Circles Are Squares

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We have been working together in a small rural school to improve our understanding using reform-based mathematics teaching strategies.

Last year, we tried a lesson in which we asked Grades 2 and 3 students to draw on 1 cm grid paper as many shapes as they could with an area of 16 cm^2, by counting the number of complete grid squares inside each shape. During this activity, one student became interested in trying to draw a circle with an area of 16 cm^2. Knowing that calculating or estimating the area of a circle is difficult, we were unsure of what to do with this sudden and unexpected interest.

Before long, most students were searching the classroom for objects with a circular face, trying to find one with an area close to 16 cm^2, and we observed significant mathematical thinking. For example, several students noted that small changes in the radius of the circle resulted in large changes in the area, which is an informal realization that area is a function of the square of a length (the relationship is quadratic, not linear).

After the lesson, we wondered whether we could create lessons to scaffold students' learning in order to help them acquire the skills to successfully determine the area of a circle. This year, we developed and implemented a series of inquiry-based lessons on area for Grades 2 and 3; the lessons were designed to reproduce the kinds of thinking we observed during the lesson described above. In this article, we describe those lessons.

Inquiry and Mathematics

Inquiry is a reform-based approach to teaching mathematics. It is premised on the idea that children can construct meaningful representations of mathematics, given a learning community that validates the thinking of all children (Cobb and Bauersfeld 1995).

We reject the idea that children must be told about mathematics. Rather, we wish to establish a learning environment in which students and the teacher work together to communicate ideas and use reasoning to jointly establish the validity of mathematical ideas. In order to foster individual and classroom mathematical engagement, mathematics educators often recommend the use of rich mathematical tasks. Such tasks are characterized as open-ended, with multiple possible entry points, trajectories and end points.[1]

Inquiry can be difficult to implement because of the tension between validating the thinking of all students and achieving the outcomes of the mathematics curriculum. The process for solving a rich mathematical task is not specified in advance; students must develop their own strategies, and teachers cannot control the activity (Smith, Bill and Hughes 2008). This lack of control means that teachers will make many decisions during an activity, trying to strike a balance between scaffolding student thinking and meeting broader mathematics teaching goals (such as curricular goals and the big ideas of mathematics).

The Mathematical Horizon

The mathematical horizon is a specific kind of broader mathematics teaching goal that teachers can attend to during inquiry lessons. Ball and Bass (2009, 5–6) define *horizon knowledge of mathematics* as "a sense of how the mathematics at play now [within a specific lesson] is related to larger mathematical ideas, structures, and principles." For example, knowing the relationship between area and radius for circles would be an example of horizon knowledge for lessons that explore finding the area of a circle by counting grid paper squares.

We agree with Ball and Bass (2009) that mathematics activities have greater potency when teachers attend to the mathematical horizon to inform their in-the-moment decision making. Thus, we have noted potential mathematical horizons for the lessons on area that we describe below. These should not be interpreted as the only mathematical horizons, teaching or curriculum goals, or learning goals that could guide teacher decision making.

Outline of Lessons

The overall mathematical horizon of this series of lessons is for students to develop a rich understanding of the meaning of area, including conservation of area and the informal precursors of area formulas.[2]

In this case, conservation of area is expressed by realizing that transforming squares (by moving or exchanging them, for example) does not change the area. We have found that many Grades 2 and 3 students need several experiences with conservation of area before it becomes meaningful. At this age, a child's apparent understanding of area conservation is not necessarily permanent and may be unstable.

We did not expect students to develop formulas for area. Rather, the intention was for students to experience the relationships evident in area formulas, and perhaps informally express those relationships. For example, for rectangles, area is a representation of multiplication. For circles, radius and area are not linearly related (rather, area is a function of the square of the radius).

The lessons outlined below are based on 40-minute periods. Each lesson begins on the learning carpet, a permanently gridded, 100-square floor carpet.[3] The focus is on exploring the notion of area, not formulas for calculating area, where the unit of area would be learning carpet squares.[4]

The mathematical horizon given for each lesson is intended to incrementally build toward the overall horizon described above. We also briefly describe the task and potential scaffolds to indicate some of the teacher decision making needed to ensure that the lessons remain inquiries, rather than devolving into traditional lessons where the teacher tells the students what to do. Possible scaffolds are questions that the teacher could ask students to restart their thinking when it appears to be stalled, when they are seeking teacher guidance, or when they appear to be frustrated with the task.

It should be expected that these lessons will play out differently when implemented, and teachers should be prepared to adapt in the moment.[5] We describe the lessons in further detail in a later section, where we also note specific events that occurred during our implementation.

Lesson 1

Mathematical Horizon

- Introducing area as covering

Task

- Make several swimming pools (rectangles) with 16 paper squares, so that the area of each pool is 16 squares.

Possible Scaffolds

- Should we remove squares or get extra squares?
- Do the various pool shapes still have the same area?

Materials

- Learning carpet
- 16 paper squares, plus extras (each paper square should be congruent to a square on the learning carpet)
- 1 cm grid paper (at least one sheet per student)

Lesson 2

Mathematical Horizon

- Experiencing area conservation

Task

- Make a triangle with an area of 16 paper squares.

Possible Scaffolds

- Is the shape really a triangle?
- Would it help to find area by cutting a paper square in half?
- What is the area of two of these half pieces of paper?

Materials

- Learning carpet
- 16 paper squares, plus extras
- Scissors
- 1 cm grid paper (at least one sheet per student)

Lesson 3

Mathematical Horizon

- Further experiencing area conservation

Task

- Make interesting shapes with an area of 16 paper squares.

Possible Scaffolds

- What about using pieces of paper of other sizes?
- Which pieces together make a square?

Materials

- Learning carpet
- 16 paper squares, plus extras
- Extra half pieces and precut pieces of various sizes (see Figures 3 and 4)
- "Paired Shapes" handouts (see the appendix for an example)
- Scissors

Lesson 4

Mathematical Horizon

- Consolidating area conservation
- Estimating area

Task

- Find the area of a shape bounded by string.

Possible Scaffolds

- When do pairs of pieces form a square?
- What about this little gap or overhang?

Materials

- Learning carpet
- 16 paper squares, plus extras
- Precut half pieces, one-quarter pieces, three-quarter pieces, one-sixth pieces, three-sixth pieces and five-sixth pieces
- Scissors
- String about 4 m long[6]

Lesson 5

Mathematical Horizon

- Estimating area

Task

- Find a circle with an area close to 16 cm^2.

Possible Scaffolds

- Is this an accurate circle?
- What about this gap or overlap?

Materials

- Learning carpet
- Plastic hoop
- Various objects with a circular base with a radius of up to 6 cm (mugs, plastic cups, cylinders, tape rolls, pop cans, bottle caps, Tupperware containers and lids, yogurt containers)[7]
- 1 cm grid paper

Detailed Lesson Descriptions

Lesson 1

Our first lesson was an introduction to area. On the learning carpet, we formed a rectangle with an area of 16 squares by placing 16 paper squares (each congruent to a learning carpet square) in two rows of eight (see Figure 1).

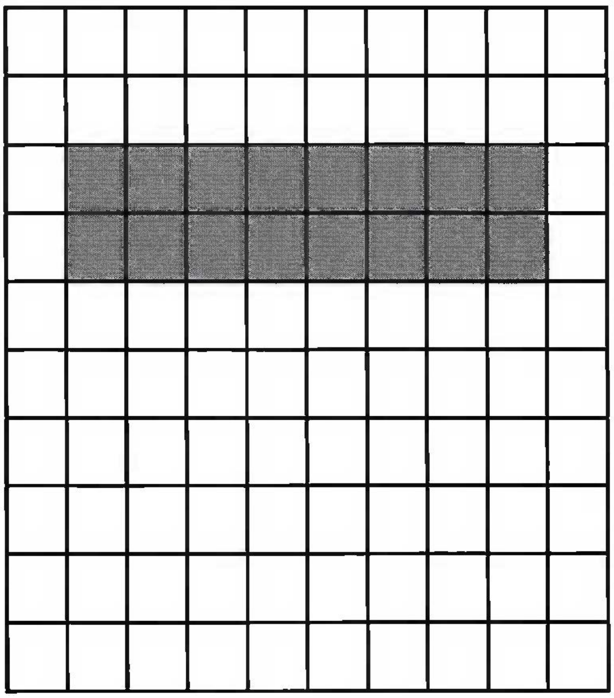
We asked the students, "If this rectangle is a pool, and we have to build a cover for the pool, how many squares big will that cover have to be?" After all the students had successfully counted the squares and mouthed the number to us,[8] we introduced the term *area*. For the rest of the lesson, we always used the terms *area* and *covering* to reinforce this new terminology.

We then began an inquiry into the core task of this lesson—making as many rectangular pools as possible with an area of 16 squares (using the 16 paper squares). Extra squares were in view of students, and we told them that they could have more squares or take squares away. One by one, students were selected to create new pools with the 16 paper squares, and with each new shape all students were asked to determine and then mouth the area.

As this was repeated several times, some of the students began to recognize that the area was unchanged, provided that no squares were added or removed. Other students continued to count squares, so we prompted them with the question, "Do we need any of these extra squares?" To provoke a deeper understanding, we said, "But it looks different. Are you sure it still has an area of 16 squares?" At first this question led students to recount the squares. However, later some students responded, "There are still the same amount of squares. I just moved them around," which indicated their recognition that the area was conserved.

To consolidate the lesson, we gave the students 1 cm grid paper (one piece for each student) and asked them to draw as many pool shapes as they could with an area of 16 squares (that is, square centimetres). We deliberately chose the word *shapes* rather than *rectangles* to allow for other possibilities, thereby maintaining an open-ended task, even though our intent was to consolidate the lesson.

Figure 1



Lesson 2

We began Lesson 2 by placing on the learning carpet a 4×4 rectangle with an area of 16 squares. We then asked students to determine and mouth the area. At this point, the students seemed comfortable with the term *area,* so we dropped the pool covering language. We then asked them for an explanation. They helped each other accurately count the number of squares. We repeated this task with one more block-like shape (a composite of two 2×4 rectangles in the shape of an L) to ensure an adequate review of area.

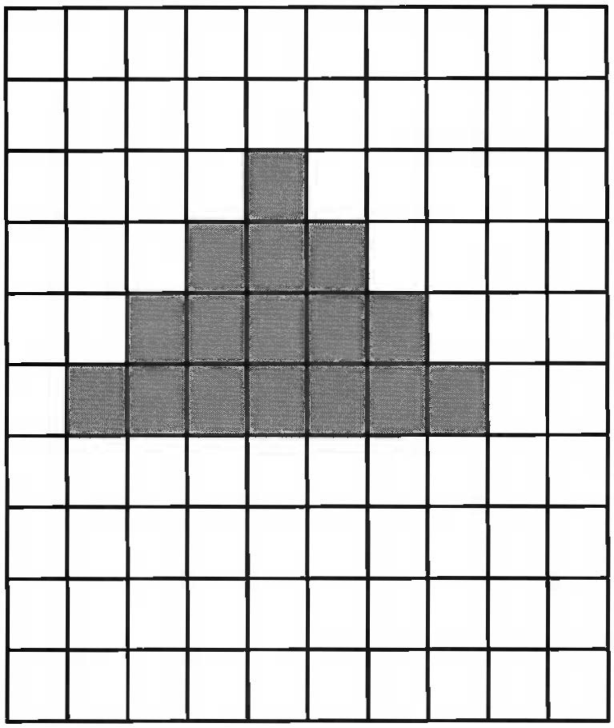
Then we began our next inquiry: "Using 16 paper squares, can we make a triangle with an area of 16 squares?" At first the students manipulated whole paper squares to create a stepped triangle-like shape (see Figure 2). We asked, "Is this really a triangle? Can we make the sides smooth?" Then, as hoped, one student suggested cutting a paper square. (It should be noted that throughout our questioning, scissors were placed near the learning carpet and were visible to students, but we did not refer to them.)

Upon cutting a paper square in half along its diagonal, the students placed both pieces back together on the learning carpet. When questioned about the area, they assured us that the shape still had an area of 16 squares. After moving the cut pieces into different locations, the students realized that they could cut more paper squares in half to make more of the "slanting part of a triangle."

As the students placed these half pieces in various locations, not all of them were assured that the area remained 16 squares. Those students counted pieces rather than whole squares (in other words, they counted a half square as one), and decided that the area was more than 16 squares. We resisted the urge to tell them what they were doing wrong. Instead, we acted unsure about what the area really was and asked, pointing to a half piece, "What is the area of one of these pieces?" We allowed students to move half pieces back together and then count whole squares to determine the area. Later we encouraged students to point to a pair that was counted as one. At this point, they discussed how to keep track of which pieces had already been counted, to avoid counting a piece more than once or not at all.

To consolidate, we asked students to draw on 1 cm grid paper at least two more shapes with an area of 16 cm^2—one that they had seen on the learning carpet and one new one. They used half squares in many of their new shapes.

Figure 2



Lesson 3

Lesson 3 began with a familiar shape on the learning carpet. The shape, which had an area of 16 squares and used half pieces, looked like a house (a 3×4 rectangle topped with a triangle made of four half pieces and two whole squares). The shape was familiar because it had been formed on the learning carpet in the previous lesson, and many students had also drawn it on the 1 cm grid paper.

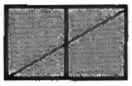
We reviewed what area is (that is, covering) and then calculated the area of the shape, remembering to match half pieces to create one full square. We motivated a new inquiry by stating, "This shape is boring. What other interesting shapes can we make?" Students took turns making new shapes, sometimes with help from others, and named them based on what they looked like (for example, rocket, tree and house). With each new shape, we determined its area, ensuring that it was still 16 squares.

Then, we again asked, "Can we make a more interesting shape?" The number of shapes that could be made with only whole and half squares was limited, so we were hoping that students would suggest using pieces of different sizes. Again, scissors were in view of students but were not referred to. Also in view were precut half pieces; one-quarter and three-quarter pieces; and one-sixth, three-sixth and five-sixth pieces.

To make one-quarter and three-quarter pieces, place two paper squares side by side, and cut from one corner to the opposite corner (see Figure 3). This produces four pieces—two small congruent pieces and two large congruent pieces. We will call these small and large pieces one-quarter pieces and three-quarter pieces, respectively.

Figure 3

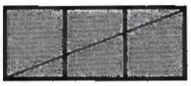
How to Cut One-Quarter and Three-Quarter Pieces



To make one-sixth, three-sixth and five-sixth pieces, place three paper squares side by side, and cut from one corner to the opposite corner (see Figure 4). This produces six pieces—two small congruent pieces, two medium congruent pieces and two large congruent pieces. We will call these small, medium and large pieces one-sixth, three-sixth and five-sixth pieces, respectively.[9]

Figure 4

How to Cut One-Sixth, Three-Sixth and Five-Sixth Pieces



Students started replacing full squares with two matching pieces that were not halves. At first, each time we replaced one full square with a new pair, we asked a student to create a new shape and then determine if the area was still 16 squares. This was repeated several times. Later, students wanted to try multiple replacements. We watched to make sure that they were pairing the pieces correctly (a pair is congruent to one whole square). For some students, the pairing was less obvious when the pairs were far apart, and they had to physically move the pieces together to confirm that the shape still had an area of 16 squares.

To consolidate their learning, students completed three "Paired Shapes" handouts (Appendix A is the first page). On these worksheets, students were asked to determine which shape in each pair had an area closest to 16 squares.

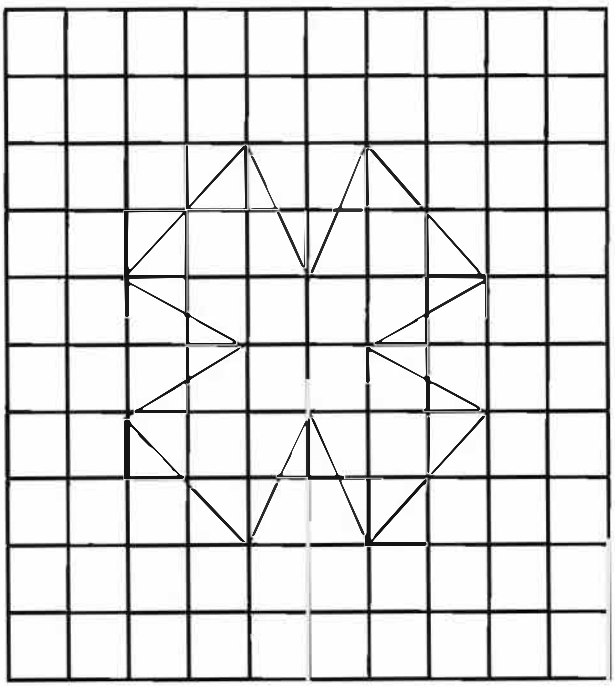
Lesson 4

Lesson 4 began with a 2 × 8 rectangle, formed on the learning carpet with paper squares and string along the perimeter. We asked the students to mouth the area of the rectangle. Then we removed the squares, leaving the string in place, and again asked the students to mouth the area. We repeated this review of area with two more shapes: an isosceles triangle with a base of 8 and a height of 4 (12 whole squares and 8 half pieces), and a right-angle triangle with a base of 4 and a height of 8 (12 whole squares, 4 one-quarter pieces and 4 three-quarter pieces). We reviewed these three basic shapes with an area of 16 squares to ensure that students were ready for the following task, which we prompted by asking, "What is the area of this four-leafed clover?" With string, we traced out the shape shown in Figure 5.[10]

Students worked together to fill in the shape, and we refrained from commenting except to encourage them to check their thinking. Finally, we began our inquiry: "Using the string, make any shape you want, provided that it is interesting in some way. The area does not have to be 16 squares. What do you think its area is?" As students tried to fill in their shapes with the available pieces (squares, halves, quarters and so on), they soon realized that there would be gaps smaller than any of the pieces. No students cut new shapes, which we would have allowed. Rather, they added more one-quarter pieces, trying to balance the remaining gaps (inside the shape) with the overhangs (the pieces going outside of the string).

After this activity, we were confident that most students were comfortable with area conservation and estimation of area. But we could have added another lesson similar to Lesson 4, to make sure that students were ready for Lesson 5.

Figure 5



Lesson 5

Our fifth and final lesson began with a plastic hoop on the learning carpet. We asked, "What do you think is the area of this circle?" Students used an approach similar to that used in Lesson 4 to cover the circle with paper pieces and estimate its area, by balancing gaps with overhangs. We were ready to repeat this review with a second plastic hoop if students did not appear ready for the subsequent inquiry.

We then challenged students to draw on 1 cm grid paper a circle with an area of 16 cm^2. Students were put into groups and given an assortment of objects with circular faces. The two main concerns that arose were related to drawing an accurate circle and accurately determining its area. For example, some students did not trace the circular face accurately, which called into question whether it was a candidate for having an area of 16 cm^2. At other times, students were not careful when matching gaps with overlaps when estimating the area. We saw both of these issues as opportunities rather than problems, as they gave students a chance to discuss the importance of accuracy in mathematics while also recognizing the need to estimate and improve estimates.

Final Thoughts and Considerations

The series of lessons above was triggered by an unexpected exploration of circles, which we initially did not know how to respond to. After implementing the lessons, we no longer wondered if exploring circles would be a waste of time. We had generated a series of inquiries leading to a productive exploration of the area of circles. Throughout, we attended to our mathematical horizon while still respecting the contributions and thinking of all students.

We have tried these activities several times (with some modifications and with different groups of children), and we have seen varying results. These differences can be navigated by teacher scaffolds that maintain student momentum for the inquiry while keeping the mathematical horizon in sight. For example, sometimes children will suggest cutting the shapes, while on other occasions a prompt from the teacher is needed ("What if we use scissors?"). The decision to cut leads to questions about how to cut (diagonally, lengthwise), which may require a teacher prompt such as "Can we cut to make two triangles?" Our rule of thumb is to be patient, building on what children do and say, rather than telling them what to do. We allow students to wind their way through an inquiry, and we provide a prompt only if they get stuck or if they are moving too far astray of our mathematical horizon.

Math classes are usually 40 minutes long. We have been surprised at how long students are willing to stay focused on these activities. We thought they would stay engaged for a much shorter time, leaving time for regular math class routines. On the contrary, each activity fills the entire math class, and sometimes goes over.

When we see student engagement waning, we wrap up the inquiry with a consolidation activity. This always occurs after significant mathematical thinking has been demonstrated by most students.

The greatest difficulty we have found is getting to all students so that they can share their thinking. To address this concern, we move back and forth from individual or small-group exploration to whole-class discussion. This teaching method provides just enough structure so that all students can participate and stay engaged, but is also flexible enough to take into account the diversity of thinking that can occur during an inquiry.

Why did we choose an area of 16 squares? Selecting this number was an inquiry in itself for us as teachers. After exploring various areas (12, 15 and 16), we decided that an area of 16 squares allows for generating many different shapes. For example, there are whole-number-dimension rectangles with an area of 16, as well as several triangles. Using an area of 12 or 15 would allow for only rectangular shapes, which would provide less opportunity to scaffold into half and other part pieces. That opportunity is critical to the success of the activities; it provides the openness needed for initiating and sustaining an inquiry, because there are more opportunities for mathematical thinking by students.

Beyond the specific decision to consider shapes with an area of 16 squares, planning the inquiries described above was much different from planning for a traditional, direct-instruction lesson. Our main goal was to explore area, rather than to target specific outcomes. It was only after the lessons that we recognized some of the specific outcomes and processes that had been met. This outcome openness fosters the flexibility needed for an inquiry to be successful. We also tried to anticipate several possible trajectories of student thinking, rather than creating a linear description of what should happen. This allowed us to stay flexible while also being ready with several possible scaffolds to keep the inquiry going. In a way, planning for an inquiry has in itself the elements of an inquiry.

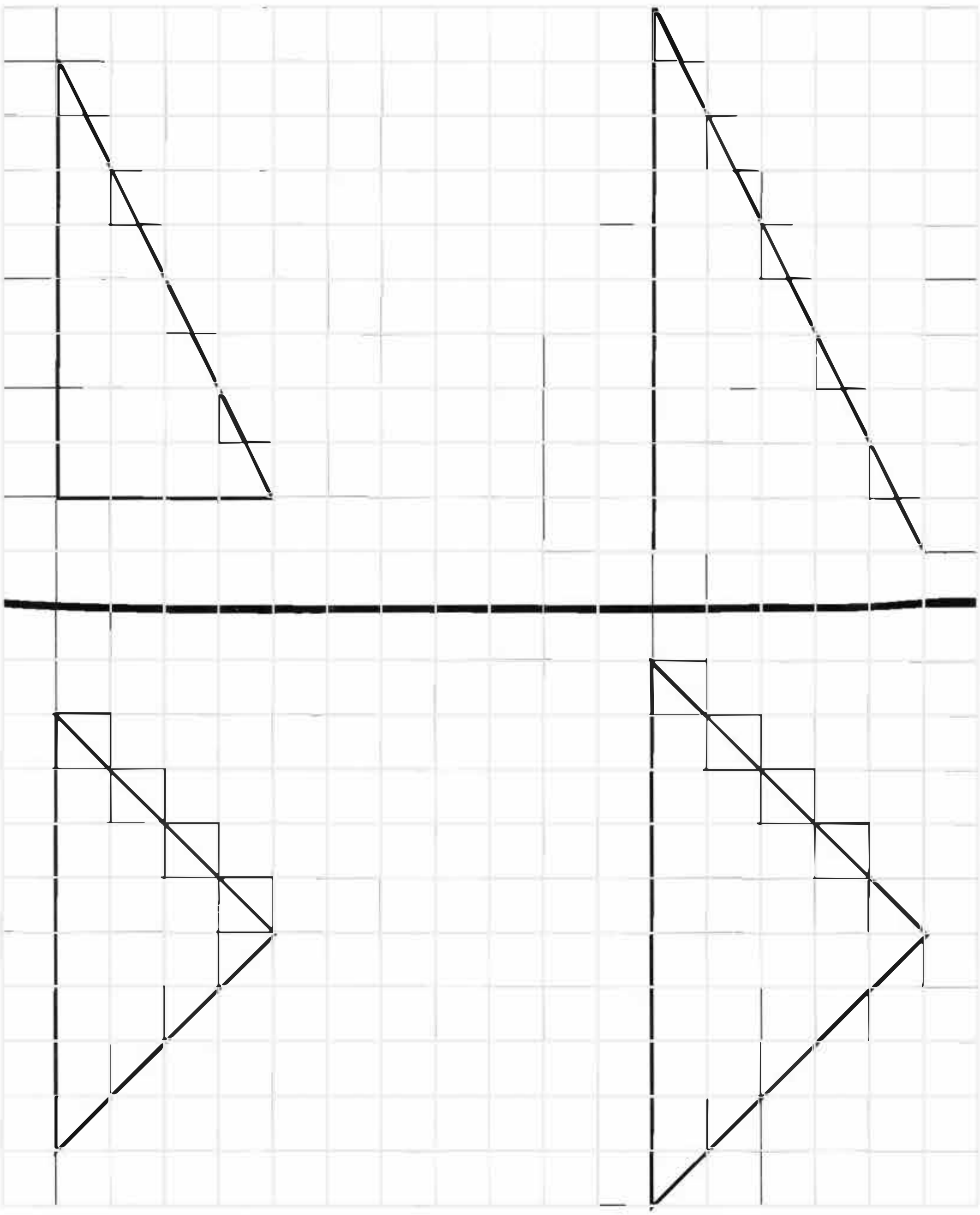
Unfortunately, it is impossible to anticipate everything that can happen. For example, we did not expect students to struggle with matching pairs to form a square: some students could not match pairs when the pieces were far apart, even though they could do so when the pieces were adjacent. We still are not sure why this would be the case. Further, we did not achieve all of our goals explicitly. For example, in Lesson 5, no students formally and explicitly noted that the area of a circle can change a lot when the radius changes only a little.

Nevertheless, we believe that significant mathematical thinking occurred during these inquiries, and we have laid an informal groundwork for lessons in later grades concerning area formulas.

Appendix A

Paired Shapes Handout

In pairs, which shape is closer to 16?



Notes

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1. For two of many examples of rich mathematical tasks that promote inquiry, see Smith et al (2009) and Mooney (2008).

2. There are other, larger mathematical concepts that teachers could attend to during these lessons. For example, the idea of leaving no gaps or overlaps when covering an area is related to tessellation. In Lesson 4, connections between perimeter and area are possible.

3. The Learning Carpet can be ordered at www.thelearningcarpet.ca. Alternatively, use masking tape to create a 10×10 grid on a tile floor and cut paper squares of the same size as the tiles.

4. Throughout this article, unless otherwise noted, the unit for all areas is learning carpet squares (and is noted as square units or squares), and one unit of length is the side of a learning carpet square.

5. Smith, Bill and Hughes (2008) provide a framework for planning for inquiry lessons, which is premised on the need to anticipate possible student trajectories before implementing a lesson.

6. The string must be about 4 m long to ensure that there is enough string to surround various shapes with an area of 16 squares. For example, the 2×8 rectangle has an area of 16 squares and a perimeter equal to 20 sides of a square, but a square on a learning carpet has a side length of about 18 cm, so the rectangle's perimeter is 20×18 cm, or 360 cm. It is possible to make extremely irregular shapes with an area of 16 squares and with much larger perimeters, but with such shapes it is difficult to confirm that the area is 16 squares.

7. We wanted to make sure that circles of many sizes, with areas less than and greater than 16 cm^2, were available to students. On 1 cm grid paper, a circle with an area of 16 cm^2 will have a radius of approximately 2.26 cm.

8. Trevor Calkins (creator of the Power of Ten system for learning numeracy skills) recommends this questioning method to encourage all students to safely participate in whole-class discussions. When the teacher asks a question, rather than having students raise their hands or verbalize an answer, the teacher watches for each student to silently mouth the answer.

9. During the lesson, students did not use a consistent language to refer to these pieces. We chose fraction terminology for this article to avoid confusion and because the name of each piece is its area. This terminology was not used with students, as it would have confused them and detracted from the goals of the lesson.

10. We deliberately chose a shape with an area of 16 squares that hadn't arisen in previous lessons and that could be filled in only by using pieces of more than one size (in this case, half pieces, one-quarter pieces and three-quarter pieces).

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