and prompted her to attend to her directions. She focused on guiding her students to have two separate teams and reminding them that the sweaters needed to be different (but that one team needed to look the same and the other team needed to look the same). Lesley was missing almost a third of her class for the lesson, but because colleague visits had been arranged, she borrowed some students from another classroom. She didn't have enough sweaters, so she tried groups of three. She found that groups of two worked really well but groups of three were more challenging. One change Lesley would make the next time she taught the lesson would be to state her criteria and show the students the rubric and checklist. She felt this would provide students with a guide for what she was looking for.

Her assessment of student learning focused on strategies students used to meet the outcomes of the lesson. She noted that some groups of students counted and recounted the number of cards when they arrived at different totals. A couple of groups were able to figure out how many more they needed without recounting. For example, students in one group counted 17 jerseys and were able to figure out that they needed three more. Students in another group had 15 cards, and they instantly knew they needed 5. Lesley had been working with her students on double facts (2 + 2, 5 + 5) and noted that most groups figured out quite quickly that if there were 2 teams and 20 players, it was 10 and 10, and they were able to make equal groups. Another thing she was interested in observing was number formation and making sure they could form numbers. Some students doing this task were only comfortable with numbers to 20 and only formed numbers from 1–20. Other students were comfortable with numbers to 10 and wrote 1–10 on one team and 1–10 on the other team. Other kids who were excited about numbers and who had lots of experience with numbers were making up numbers up to a million.

Brenda was impressed with the students' successful demonstration of the outcomes. Drawing on their knowledge of doubles, most of the students knew that 10 + 10 equals 20, so they knew that each team needed that many jerseys. She noted that they were able to count at this point, they were able to count back from 0 to 20 and they were able to identify that each sweater needed a different number. She found it interesting to see what numbers they chose: some went 1–10, some decided they were going to skip count, some skip counted using odd numbers. One challenge was "that art became the primary process and that they wanted to decorate, so a few needed to be reminded to please choose the numbers first."

When teaching this next time, Brenda would emphasize the numbers that students needed to figure out to identify the team sweaters. She wondered if there would be a difference between the students' actions if she did the lesson earlier in the year. In particular, she was interested in knowing if they knew 10 + 10 was 20 earlier, or if they had developed that through their recent math units.

Brenda noticed that all her students counted and used one-to-one correspondence; that is, they all physically moved the sweaters and counted "1, 2, 3, 4, ..." They all knew that if they had 10 sweaters, then their set equals 10. She observed that they knew the double strategy (10 + 10 = 20), so they didn't have to stop and do "one for me, one for you, one for me" but instead they were able to figure this out in their heads. She also noticed that if they had 10, some children counted forward to 10, whereas other children counted backward and started at 10 to make sure that they used each digit only once.

Lesley Ross has been teaching Grade 1 at Dr Clark School for three years and is currently teaching Grade 1 at Beacon Hill School in Fort McMurray. Lesley enjoys teaching mathematics and especially enjoys exploring with students math concepts through a hands-on approach.

Brenda Wells is a Grade 1 teacher at St Martha School in Fort McMurray. Brenda has been teaching Grade 1 for five years and has previously taught Grades 2 and 5. Over the years, she has taught in various urban and rural communities around Alberta. She has also experienced teaching in isolated settlements in the Northwest Territories. With the development of the new math curriculum, Brenda has participated in locally offered professional development projects.

Representing Numbers to 20

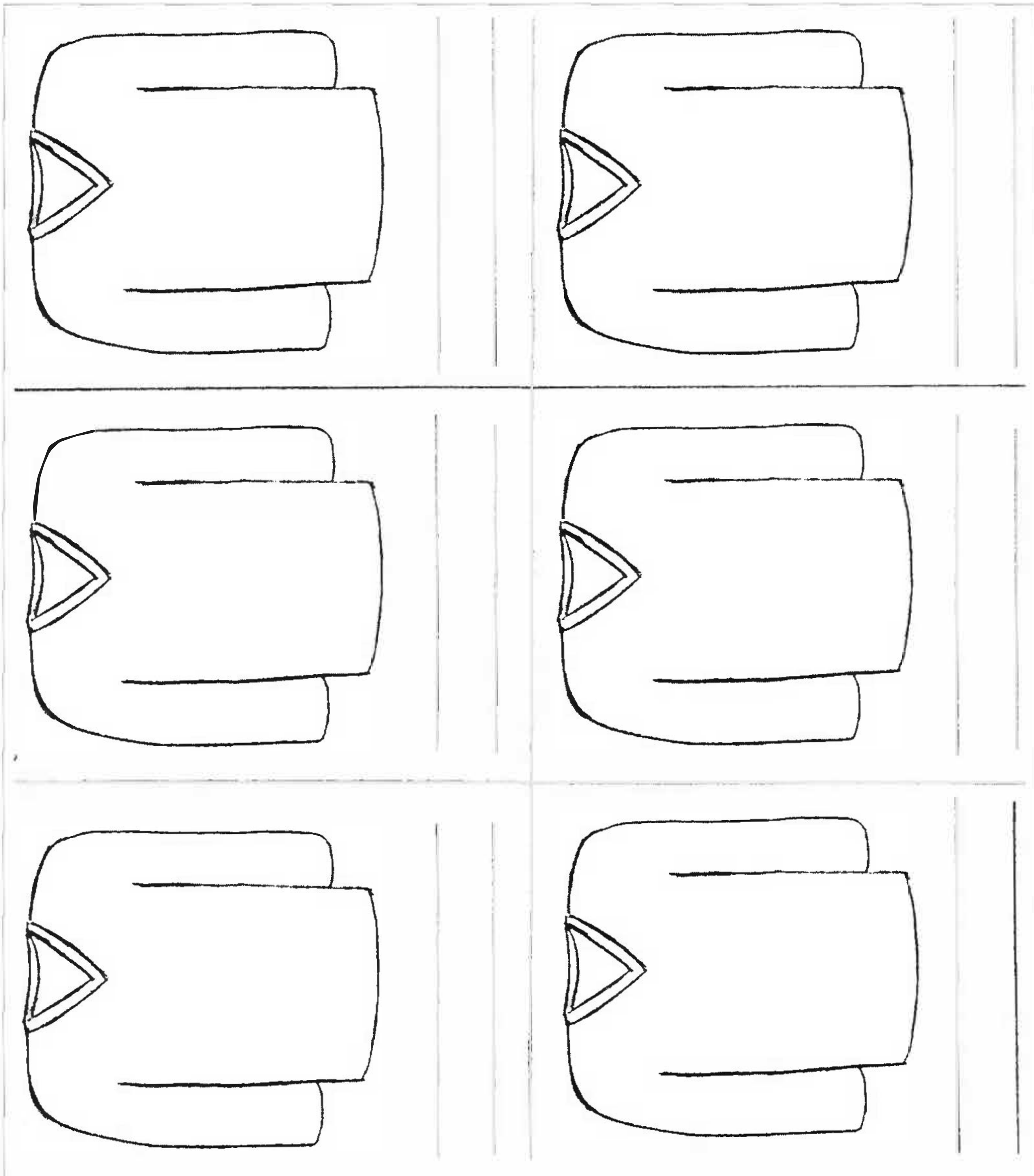
Student Partners ______________________

	Excellent	Proficient	Adequate	Not yet adequate
Counts a set with one to one correspondence	Counts to 20 with no errors	Counts to 20 with 1–2 errors	Counts to 20 with 3–5 errors	Counts to 20 with greater than 5 errors
Shows that the set has only one count— the last number is how many.	Accurately identifies each set without assistance	Accurately identifies each set with limited assistance	Requires several attempts to identify each set with assistance	Unable to demonstrate that each set has only one count
Prints the numerals legibly without reversals	All numerals are legible with no reversals	1 reversal and legible	2 reversals and legible	3 or more reversals and/or illegible
Produces a potential solution to the problem	Identifies a plausible solution independently	Identifies a plausible solution with limited assistance	Identifies a plausible solution with direct guidance	Unable to identify a plausible solution
Demonstrates how 20 items will be divided into 2 teams	Accurately produces 2 equal groups	Accurately produces 2 equal groups with assistance	Accurately produces 2 equal groups with direct guidance	Unable to produce 2 equal groups
Describes how the teams are balanced or unbalanced	Accurately explains their solution to others independently	Accurately explains their solution to others with limited assistance	Accurately explains their solution to others with direct guidance	Unable to provide an explanation
Uses math language appropriately (for example, number words zero to twenty, count, number, equal groups, singles, estimate, more, fewer, same as, equal, as many as)	Uses math language independently	Uses math language with limited prompting	Uses math language with direct prompting	Unable to use math language

Representing Numbers to 20 Checklist
Uniform Finished Product

Student Name	Two Equal Teams	All Uniforms Numbered	Two Different Team Sweaters	Appearance/Neat Work

Hockey Jersey Template



The Alberta High School Mathematics Competition
Part I, November 17, 2009

1. If $2^x = 3^y$, then 4^x is equal to

   (a) 5^y (b) 6^y (c) 8^y (d) 9^y (e) none of these

2. Caroline bought some bones for her 7 dogs. Had she owned 8 dogs, she could have given each the same number of bones. As it was, she needed two more bones to give each dog the same number of bones. The number of bones she could have bought was

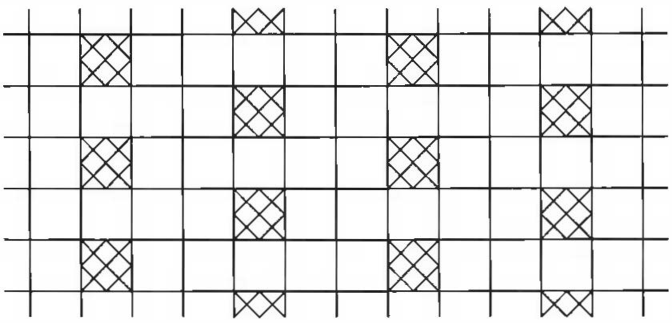
   (a) 16 (b) 24 (c) 32 (d) 40 (e) 48

3. Ace calculates the average of all the integers from 1 to 100. Bea calculates the average of all the integers from 1001 to 1100 and subtracts 1000. Cec calculates the average of all the integers from 1000001 to 1000100 and subtracts 1000000. The largest answer is given by

   (a) Ace only (b) Bea only (c) Cec only (d) exactly two of them
   (e) all three of them

4. A large rectangular gymnasium floor is covered with unit square tiles, most of them blank, in the pattern shown in the diagram below. Of the following fractions, the one nearest to the fraction of tiles which are not blank is

   (a) $\frac{1}{12}$ (b) $\frac{1}{8}$ (c) $\frac{1}{6}$ (d) $\frac{1}{5}$ (e) $\frac{1}{4}$



5. The number of integers between 1 and 2009 inclusive which can be expressed as the difference of the squares of two integers is

   (a) 1 (b) 502 (c) 1005 (d) 1507 (e) 2009

6. Among the positive integers with six digits in their base-10 representation, the number of those whose digits are strictly increasing from left to right is

   (a) between 1 and 50 (b) between 51 and 100 (c) between 101 and 500

   (d) between 501 and 1000 (e) greater than 1000

7. The number of arrangements of the letters AABBCC in a row such that no two identical letters are adjacent is

(a) 30 (b) 36 (c) 42 (d) 48 (e) none of these

8. If 2^{2009} has m digits and 5^{2009} has n digits in their base-10 representations, then the value of $m+n$ is

(a) 2007 (b) 2008 (c) 2009 (d) 2010 (e) 2011

9. An equilateral triangle has area $2\sqrt{3}$. From the midpoint of each side, perpendiculars are dropped to the other two sides. The area of the hexagon formed by these six lines is

(a) $\frac{\sqrt{3}}{2}$ (b) 1 (c) $\sqrt{3}$ (d) 2 (e) none of these

10. Two sides of an *obtuse* triangle of positive area are of length 5 and 11. The number of possible integer lengths of the third side is

(a) 3 (b) 4 (c) 6 (d) 8 (e) 9

11. $Q(x)$ is a polynomial with integer coefficients such that $Q(9) = 2009$. If p is a prime number such that $Q(p) = 392$, then p can

(a) only be 2 (b) only be 3 (c) only be 5
(d) only be 7 (e) be any of 2, 3, 5 and 7

12. A parallelogram has two opposite sides 5 centimetres apart and the other two opposite sides 8 centimetres apart. Then the area, in square centimetres, of the parallelogram

(a) must be at most 40 and can be any positive value at most 40
(b) must be at least 40 and can be any value at least 40
(c) must be 40 (d) can be any positive value (e) none of these

13. The number of positive integers n such that $\sqrt{n+\sqrt{n+\cdots+\sqrt{n}}} < 10$ for any finite number of square root signs is

(a) 10 (b) 90 (c) 91 (d) 99 (e) 100

14. A chord of a circle divides the circle into two parts such that the squares inscribed in the two parts have areas 16 and 144 square centimetres. In centimetres, the radius of the circle is

(a) $2\sqrt{10}$ (b) $6\sqrt{2}$ (c) 9 (d) $\sqrt{85}$ (e) 10

15. The number of prime numbers p such that $2^p + p^2$ is also a prime number is

(a) 0 (b) 1 (c) 2 (d) 3 (e) more than 3

16. Suppose that $2-\sqrt{99}$ is a root of x^2+ax+b where b is a negative real number and a is an integer. The largest possible value of a is

(a) -4 (b) 4 (c) 7 (d) 8 (e) none of these

Alberta High School Mathematics Competition

Solution to Part I – 2009

1. If $2^x = 3^y$, then $4^x = (2^x)^2 = (3^y)^2 = 9^y$. The answer is **(d)**.

2. The number of bones Caroline bought is a multiple of 8 but 2 less than a multiple of 7. The answer is **(d)**.

3. The calculation is $\frac{(n+1)+(n+2)+\cdots+(n+100)}{100} - n = \frac{100n+(1+2+\cdots+100)}{100} - n = \frac{1+2+\cdots+100}{100}$, with $n = 0$ for Ace, $n = 1000$ for Bea and $n = 1000000$ for Cec. The answer is **(e)**.

4. Almost the entire gymnasium floor may be divided into 2×3 non overlapping rectangles each with exactly one non-blank square at the lower left corner. The answer is **(c)**.

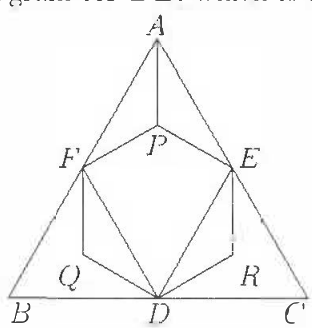
5. Observe that $x^2 - y^2 = (x-y)(x+y)$ is the product of two integers of same parity. Hence $x^2 - y^2$ is either odd or divisible by 4. Thus a number which is neither odd nor divisible by 4 cannot be expressed as a difference of two squares. On the other hand, if n is odd, then $n = 2k+1 = (k+1)^2 - k^2$. If n is divisible by 4, then $n = 4k = (k+1)^2 - (k-1)^2$. Between 1 and 2009 inclusive, there are 1005 numbers that are odd and 502 that are divisible by 4. The answer is **(d)**.

6. We can choose any six of the nine non-zero digits. The number of choices is $\binom{9}{6} = 84$. Each choice gives rise to a unique number. The answer is **(b)**.

7. Assume that the first A appears before the first B, and the first B before the first C. Then we must start with AB and continue with A or C. If we continue with A, the last three letters must be CBC. If we start with ABC, we must continue with A or B. In either case, either of the last two letters can appear before the other. So the total is $1 + 2 \times 2 = 5$. Relaxing the order of appearance, the total becomes $5 \times 3! = 30$. The answer is **(a)**.

8. Since $10^{n-1} < 2^{2009} < 10^n$ and $10^{m-1} < 5^{2009} < 10^m$, we have $10^{m+n-2} < 2^{2009}5^{2009} < 10^{m+n}$. It follows that $10^{m+n-1} = 2^{2009}5^{2009} = 10^{2009}$. Hence $m + n - 1 = 2009$. The answer is **(d)**.

9. Let ABC be the triangle and $DREPFQ$ be the hexagon, as shown in the diagram below. Triangles APE, APF, ERD and FQD are all congruent to one another. Hence $DREPFQ$ has the same area as the parallelogram $AFDE$, which is one half of $2\sqrt{3}$. The answer is **(c)**.



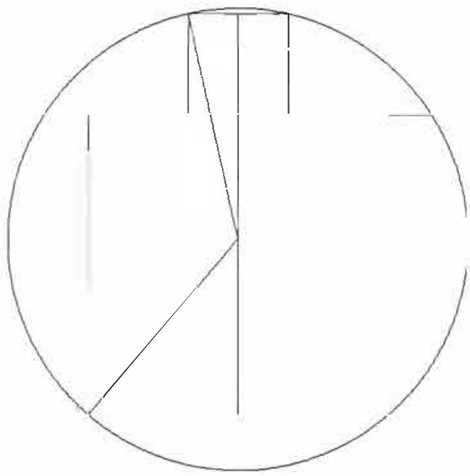
10. By the Triangle Inequality, the third side must be from 7 to 15. Now $9^2 < 11^2 - 5^2 < 10^2$, so that (5,7,11), (5,8,11) and (5,9,11) are obtuse triangles. Also, $12^2 < 11^2 + 5^2 < 13^2$, so that (5,11,13), (5,11,14) and (5,11,15) are obtuse triangles. The other three are not. The answer is **(c)**.

11. Since $Q(9) = 2009$, $x - 9$ is a factor of $Q(x) - 2009$. Since $Q(x)$ has integral coefficients, $p - 9$ divides $Q(p) - 2009 = 392 - 2009$. Since this number is odd, $p - 9$ must be odd. Since p is prime, we must have $p = 2$. Now $392 - 2009 = -1617 = -231 \times 7$, so that $Q(x)$ does exist. For instance, we may have $Q(x) = 231x - 70$. The answer is **(a)**.

12. Let 5 centimetres be the height of the parallelogram. Its base is a segment intercepted by the other pair of parallel lines 8 centimetres apart. Hence its length is at least 8 centimetres and can be arbitrarily large. The answer is **(b)**.

13. With one square root sign, $\sqrt{n} < 10$ is equivalent to $n < 100$. With two square root signs, $\sqrt{n + \sqrt{n}} < 10$ is equivalent to $n + \sqrt{n} < 100$, which is in turn equivalent to $n < 91$. With three square root signs and $n < 91$, we have $\sqrt{n + \sqrt{n + \sqrt{n}}} < \sqrt{n + 10} \leq 10$. With more square root signs, the same inequality will hold. There are 90 positive integers which satisfy $n < 91$. The answer is **(b)**.

14. Let x be the distance from the centre of the circle to the bottom edge of the larger square. The square of the radius of the circle is given by $6^2 + x^2 = 2^2 + (4 + 12 - x)^2$. This yields $x = 7$ so that the radius of the circle is $\sqrt{6^2 + 7^2} = \sqrt{85}$. The answer is **(d)**.



15. If $p = 2$, $2^p + p^2 = 8$ is not prime. If $p = 3$, $2^p + p^2 = 17$ is prime. If $p > 3$, then $p = 6k \pm 1$ for some integer k, so that $p^2 = 36k^2 \pm 12k + 1$ is 1 more than a multiple of 3. On the other hand, when divided by 3, successive powers of 2 leave remainders of 2 and 1 alternately. Since $p > 3$ is odd, 2^p is 2 more than a multiple of 3. Hence $2^p + p^2$ is divisible by 3, and cannot be prime. The answer is **(b)**.

16. Let the other root be t. Then $-a = t + 2 - \sqrt{99}$. Now $b = t(2 - \sqrt{99}) < 0$, so that $t = \sqrt{99} - 2 - a > 0$. Since a is an integer, $a \leq 9 - 2 = 7$. The answer is **(c)**.

The Alberta High School Mathematics Competition
Part II, February 3, 2010

1. Of Melissa's ducks, $x\%$ have 11 ducklings each, $y\%$ have 5 ducklings each and the rest have 3 ducklings each. The average number of ducklings per duck is 10. Determine all possible *integer* values of x and y.

2. (a) Find all real numbers $t \neq 0$ such that $tx^2 - (2t-1)x + (5t-1) \geq 0$ for all real numbers x.

   (b) Find all real numbers $t \neq 0$ such that $tx^2 - (2t-1)x + (5t-1) \geq 0$ for all $x \geq 0$.

3. Points A, B, C and D lie on a circle in that order, so that $AB = BC$ and $AD = BC + CD$. Determine $\angle BAD$.

4. Let n be a positive integer. A $2^n \times 2^n$ board, missing a 1×1 square anywhere, is to be partitioned into rectangles whose side lengths are integral powers of 2. Determine in terms of n the smallest number of rectangles among all such partitions, wherever the missing square may be.

5. Let f be a non-constant polynomial with non-negative integer coefficients.

   (a) Prove that if M and m are positive integers such that M is divisible by $f(m)$, then $f(M+m)$ is also divisible by $f(m)$.

   (b) Prove that there exists a positive integer n such that each of $f(n)$ and $f(n+1)$ is a composite number.

The Alberta High School Mathematics Competition
Solution to Part II, 2010.

1. We have $11x + 5y + 3(100 - x - y) = 1000$ or $4x + y = 350$. Since $y \geq 0$, we get $x \leq 87$. Since $x + y \leq 100$, we also have that $3x \geq 250$, so $x \geq 84$. Thus the only solutions are $(x, y) = (84, 14)$, $(85,10)$, $(86,6)$ and $(87,2)$.

2. For either (a) or (b), clearly the leading coefficient t of the quadratic must be positive.

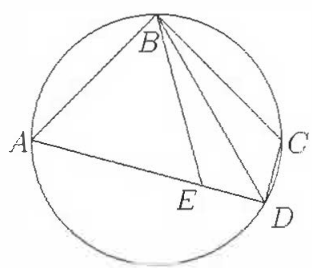
   (a) For the inequality to hold for all real x, the discriminant must be non-positive, that is,

   $$0 \geq (2t-1)^2 - 4t(5t-1) = 1 - 16t^2 = (1-4t)(1+4t).$$

   Since $t > 0$, $1 + 4t > 0$, so we need $1 - 4t \leq 0$. Thus $t \geq \frac{1}{4}$.

   (b) We now have the additional possibility that the two roots of the quadratic are real and non-positive. This holds if and only if $0 < t \leq \frac{1}{4}$, $2t - 1 \leq 0$ and $5t - 1 \geq 0$. This is equivalent to $\frac{1}{5} \leq t \leq \frac{1}{4}$. Combining with the answer to (a), we have $t \geq \frac{1}{5}$.

3. **First Solution:**
   Putting $AB = BC = b$ and $CD = c$, we get $AD = b + c$. Let $\angle BAD = \alpha$. Since $ABCD$ is cyclic, $\angle BCD = 180° - \alpha$. Applying the cosine law to triangles BAD and BCD, we have $BD^2 = b^2 + (b+c)^2 - 2b(b+c)\cos\alpha$ and $BD^2 = b^2 + c^2 - 2bc\cos(180° - \alpha) = b^2 + c^2 + 2bc\cos\alpha$. Hence $b^2 + (b+c)^2 - 2b(b+c)\cos\alpha = b^2 + c^2 + 2bc\cos\alpha$, so that $b^2 + 2bc = (2b^2 + 4bc)\cos\alpha$. This yields $\cos\alpha = \frac{1}{2}$, so that $\alpha = 60°$ is the only possibility.



   Second Solution:
   Let E be the point on AD such that $DE = DC$, so that $AE = AD - DE = BC = AB$. Now $\angle BDE = \angle BDC$ since they are subtended by the equal arcs BA and BC. It follows that triangles BED and BCD are congruent, so that $BE = BC = BA = AE$, triangle BAE is equilateral and $\angle BAD = 60°$.

4. **First Solution:**
   The area of the punctured board is $2^{2n} - 1$. The base-2 representation of this number consists of $2n$ 1s. Since the area of each rectangle in the partition is a power of 2, we must have at least $2n$ rectangles. There exist such partitions with exactly $2n$ rectangles. Divide the board in halves by a horizontal grid line. Set aside the one with the missing square and cover the other with a rectangle of height 2^{n-1}. Repeating the process with the strips set aside, we obtain rectangles with decreasing heights 2^{n-2}, 2^{n-3}, $\ldots$, 2^1 and 2^0, a total of n rectangles. We now divide the resulting $2^n \times 1$ board in halves by a vertical line. Set aside the one with the missing square and cover the other with a rectangle of width 2^{n-1}. Repeating the process with the strips set aside, we obtain another n rectangles with decreasing widths, for a total of $2n$ rectangles in the overall partition.

Second Solution:

Divide the board into four congruent quadrants. Set aside the one with the missing square. Merge two of the other quadrants into one rectangle and keep the third quadrant as the second rectangle. In reducing a $2^n \times 2^n$ board down to a $2^{n-1} \times 2^{n-1}$ board, we use two rectangles. It follows that we will use exactly $2n$ rectangles in the overall partition. We now prove that we cannot get by with a smaller number. The area of a rectangle of the prescribed type is a power of 2. The smallest has area 1, and the largest has area 2^{2n-1}. Thus there are $2n$ different sizes. If we use one of each size, the total area of these $2n$ rectangles is $1 + 2 + \cdots + 2^{2n-1} = 2^{2n} - 1$, exactly the size of the punctured chessboard. Consider any other collection of rectangles whose areas are powers of 2 and whose total area is $2^{2n-1} - 1$. Replace any pair of rectangles of equal area by one with twice the area. Repeat until no further replacement is possible. The resulting collection consists of rectangles of distinct areas which are powers of 2, and with total area $2^{2n-1} - 1$. It can only be our collection, and since mergers only reduce the number of rectangles, $2n$ is indeed minimum.

5. (a) Note that $f(M+m) - f(m)$ is a sum of terms of the form $a_k((M+m)^k - m^k)$ where a_k is the coefficient of the term x^k in $f(x)$. Since each term is divisible by $M = (M+m) - m$, so is $f(M+m) - f(m)$. Since M is divisible by $f(m)$, $f(M+m) - f(m)$ is divisible by $f(m)$. It follows that $f(M+m)$ is divisible by $f(m)$.

   (b) Since all the coefficients of f are non-negative and f is non-constant, it is strictly increasing. Let $M = f(2)f(3)$ and $n = M + 2$. By (a), $f(n)$ is divisible by $f(2)$ and $f(n+1)$ is divisible by $f(3)$. Since $f(n+1) > f(n) > f(3) > f(2) > f(1) \geq 1$, both $f(n)$ and $f(n+1)$ are composite.

Print ID # __________ (preprinted) /100

School Name ____________________ Student Name ____________________ (Print First, Last)

2010 Edmonton Junior High Math Contest

Part I: Multiple Choice (PRINT neatly, use CAPITAL letters, 4 points each)

1.	2.
3.	4.
5.	6.
7.	8.
9.	10.

Part II: Numeric Response (PRINT small but legibly, 6 points each)

11.	12.
13.	14.
15.	16.
17.	18.
19.	20.

Part I: ________ $\times 4 +$ ________ $\times 2 =$ ________ (be sure blanks ≤ 3)

Correct Blank

Part II: ______ $\times 6 =$ ________

Correct

Total = ________ (enter the total score above)

MARKER ONLY

Instructions

1. Calculator, grid paper and scrap paper are permitted. You may write on the booklet.
2. Programmable calculators and cellphones are not allowed.
3. Don't write your answers too LARGE because others might see your answers. COVER your answers at all times.
4. All fractions must be proper and reduced to lowest terms.
5. Each correct answer is worth 4 points for multiple choice and 6 points for numeric response.
6. Each incorrect answer is worth 0 points.
7. Each unanswered question in Part I is worth 2 points up to a maximum of 6 points.
8. Unanswered questions in Part II are worth 0 points.
9. You have 90 minutes of writing time.
10. When done, carefully REMOVE and HAND IN only page 1.

Multiple-Choice Questions

1. The side lengths of a rectangle are whole numbers and the perimeter is 202 cm. What is the largest area possible?
   A) 1000 cm^2
   B) 2050 cm^2
   C) 2550 cm^2
   D) 3050 cm^2
   E) none of these

2. x is a positive number with the property that $x^2 + \frac{1}{x^2} = 23$. The value of $x + \frac{1}{x}$ is
   A) 5
   B) 7
   C) $\sqrt{23}$
   D) $\sqrt{46}$
   E) none of these

3. How many whole numbers, n, can be found so that $\frac{2}{7} < \frac{n}{11} < \frac{2}{3}$?
   A) 3
   B) 4
   C) 5
   D) 6
   E) none of these

4. If p, $p + 10$ and $p + 14$ are all prime numbers, how many possible values are there of p?
   A) 0
   B) 1
   C) 2
   D) 3
   E) infinitely many

5. In how many ways can two As, two Bs and two Cs be arranged in a row so that no two adjacent letters are the same?
   A) 120
   B) 90
   C) 60
   D) 45
   E) 30

6. Given: $3^x = 9^{y-2}$ and $2^{x-3} = 8^y$, find the value of xy.
   A) -126
   B) -60
   C) 0
   D) 60
   E) 126

7. If x represents the units digit in the product of the first 100 prime numbers, and y represents the sum of the units digits of the first 10 multiples of 3, find $x + y$.
   A) 5
   B) 15
   C) 18
   D) 45
   E) 48

8. Twenty-four teens washed one-third of the cars lined up for a charity car wash in 4 hours. Then more teens joined the car-washing team, and the remaining cars were washed in another 6 hours. How many teens joined the team?
   A) 4
   B) 8
   C) 16
   D) 32
   E) 64

9. Kristoff earned \$10.00 each day for a week shovelling walks. On the eighth day he earned \$1.75 more than his average earnings for all 8 days. How much did Kristoff make on the eighth day?
   A) \$11.00
   B) \$11.75
   C) \$12.00
   D) \$13.75
   E) \$15.00

10. $\frac{2}{f}$ and $\frac{5}{2f}$ represent positive fractions. Which expression represents the sum of the two fractions that are evenly spaced between $\frac{2}{f}$ and $\frac{5}{2f}$?
   A) $\frac{9}{2f}$
   B) $\frac{10}{f}$
   C) $\frac{1}{6f}$
   D) $\frac{55}{12f}$
   E) $\frac{25}{6f}$

Numeric Response (Record answers on page 1.)

11. ABCD is a trapezoid, with side AB parallel to side CD. Sides: AB, BC and DA are equal and are of length 2 cm. Side CD has length of 4 cm. The measure of angle ADC is ________°.

12. The product of the digits of the number 176 is 42. How many other three-digit numbers have 42 as the product of their digits?

13. Alyssa said to Bryan, "If you give me half of your money, I will have just enough to buy that horse." Bryan replied, "If instead you give me two-thirds of your money, I will have just enough to buy that same horse." Neither gave, and instead spent all their money buying pigs, each of which cost the same. If Alyssa bought 30 pigs, how many did Bryan buy?

14. The length and area of a rectangle can be expressed as x^2y and $x3y^2$, respectively, where x and y are natural numbers. If the area is 1323 m^2, what is the perimeter of the rectangle?

15. The sum of three positive integers is 9. What is the least possible product of their reciprocals? Express your answer to the nearest thousandth.

16. A large rectangular field is subdivided into 4 congruent rectangular fields as shown at the right. The area of the large rectangular field is 3468 m^2. Find the area of a square field that has the same perimeter as the large field before it was subdivided. Express your answer to the nearest whole square metre.

17. The height of a cone is increased by 10% and the radius is decreased by 10%. By what per cent, expressed to the nearest tenth of a per cent, will the volume of the cone change? (Volume of cone = $\frac{\pi R^2 H}{3}$, where R is the radius and H is the height)

18. If you randomly select 3 vertices of a regular hexagon, and then connect them, what is the probability that a right triangle will be formed? Express your answer as a decimal, to the nearest hundredth.

19. Kylee is a cheerleader and has a drawer that contains four colours of poms. Ninety are gold, 70 are green, 50 are blue and 40 are red. She randomly pulls out poms, one at a time, without looking at the colours. What is the fewest number of poms that Kylee must remove to be certain that she has at least 10 pairs of matching poms?

20. Travis left home, travelled 12 km west to the grocery store, then 6 km south to the post office, then 2 km west to the bank, then x km north to the library. The shortest distance from the library to Travis's home is 15 km. What is the farthest distance that the library can be from the bank? Express your answer to the nearest tenth of a kilometre.

Print ID # ____________ /100

School Name ________________________ Student Name ________________________ (Print First, Last)

2010 Edmonton Junior High Math Contest Solutions

Part I: Multiple Choice (PRINT neatly, use CAPITAL letters, 4 points each)

1. C	2. A
3. B	4. B
5. E	6. E
7. D	8. B
9. C	10. A

Part II: Numeric Response (PRINT small but legibly, 6 points each)

11. 60°	12. 11
13. 20	14. 168
15. 0.037	16. 3540
17. 10.9	18. 0.60
19. 23	20. 11.4

Part I: ________ × 4 + ________ × 2 = ________ (be sure blanks ≤ 3)
Correct Blank

Part II: ________ × 6 = ________
Correct

Total = ________ (enter the total score above)

MARKER ONLY

Instructions

1. Calculator, grid paper and scrap paper are permitted. You may write on the booklet.
2. Programmable calculators and cellphones are not allowed.
3. Don't write your answers too LARGE because others might see your answers. COVER your answers at all times.
4. All fractions must be proper and reduced to lowest terms.
5. Each correct answer is worth 4 points for multiple choice and 6 points for numeric response.
6. Each incorrect answer is worth 0 points.
7. Each unanswered question in Part I is worth 2 points up to a maximum of 6 points.
8. Unanswered questions in Part II are worth 0 points.
9. You have 90 minutes of writing time.
10. When done, carefully REMOVE and HAND IN only page 1.

Multiple-Choice Questions

1. The side lengths of a rectangle are whole numbers and the perimeter is 202 cm. What is the largest area possible?

A) 1000 cm^2
B) 2050 cm^2
C) 2550 cm^2
D) 3050 cm^2
E) none of these

Solution: C) 2550 cm^2

The largest area is obtained when the rectangle's shape is closest to a square. Since $202 \div 4 = 50.5$ and the sides must be whole numbers, the shape closest to a square would have dimensions of 50 and 51. The area is $50(51) = 2550$. Thus, the answer is C) 2550 cm^2.

2. x is a positive number with the property that $x^2 + \frac{1}{x^2} = 23$. The value of $x + \frac{1}{x}$ is

A) 5
B) 7
C) $\sqrt{23}$
D) $\sqrt{46}$
E) none of these

Solution: A) 5

The square of $x + \frac{1}{x}$ is

$(x + \frac{1}{x})^2 = x^2 + 2x(\frac{1}{x}) + \frac{1}{x^2}$

$= x^2 + \frac{1}{x^2} + 2$

Replace $x^2 + \frac{1}{x^2}$ with 23

$= 23 + 2$
$= 25$

Since, $(x + \frac{1}{x})^2 = 25$, the square root of $(x + \frac{1}{x})^2$ is $x + \frac{1}{x}$ and $x + \frac{1}{x} = \sqrt{25}$ and $\sqrt{25} = 5$. This is answer A).

3. How many whole numbers, n, can be found so that $\frac{2}{7} < \frac{n}{11} < \frac{2}{3}$?

A) 3
B) 4
C) 5
D) 6
E) none of these

Solution is B) 4

Multiply $\frac{2}{7} < \frac{n}{11} < \frac{2}{3}$ by 11.

$11\,(\frac{2}{7} < \frac{n}{11} < \frac{2}{3})$ which is equivalent to

$\frac{22}{7} < n < \frac{22}{3}$ which is equivalent to

$3.14 < n < 7.33$

Therefore, there are four whole numbers: 4, 5, 6 and 7, which is answer B).

4. If p, $p + 10$ and $p + 14$ are all prime numbers, how many possible values are there of p?
   A) 0
   B) 1
   C) 2
   D) 3
   E) infinitely many

   Solution is B)
   The pair wise differences of the three numbers are 4, 10 and 14. No two of these numbers leave the same remainder when divided by 3. Therefore, one of them will be divisible by 3. Since they are all prime numbers, one of them is 3, which means $p = 3$. Therefore, $p + 10 = 13$, and $p + 14 = 17$, which are all prime. Therefore, there is only 1 possible value for p. This is answer B).

5. In how many ways can two As, two Bs and two Cs be arranged in a row so that no two adjacent letters are the same?
   A) 120
   B) 90
   C) 60
   D) 45
   E) 30

   Solution is E) 30
   There are six ways to arrange the two As and the two Bs without worrying about adjacent letters being the same. These six arrangements are ABAB, BABA, AABB, BBAA, ABBA, BAAB. Now determine how many ways the two Cs can be added. In AABB and BBAA, the two Cs must be added between the two As and the two Bs. So, there are two arrangements. In ABBA and BAAB, one of the two Cs must be between the two Bs in the first case and the two As in the second case. The other C can go in front or at the back, or between an A and a B. So, there are eight arrangements. In ABAB and BABA, there are five places where C can go, and the two Cs cannot go in the same place. So, they can be in places 1-2, 1-3, 1-4, 1-5, 2-3, 2-4, 2-5, 3-4, 3-5 and 4-5. So, there are 20 arrangements of this. The final total is 2 + 8 + 20 = 30. Which is answer E).

6. Given: $3^x = 9^{y-2}$ and $2^{x-3} = 8^y$, find the value of xy.
   A) -126
   B) -60
   C) 0
   D) 60
   E) 126

   Solution is E) 126
   Given: $3^x = 9^{y-2}$, rewrite 9^{y-2} as base 3
   $3^x = 3^{2(y-2)}$
   Therefore, $x = 2y - 4$
      Given: $2^{x-3} = 8^y$, rewrite 8^y as base 2
      $2^{x-3} = 2(3)^y$
      Therefore, $x - 3 = 3y$,
      Since, $x = 2y - 4$, replace x with $2y - 4$ in the equation $x - 3 = 3y$
   $x - 3 = 3y$
   $2y - 4 - 3 = 3y$
   Solve for y:
   $y = -7$
   Substitute $y = -7$ into the equation: $x = 2y - 4$
   $x = 2y - 4$
   $x = 2(-7) - 4$
   $x = -18$
   Therefore, $xy = (-18)(-7)$
   $xy = 126$ which is answer E)

7. If x represents the units digit in the product of the first 100 prime numbers, and y represents the sum of the units digits of the first 10 multiples of 3, find $x + y$.
   A) 5
   B) 15
   C) 18
   D) 45
   E) 48

   The solution is D) 45.
   In the list of the first 100 prime numbers, 2 and 5 appear. Therefore the unit's digit of the product will be a 0. The first 10 multiples of 3 are 3, 6, 9, 12, 15, 18, 21, 24, 27 and 30. The sum of these numbers is 45. So, $x = 0$ and $y = 45$. $x + y = 0 + 45$
   Therefore, the sum is 45, which is answer D).

8. Twenty-four teens washed one-third of the cars lined up for a charity car wash in 4 h. Then more teens joined the car-washing team, and the remaining cars were washed in another 6 hours. How many teens joined the team?
   A) 4
   B) 8
   C) 16
   D) 32
   E) 64

   Solution: The answer is B) 8.
   Let C represent the number of cars lined up to be washed.
   Use the formula: Work = Rate × Time
   $\frac{1}{3}C = 24(4)$ where 24 (the number of workers) represents the Rate and 4 represents the Time
   Solve for C.
   $C = 288$.
   Therefore, there were 288 cars lined up to be washed.
   Let N represent the number of workers joining after 4 hours.
   Two-thirds of the cars remain to be washed by 24 + N workers in 6 hours. Substitute into the work formula.
   $W = R \times T$
   $\frac{2}{3}C = (24 + N)6$
   Replace C with 288 and solve for N.
   $\frac{2}{3}(288) = (24 + N)6$
   $192 = 144 + 6n$
   $48 = 8N$
   $8 = N$
   $N = 8$, which is answer B).

9. Kristoff earned $10.00 each day for a week shovelling walks. On the eighth day he earned $1.75 more than his average earnings for all 8 days. How much did Kristoff make on the eighth day
   A) $11.00
   B) $11.75
   C) $12.00
   D) $13.75
   E) $15.00

   Solution is: C) $12.00
   The earnings for 7 days at $10.00/day is $70. Add this to a, which represents the average of all eight days plus the $1.75 increase. Divide by 8 and set equal to a.

   $\frac{70 + a + 1.75}{8} = a$
   $71.75 + a = 8a$
   $71.75 = 7a$
   $10.25 = a$
   Add $1.75 to $10.25 to find his earnings on the 8th day.
   $1.75 + $10.25 = $12.00, which is answer C).

10. $\frac{2}{f}$ and $\frac{5}{2f}$ represent positive fractions. Which expression represents the sum of the two fractions that are evenly spaced between $\frac{2}{f}$ and $\frac{5}{2f}$?
   A) $\frac{9}{2f}$
   B) $\frac{10}{f}$
   C) $\frac{1}{6f}$
   D) $\frac{55}{12f}$
   E) $\frac{25}{6f}$

   Solution is A) $\frac{9}{2f}$
   Find the difference of the two fractions:
   $\frac{5}{2f} - \frac{2}{f} = \frac{5}{2f} - \frac{4}{2f} = \frac{1}{2f}$

   Divide the difference by 3:
   $\frac{1}{2f} \div 3 = \frac{1}{6f}$

   Add this quotient to the first fraction:
   $\frac{2}{f} + \frac{1}{6f} = \frac{12}{6f} + \frac{1}{6f} = \frac{13}{6f}$

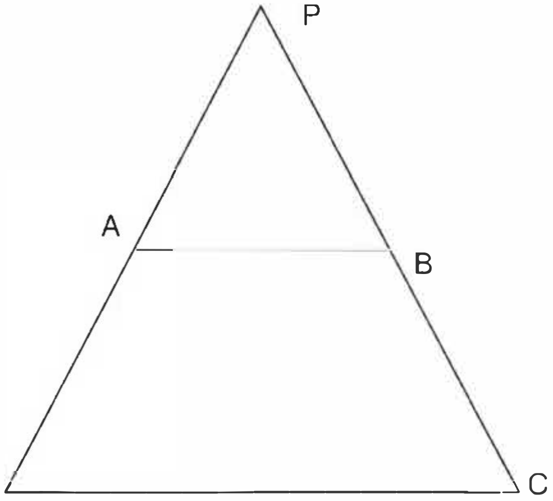
   Add the quotient to the sum.
   Second fraction: $\frac{13}{6f} + \frac{1}{6f} = \frac{14}{6f}$
   The two fractions are: $\frac{13}{6f}$ and $\frac{14}{6f}$.

   Find their sum: $\frac{13}{6f} + \frac{14}{6f} = \frac{27}{6f} = \frac{9}{2f}$
   This is answer A).

Numeric Response

11. ABCD is a trapezoid, with side AB parallel to side CD. The sides: AB, BC and DA are equal and are of length 2 cm. Side CD has length of 4 cm. The measure of angle ADC is _____.

The solution is 60°.



Let P be the point of intersection of AD and BC. Since AB is parallel to CD and AB is half of CD, AB is the midline of triangle PCD. Therefore, A is the midpoint of PD and B is the midpoint of PC. And PC = PD = CD = 4 cm. Therefore, triangle PCD is equilateral, and angle ADC has a measure of 60°.

12. The product of the digits of the number 176 is 42. How many other three-digit numbers have 42 as the product of their digits?

The solution is 11.

A product of 42 can be formed two ways using three whole numbers:

$1 \times 6 \times 7$ or $2 \times 3 \times 7$

Rearranging these numbers, the following three-digit numbers can be formed:

167, 176, 671, 617, 761, 716, 237, 273, 327, 372, 723 and 732. Since, the number 176 was given, there are 11 other ways.

13. Alyssa said to Bryan, "If you give me half of your money, I will have just enough to buy that horse." Bryan replied, "If instead you give me two-thirds of your money, I will have just enough to buy that same horse." Neither gave, and instead spent all their money buying pigs, each of which cost the same. If Alyssa bought 30 pigs, how many did Bryan buy?

The solution is 20.

Alyssa's money plus half of Bryan's was equal to Bryan's money plus two-thirds of Alyssa's.

Therefore, one third of Alyssa's money was equal to half of Bryan's. Since Alyssa bought 30 pigs, she bought 10 pigs with one third of her money. And, Bryan bought 10 pigs with half of his money. Or he bought 20 pigs total.

14. The length and area of a rectangle can be expressed as x^2y and x^3y^2, respectively, where x and y are natural numbers. If the area is 1323 m^2, what is the perimeter of the rectangle?

The solution is 168 m.

Since the area is 1323 m^2 , and x and y are natural numbers, 1323 must be written as the product of a perfect square number and a perfect cube number. The prime factorization of 1323 is $3 \times 3 \times 3 \times 7 \times 7$. Therefore, x = 3 and y = 7.

Area of a rectangle: A = lw

$$x^3y^2 = (x^2y)w$$

$$\frac{x^3y^2}{x^2y} = w$$

This simplifies to xy = w

The perimeter of a rectangle is 2l + 2w

P = 2l + 2w

P = $2(x^2y) + 2xy$, replace x with 3 and y with 7.

P = 2(327) + 2(3)(7)

P = 168.

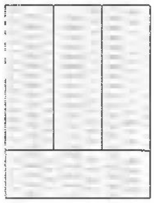
The perimeter of the rectangle is 168 m.

15. The sum of three positive integers is 9. What is the least possible product of their reciprocals? Express your answer to the nearest thousandth.

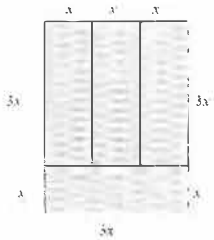
The solution is 0.037.

Possible Sums	Product of Reciprocals
1 + 1 + 7	$1 \times 1 \times \frac{1}{7} = \frac{1}{7}$
1 + 2 + 6	$1 \times \frac{1}{2} \times \frac{1}{6} = \frac{1}{12}$
1 + 3 + 5	$1 \times \frac{1}{3} \times \frac{1}{5} = \frac{1}{15}$
1 + 4 + 4	$1 \times \frac{1}{4} \times \frac{1}{4} = \frac{1}{16}$
2 + 2 + 5	$\frac{1}{2} \times \frac{1}{2} \times \frac{1}{5} = \frac{1}{20}$
2 + 3 + 4	$\frac{1}{2} \times \frac{1}{3} \times \frac{1}{4} = \frac{1}{24}$
3 + 3 + 3	$\frac{1}{3} \times \frac{1}{3} \times \frac{1}{3} = \frac{1}{27}$

The smallest product = $\frac{1}{27}$ = 0.037



16. A large rectangular field is subdivided into 4 congruent rectangular fields as shown at the right. The area of the large rectangular field is 3468 m^2. Find the area of a square field that has the same perimeter as the large field before it was subdivided. Express your answer to the nearest whole square metre.



The solution is 3540 m^2.

Since the rectangle is divided into 4 congruent rectangles, the original rectangle has a length to width ratio of 4 to 3. So, the perimeter of the original rectangle can be expressed as 14x and the area as $12x^2$.

Therefore, $12x^2 = 3468$. and x = 17.

The perimeter of the original rectangle is 2(4x) + 2(3x)

P = 2(4)(17) + 2(3)(17)

P = 238 m

Therefore, a square with perimeter 238 m would have side length of 238 ÷ 4 or 59.5 m.

The area of the square is 59.52 or 3540.25 m^2. Rounded to the nearest whole metre squared: 3540 m^2.

17. The height of a cone is increased by 10% and the radius is decreased by 10%. By what per cent, expressed to the nearest tenth of a per cent, will the volume of the cone change? (Volume of cone = $\frac{\pi R^2 H}{3}$, where R is the radius and H is the height)

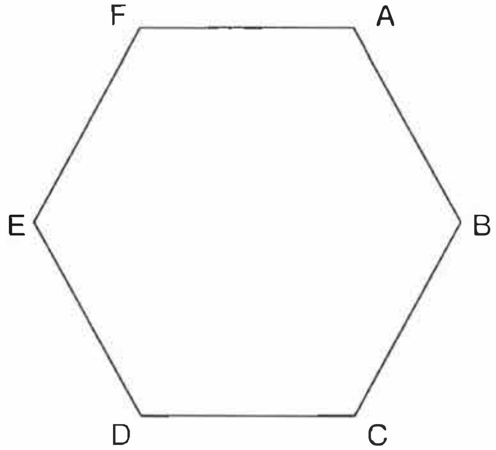
The solution is 10.9%.

Let the R = 10 units and the H = 10 units.

Substitute into the formula: V = $\frac{\pi R^2 H}{3}$

$$V = \frac{\pi 10^2 10}{3}$$

$$V = \frac{1000\pi}{3}$$



The altered cone has a height increase of 10%, so the new H is 11. The altered cone has a radius decrease of 10%, so the new R is 9. Substitute into the volume formulas.

$$V = \frac{\pi R^2 H}{3}$$

$$V = \frac{\pi 9^2 11}{3}$$

$$V = \frac{891\pi}{3}$$

This is a decrease in volume of $\frac{1000\pi}{3} - \frac{891\pi}{3}$ or $\frac{109\pi}{3}$ which is 10.9 % of the original volume.

18. If you randomly select 3 vertices of a regular hexagon, and then connect them, what is the probability that a right triangle will be formed? Express your answer as a decimal, to the nearest hundredth.

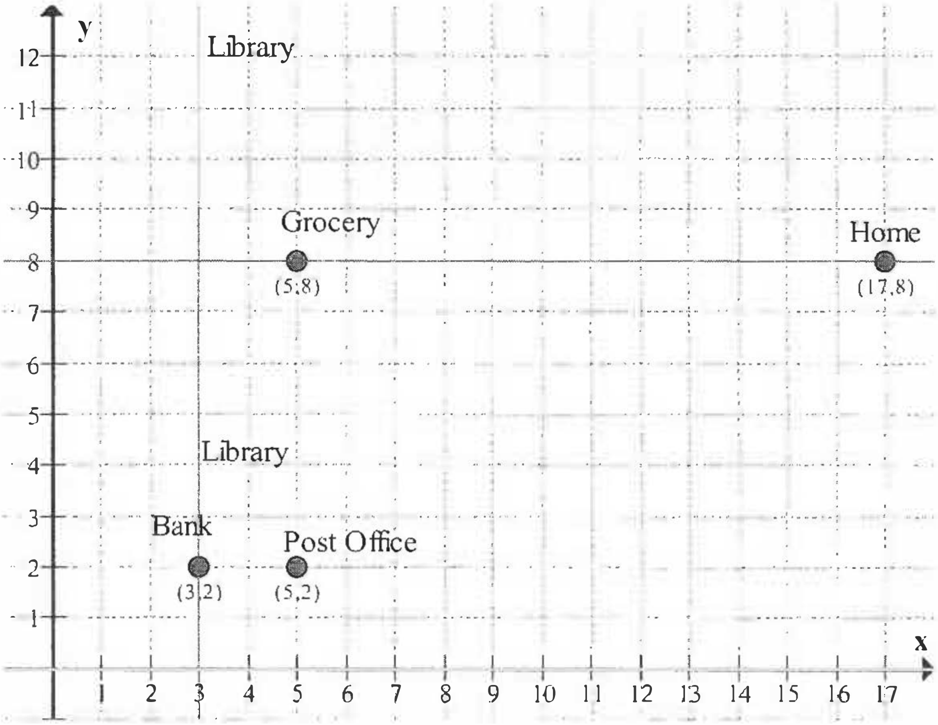
The solution is 0.60

There are 20 triangles that can be formed: ABC, ABD, ABE, ACE, ADC, ADE, AFC, AFD, BCD, BCE, BCF, BDF, CDE, CDF, DEF, EDB, EFA, EFB, FAB, FEC. Of these, 12 are right triangles: ABD, ABE, AFC, AFD, ADE, ADC, BCF, BCE, CDF, EDB, EFB, FEC. Therefore, the probability of selecting a right triangle is $\frac{12}{20}$ or 0.60

19. Kylee is a cheerleader and has a drawer that contains four colours of poms. Ninety are gold, 70 are green, 50 are blue and 40 are red. She randomly pulls out poms, one at a time, without looking at the colours. What is the fewest number of poms that Kylee must remove to be certain that she has at least 10 pairs of matching poms? The solution is 23.

Look at the worst-case scenario, which is that Kylee pulls out 18 poms of the same colour. She will still have 9 pairs of matching poms. She could then pull out 4 more poms that are one of each colour. Still, she would have only 9 pairs of matching poms from the 22 poms removed. However, the next pom removed, her 23rd pull, must match with an existing pom, thereby forming her 10th pair of matching poms.

20. Travis left home, travelled 12 km west to the grocery store, then 6 km south to the post office, then 2 km west to the bank, then x km north to the library. The shortest distance from the library to Travis's home is 15 km. What is the farthest distance that the library can be from the bank? Express your answer to the nearest tenth of a kilometer.



Suppose home is at the point (17, 8) and the grocery is 12 units west (left) at the point (5, 8). Then the post office would be 6 units to the south (down) at the point (5, 2). The bank would be 2 units west (left) at the point (3, 2). The library is located on the vertical line passing through the point (3, 2). Form a right triangle with hypotenuse of 15 km which is the shortest distance from the library to Travis's home. One leg is the horizontal distance from home (17, 8) to the vertical line with x-coordinate of 3, which has a length of 14. Find the other leg using the Pythagorean property.

$a^2 = c^2 - b^2$

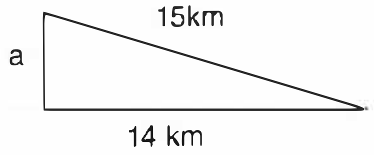
$a^2 = 15^2 - 142$

$a^2 = 225 - 196$

$a^2 = 29$

$a = \sqrt{29}$

$a = 5.385$



The library could either be below the horizontal line joining home at (17, 8) to the vertical line with x-coordinate of 3 or it could be above the line. If it is below the horizontal line the distance would be $6 - 5.4 = 0.6$ km. If above the horizontal line, the distance would be $6 + 5.4 = 11.4$ km. Since the farthest distance was requested, the answer is $6 + 5.4$ or 11.4 km.

34^{th} CALGARY JUNIOR HIGH SCHOOL MATHEMATICS CONTEST

April 28, 2010

NAME: SOLUTIONS ______ GENDER: ☐ M ☐ F

PLEASE PRINT (First name Last name)

SCHOOL: ______________________ GRADE: ______ (7,8,9)

- You have 90 minutes for the examination. The test has two parts: PART A — short answer; and PART B — long answer. The exam has 9 pages including this one.
- Each correct answer to PART A will score 5 points. You must put the answer in the space provided. No part marks are given.
- Each problem in PART B carries 9 points. You should show all your work. Some credit for each problem is based on the clarity and completeness of your answer. You should make it clear why the answer is correct. PART A has a total possible score of 45 points. PART B has a total possible score of 54 points.
- You are permitted the use of rough paper. Geometry instruments are not necessary. References including mathematical tables and formula sheets are **not** permitted. Simple calculators without programming or graphic capabilities **are allowed**. Diagrams are not drawn to scale. They are intended as visual hints only.
- When the teacher tells you to start work you should read all the problems and select those you have the best chance to do first. You should answer as many problems as possible, but you may not have time to answer all the problems.

MARKERS' USE ONLY	
PART A ×5	
B1	
B2	
B3	
B4	
B5	
B6	
TOTAL (max: 99)	

BE SURE TO MARK YOUR NAME AND SCHOOL AT THE TOP OF THIS PAGE.
THE EXAM HAS 9 PAGES INCLUDING THIS COVER PAGE.

Please return the entire exam to your supervising teacher at the end of 90 minutes.

PART A: SHORT ANSWER QUESTIONS

A1 The product of three different prime numbers is 42. What is the sum of the three prime numbers?

12

A2 Four athletes at the Olympic competitions are the only participants in each of eight events. For each event, three medals are awarded. Each of these four athletes wins the same number of medals. How many medals did each athlete win?

6

A3 Two sides of a triangle have lengths 5cm and 6cm. The area of the triangle is a positive integer. What is the maximum possible area of such a triangle, in cm^2?

15

A4 Rose has to write five tests for her class, where each test has a maximum possible score of 100. She averaged a score of 80 on her first four tests. What is the maximum possible average she can get on all five tests?

84

A5 Notice that $1 - 2 = -1$, $1 - (2 - 3) = 2$, and $1 - (2 - (3 - 4)) = -2$. What is

$$1 - (2 - (3 - (4 - \cdots - 100)))\ldots)?$$

-50

A6 The number

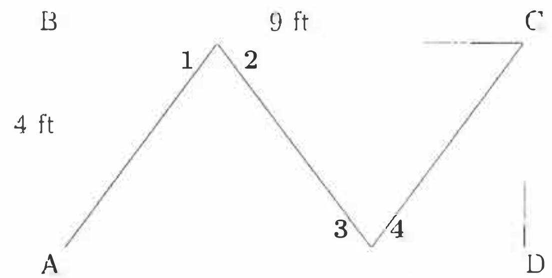
$$\frac{2^{100} + 2^{99} + 2^{98}}{14}$$

is equal to 2^n, for some positive integer n. Find n.

$\boxed{97}$

A7 A rectangular billiard table has dimensions 4 feet by 9 feet as shown. A ball is shot from A, bounces off BC so that angle **1** = angle **2**, bounces off AD so that angle **3** = angle **4** and ends up at C. What is the distance (in feet) that the ball traveled?

$\boxed{15}$



A8 Suppose that a is a certain real number so that $\frac{3x^2 - a}{x^2 + 2}$ is always the same number no matter what real number x is. What is a?

$\boxed{6}$

A9 The numbers $1, 2, 3, \cdots, 100$ are written in a row. We first remove the first number and every second number after that. With the remaining numbers, we again remove the first number and every second number after that. We repeat this process until one number remains. What is this number?

$\boxed{64}$

PART B: LONG ANSWER QUESTIONS

B1 In a video game, the goal is to collect coins and levels. A player's level is calculated by finding the number of digits of the number of coins he has collected. For example, if a player has 240 coins, then the player's level is 3, since 240 has 3 digits. Currently, Lario has 120 coins and Muigi has 9600 coins.

(a) (4 marks) What is Muigi's level? How many coins does Muigi need to collect to increase his level by 1?

Solution: Muigi has 9600 coins, which has four digits. Therefore, Muigi's level is <u>4</u>.

In order for Muigi to reach level 5, Muigi must collect $10000 - 9600 = 400$ coins to increase his level by 1.

(b) (5 marks) In their next game, Lario and Muigi each collect the same number of coins, and they end up at the same level. What is the smallest number of coins that Lario and Muigi could each have collected to accomplish this feat?

Solution: Lario needs $1000 - 120 = 880$ coins to increase his level by 1 to 4. After collecting 880 coins, Muigi will have $9600 + 880 = 10480$ coins. Hence, Muigi is at level 5. Therefore, for Lario and Muigi to end up at the same level, Lario needs to collect $10000 - 1000 = 9000$ more coins. After that, Muigi will have $10480 + 9000 = 19480$. Then both Lario and Muigi will be at the same level.

Therefore, Lario and Muigi need to collect $880 + 9000 = 9880$ coins.

B2 You are preparing skewers of meatballs, where each skewer has either 4 or 6 meatballs on it. Altogether you use 32 skewers and 150 meatballs. How many skewers have only 4 meatballs on them?

Solution 1: Suppose all of the skewers contain 4 meatballs. Then the number of meatballs used is $32 \times 4 = 128$ meatballs. Therefore, we need $150 - 128 = 22$ more meatballs. Every skewer with 4 meatballs that we change to a skewer with 6 meatballs increases the number of meatballs by 2. Therefore, in order to have 150 meatballs total, there are $22/2 = 11$ skewers with 6 meatballs. Therefore, there are $32 - 11 = 21$ skewers with 4 meatballs. The answer is 21.

Solution 2: Let x be the number of skewers with 4 meatballs each. Therefore, there are $32 - x$ skewers with 6 meatballs each. Since there are 150 meatballs total,

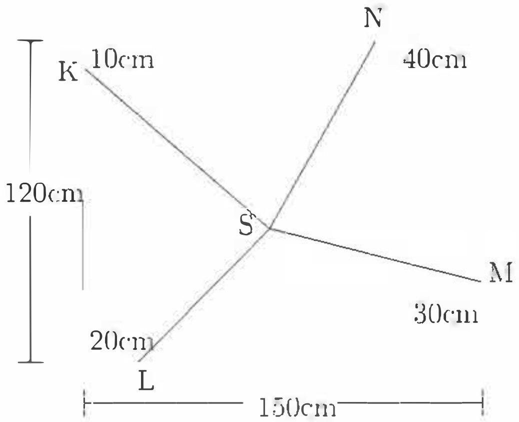
$$4x + 6(32 - x) = 150.$$

This simplifies to

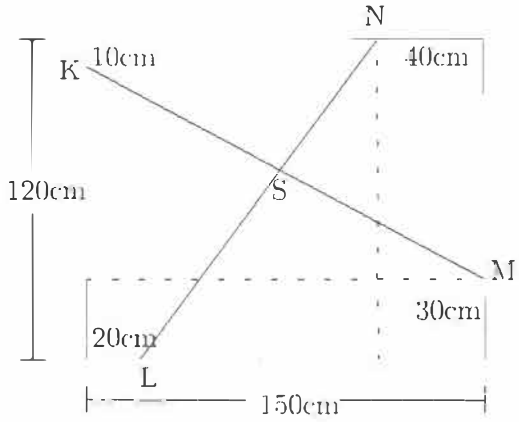
$$4x + 192 - 6x = 150 \quad \Rightarrow \quad 2x = 42.$$

Therefore, $x = 21$. The answer is 21.

B3 Khalid, Lesley, Mei and Noel are seated at 10 cm, 20 cm, 30 cm, and 40 cm, respectively, from the corners of a 120 cm by 150 cm dining table, as shown in the figure. If the salt, S, is placed so that the total distance $SK + SL + SM + SN$ is as small as possible, what is that total distance?



Solution: The sum $SK+SL+SM+SN$ is the smallest when K, S, M lie in a straight line and L, S, N lie in a straight line. In this case, $SK+SL+SM+SN = KM+LN$.



KM is the hypotenuse of a right-angled triangle with legs $120 - 10 - 30 = 80$cm and 150cm. Therefore, $KM = \sqrt{80^2 + 150^2} = 170$cm.

LN is the hypotenuse of a right-angled triangle with legs $150 - 20 - 40 = 90$cm and 120cm. Therefore, $LN = \sqrt{90^2 + 120^2} = 150$cm.

Therefore, $KM + LN = 320$cm. The answer is 320cm.

B4 ShelfCity makes shelves that hold five books each and ShelfWorld makes shelves that hold six books each.

(a) (3 marks) Jarno owns a certain number of books. It turns out that if he buys shelves from ShelfCity, he will need to buy 8 shelves to hold his books. List all of the possible numbers of books that Jarno can own.

Solution: Jarno needs more than seven shelves. Therefore, he owns at least $7 \times 5 + 1 = 36$ books. He also does not own more than $5 \times 8 = 40$ books. Therefore, the list of possible number of books that Jarno can own is $\{36, 37, 38, 39, 40\}$.

(b) (6 marks) Danny owns a certain number of books. It turns out that whether he buys shelves only from ShelfCity or buys shelves only from ShelfWorld, he will need to buy the same number of shelves. What is the largest number of books that Danny can own?

Solution 1: Note that Danny will always need at least as many shelves buying from ShelfCity than buying from ShelfWorld. If Danny owns the maximum number of books possible, then by buying books from ShelfCity, all of his shelves are full. Therefore, the answer is a multiple of 5. By trial and error, we see that if Danny owns 5, 10, 15, 20, 25, 30, 35, 40 books, then he needs to buy 1, 2, 3, 4, 5, 6, 7, 8 shelves from ShelfCity respectively and 1, 2, 3, 4, 5, 5, 6, 7 shelves from ShelfWorld, respectively. If Danny owns more than 25 books, then he will always need more shelves from ShelfCity than from ShelfWorld. Therefore, Danny owns at most 25 books.

Note: Other trial and error methods are possible and acceptable, i.e. a student does not necessarily have to considering only multiples of five.

Solution 2: Let s be the number of shelves that Danny needs from each store. Since he needs to buy s shelves from ShelfCity, the number of books he owns is from the set $\{5s - 4, 5s - 3, 5s - 2, 5s - 1, 5s\}$. He also needs to buy s shelves from ShelfWorld, so the number of books he owns is from the set $\{6s - 5, 6s - 4, 6s - 3, 6s - 2, 6s - 1, 6s\}$. The number of books he owns is in both lists, and therefore is at least $6s - 5$ and at most $5s$. Therefore, $6s - 5 \leq 5s$. Solving this yields $s \leq 5$. Since Danny needs at most five shelves from ShelfCity, the number of books Danny can own is at most $5s \leq 25$ books. If he does own 25 books, then for both companies, he will need to buy five shelves, i.e. the same number of shelves. Therefore, the largest number of books that Danny can own is 25.

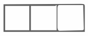
B5 Two players Seeka and Hida play a game called Hot And Cold on a row of squares. Hida starts by hiding a treasure at one of these squares. Seeka has to find out which square it is. On each of Seeka's turn, she picks a square.

• If Seeka picks the square which is where the treasure is, Hida will say "Ding!" and the game ends.

• If Seeka picks a square which is next to the square where the treasure is, Hida will say "Hot!".

• If Seeka picks a square which is not where the treasure is, and is not next to the square where the treasure is, Hida will say "Cold!".

(a) (3 marks) Suppose the game is played on three squares, as shown. Show how Seeka can pick the square with the treasure in at most two turns.



Solution: Label the squares 1, 2, 3 from left to right. Seeka first picks square 1. If it is where the treasure is, Seeka is done in one turn. If Hida says "Hot!", then Seeka knows the treasure is in square 2 and picks it on her second turn. If Hida says "Cold!", then Seeka knows the treasure is in square 3 and picks it on her second turn. In all cases, Seeka picks the square with the treasure in at most two turns.

Note: Seeka cannot accomplish this by picking the second square first.

(b) (6 marks) Suppose the game is played on nine squares, as shown. Show how Seeka can pick the square with the treasure in at most four turns.

Solution: Label the squares 1, 2, 3, 4, 5, 6, 7, 8, 9 from left to right. Seeka first picks square 5. If the square contains the treasure, Seeka is done in one turn. If Hida says "Hot!", then Seeka knows the treasure is in square 4 or 6. She picks these squares in her next two turns to guarantee finding the treasure, needing at most three turns total. If Hida says "Cold!", then the treasure is in square 1, 2, 3, 7, 8 or 9. Seeka on her second turn then picks square 2. If the square contains the treasure, Seeka found the treasure in two turns. If Hida says "Hot!", then the treasure is at square 1 or 3. Seeka picks these two squares to guarantee finding the treasure in at most four turns total. If Hida says "Cold!", then Seeka knows the treasure is at square 7, 8 or 9, with two turns remaining. We apply the strategy in (a) on these three squares to find the treasure in at most two turns. In all cases, Seeka finds the treasure in at most four turns.

Note: Other strategies are possible.

B6 Three people have identical pairs of shoes. At the end of a party, each person picks up a left and a right shoe, leaving with one shoe that is theirs and one shoe that belongs to someone else.

(a) (4 marks) In how many different ways could this happen?

Solution: Label the people 1, 2, 3 and the left and right shoes for person 1 be labeled L_1, R_1, those for person 2 be labeled L_2, R_2 and those for person 3 be labeled L_3 and R_3.

There are two choices as to which correct shoe person 1 picks, namely left or right. Person 1 then has two choices as to whose person's shoe he picks as the wrong shoe. Therefore, there are four choices of shoes for person 1, namely $(L_1, R_2), (L_1, R_3), (L_2, R_1), (L_3, R_1)$.

Suppose person 1 picks (L_1, R_2). Then person 2 must have picked L_2 (since R_2 already is in person 1's possession and person 2 must be holding one of his own shoes). If person 2 has R_1, then person 3 has L_3 and R_3, holding both of his shoes, which is not allowed. Therefore, person 2 is holding (L_2, R_3). Person 3 must then be holding (L_3, R_1). Similarly, no matter how person 1 picks the shoes, the choices for person 2 and 3 are determined.

Therefore, there are four different ways for three people to pick up a left and a right shoe leaving with one shoe that is theirs and one shoe that belongs to someone else.

(b) (5 marks) Same as part (a), except that there are four people instead of three.

Solution: We use the same labeling as in (a) and label the fourth person 4 and his left and right shoe L_4, R_4, respectively.

Consider person 1. There are two choices as to which correct shoe person 1 picks, namely left or right. There are three choices as to whose person's shoe he picks as the wrong shoe. Therefore, there are $2 \times 3 = 6$ choices of shoes for person 1, namely $(L_1, R_2), (L_1, R_3), (L_1, R_4), (L_2, R_1), (L_3, R_1), (L_4, R_1)$.

Suppose person 1 picks (L_1, R_2). Then person 2 must have picked L_2. We now consider the following cases for the right shoe that person 2 picked:

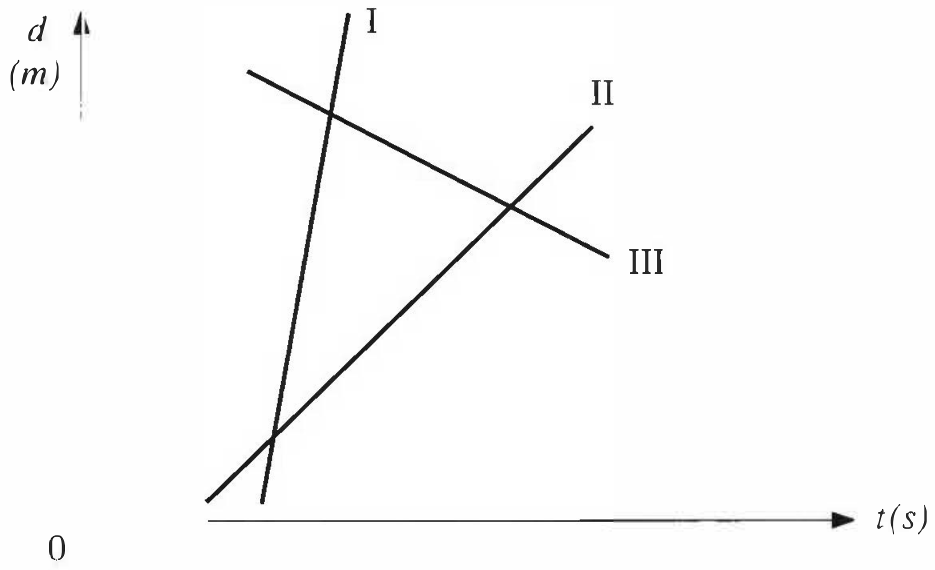
i. If person 2 is holding R_1, then either person 3 is holding (L_3, R_4) and person 4 is holding (L_4, R_3), or person 3 is holding (L_4, R_3) and person 4 is holding (L_3, R_4). Hence, there are two possibilities for this case.
ii. If person 2 is holding R_3, then person 3 must be holding L_3. Person 3 cannot be holding R_1, because then otherwise person 4 will be holding (L_4, R_4), which is both of his shoes. Therefore, person 3 is holding R_4. This leaves person 4 holding (L_4, R_1). This is the only possibility.
iii. If person 2 is holding R_4, then similar to case (ii), there is only one possibility.

Therefore, the total number of ways that all this can happen is $6 \times (2 + 1 + 1) = 24$ ways.

A Three-Speedy-Whales Problem

Gregory V Akulov

Once upon a time three whales were swimming along the north-south shoreline in the ocean near the island. Children at the island lighthouse observed the whales' motion and drew their position-time graphs in cartesian coordinates. They noticed that intersections of three graphs form an isosceles triangle (see figure below). If the first whale swam at 7 m/s [N] and the second one at 1 m/s [N], what is the velocity of the third whale?



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