Similar Triangles Versus Trigonometry

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Similar triangles are included in the shape and space strand of Alberta's Grade 9 mathematics curriculum. This provides a good footing for introducing trigonometric ratios in subsequent courses, because similar triangles familiarize students with the idea of using ratios of side lengths in a triangle. However, the premise of this article is that similar triangles provide a rich vein of mathematics when revisited from other parts of the math curriculum (in particular, Mathematics 20-1 and 30-1, and Pure Mathematics 20 and 30)—in effect, when viewed through the rear-view mirror.

This article looks at two distinct issues: (1) student approaches when using trigonometry to develop the equivalent ratios used in similar-triangle problems, and (2) the range of algebra questions that can be developed in similar-triangle questions.

Similarity by Trigonometry

The question of how similar-triangle ratios can be developed using trigonometry is fascinating—not because of the answer but, rather, because of the variety of ways the problem can be addressed. Thus, the focus here is not the answer; it is how students can explore and navigate this open-ended problem. (The answer will be provided simply to remove the focus from it. In a class, the answer would not be revealed, in the hopes that students would discover it themselves.) This topic is suited to the Mathematics 20-1 course, where the sine law is taught and students are familiar with trigonometric ratios.

A Simple Answer

Students usually perceive the problem as difficult and, in an effort to simplify it, avoid the sine law. The tendency is to start with right-angle trigonometry; however, this strategy complicates the problem, because one then needs a right angle. In fact, using the sine law is the simplest approach (using Figure 1 for clarity).

$$\frac{\sin(A)}{a}=\frac{\sin(B)}{b} \qquad \frac{\sin(A)}{c}=\frac{\sin(B)}{d}$$

$$\frac{\sin(A)}{\sin(B)}=\frac{a}{b} \qquad \frac{\sin(A)}{\sin(B)}=\frac{c}{d}$$

$$\frac{a}{b}=\frac{c}{d}$$

Why do students avoid this approach? They have difficulty accepting that they can derive useful conclusions from working with angles that have unknown values. In effect, mathematics teaching has led many students to believe that only the explicit information given to them is relevant in solving any given problem.

Use of the sine law could be encouraged by giving students a value for an angle. For instance, "I know that this problem could be done if angle A were equal to 10°." Students could then write expressions that involve that angle, and that might lead them toward the sine law. Also, the teacher could then ask students to consider what they could calculate for each triangle, a different question that would help consolidate the utility of the sine and cosine laws.

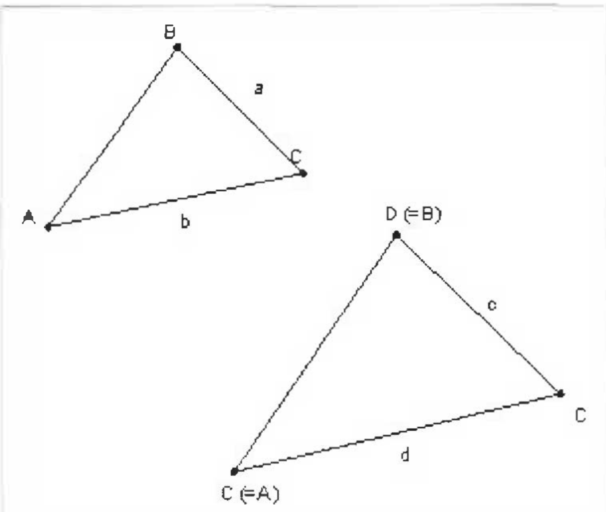
A Less Obvious Solution

Students try to keep the problem simple by using right-angle trigonometry, but that can only be done by constructing equivalent problems. The term *equivalent* here means that the given information is maintained and the information that is not specified can be changed in a manner that maintains the similarity.

One idea is to rotate a side about a point until a right angle is created; however, to keep the two triangles similar, the same degree of rotation must be

Figure 1

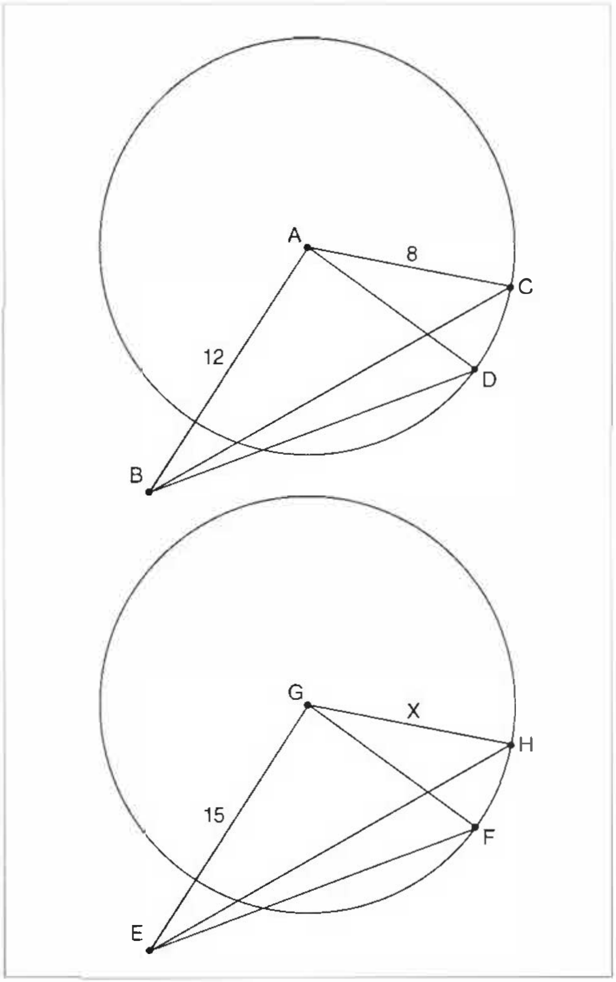
Similar Triangles



performed on the equivalent side around the equivalent point in the similar triangle (see Figure 2). The original question uses ΔBAC ~ ΔEGH, but ΔBAD ~ ΔEGF by rotating both AC and GH to AD and GF, respectively. This process preserves the lengths given in the original question but allows ∠BAD to be changed to any angle. There is a catch: changing ∠BAD to 90° changes both of the other angles, ∠ADB and ∠DBA, and that raises the question of how one can be sure that ∠ADB is equal to ∠GFE. For students, this highlights the need to provide some details.

Figure 2

Making an Equivalent Similar-Triangle Question



When the transformation is done, and ∠BAD = ∠EGF = 90°, students can use right-angle trigonometry to solve for GF, which has the same length as GH and *x*.

$$\tan(\angle\text{ABD}) = \frac{8}{12} = \tan(\angle\text{GEF}) = \frac{x}{15}$$

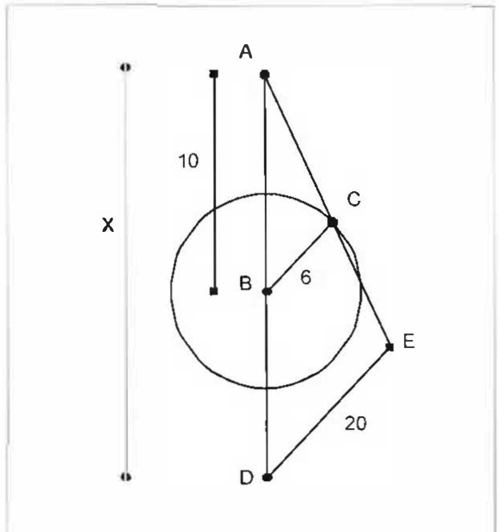
Transforming the question to an equivalent one that can be solved raises many doubts among students. This approach is not as straightforward as using the sine law, but it does arrive at a solution. Students have difficulty recognizing that changing the question is a legitimate approach, as long as the given information is preserved.

Another Approach

How can similar-triangle ratios be developed from trigonometry using the diagram in Figure 3?

Figure 3

Similar-Triangle Question



When I use this type of question, I present the problem without the circle. Discussion about how to solve this type of question leads to the addition of the circle, which promotes discussion about moving point C in a manner that allows ∠ABC to be changed. This is analogous to what was done in the previous section. However, this approach has an added benefit: keeping DE parallel to BC is a simple way of articulating the condition for equivalence of questions.

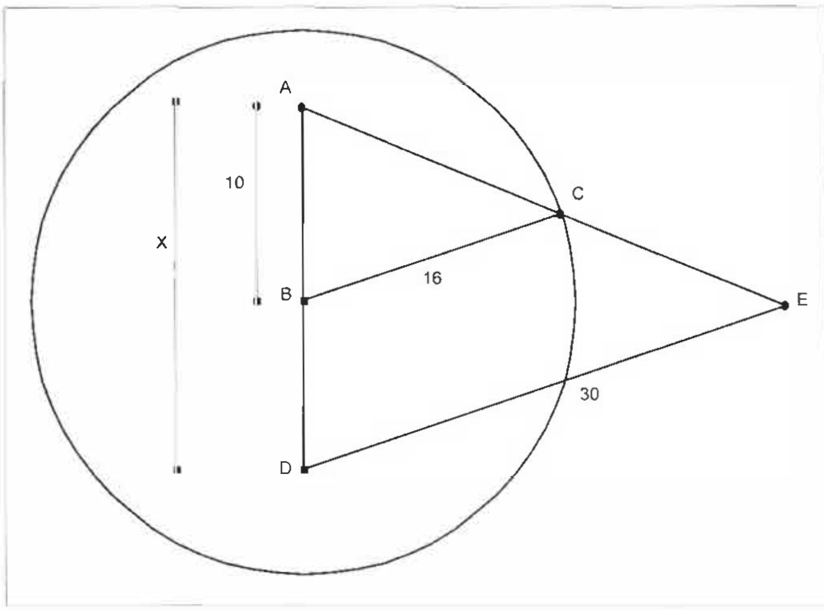
Students sometimes do the unexpected, especially when they brainstorm in small groups. While several groups will follow up on the idea of changing ∠ABC by moving point C, some groups will interpret this as rotating the line segment AE about point A. The latter approach leads to the idea that there is a maximum possible angle for A. It is then opportune to mention the concept of the tangent line and to point out that a tangent line, because of symmetry, has to be perpendicular to the radius line. Students then realize that they can move point C to the tangent point, keeping DE parallel to BC, and create a right angle at ∠ACB (and the corresponding angle, ∠AED).

This approach works only if a tangent line can be created. In one class, my students realized that with C as a tangent point and ∠ACB as 90°, they could use the sine law to determine the maximum value of angle A. They then argued that they could use the angle and sine ratio to determine the unknown length x. While I was impressed that they had come up with this solution, I felt obliged to point out that the model question (Figure 3) had BC shorter than AB, and that there could be a problem if BC were longer than AB (see Figure 4). Students then wrestled with the idea that moving point C would never make a tangent line and that there was no maximum value of angle A. In this case, their argument did not work.

Does one actually need the right angle? I suggest to students that the process shows that they can make angle A any value they choose, up to the maximum found in the earlier version of the question. If they can make angle A any value, can they solve the problem? For example, suppose angle A is 20°. Does this help solve the problem? Unfortunately, in my class, the students who were modifying ∠ABC overheard this and argued that they could make ∠ABC a specific value, allowing them to use the cosine law to find AC. This is an awkward, roundabout solution that can lead to x, but it has enough steps that the students did not get to the answer.

I was intrigued by my students' efforts and concluded that this type of question is invaluable. I had not anticipated the variety of approaches, and I had to think on my feet more than once. However, this article consolidates the major approaches students have used, helping teachers to better anticipate what may arise. It is particularly intriguing to observe what happens when the information given in the question is not sufficient. Few students recognize that they can impose new information, provided that it does not contradict the given information.

Figure 4
Modified Question with BC > AB



A Different Type of Question

The connection between trigonometry and similar triangles made me question if there are similar-triangle questions with pedagogical value that have been ignored because of the view that similar triangles are a stepping stone into trigonometry. This issue is exacerbated by a bias in teaching materials, where similar-triangle resources typically use numbers for side lengths (with three given and one unknown) and often include the same question formats.

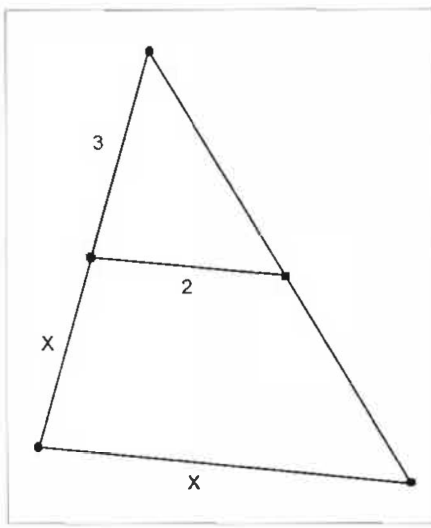
In a moment of inspiration, I formed a similar-triangle question in which the unknown, x, appeared twice. See Figure 5.

The problem is resolvable for 20-1 students as follows:

$$\frac{x+3}{3} = \frac{x}{2}$$
$$2(x+3) = 3x$$
$$2x + 6 = 3x$$
$$x = 6$$

This demonstrates a novel context for practising algebra.

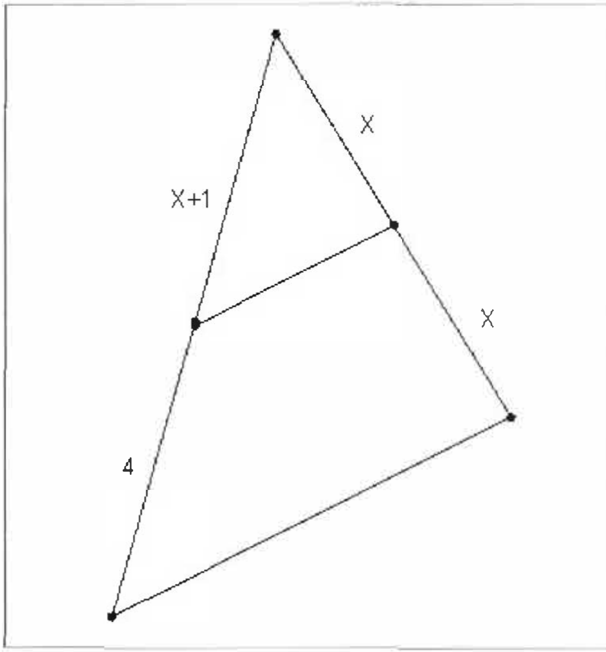
Figure 5
A Different Kind of Question?



Using other features of algebra is feasible. Consider the problem shown in Figure 6 and the following two methods of solving it.

Figure 6

Similar Triangle Requiring Algebra



The first method uses cancellation of the x values (since $x > 0$) on the left side of the equation:

$$\frac{2x}{x} = \frac{x+5}{x+1}$$
$$2 = \frac{x+5}{x+1}$$
$$2x + 2 = x + 5$$
$$x = 3$$

Another solution uses common factoring:

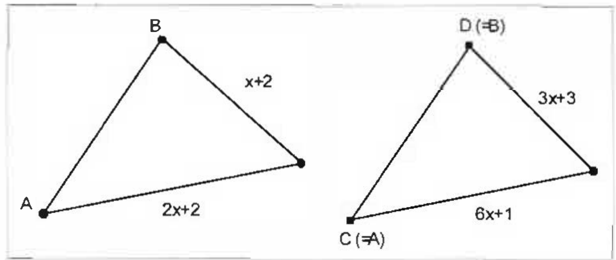
$$\frac{2x}{x} = \frac{x+5}{x+1}$$
$$2x(x+1) = x(x+5)$$
$$2x^2 + 2x = x^2 + 5x$$
$$x^2 = 3x$$
$$x(x-3) = 0$$

This could be used as a point of discussion for the domain; note that x cannot be 0 if this question is about a triangle.

It is also possible to have questions where quadratic terms arise in a manner in which they cancel. This assumes that students know how to multiply two binomials.

Figure 7

Example Requiring Algebra



Consider the example in Figure 7, which has the following solution.

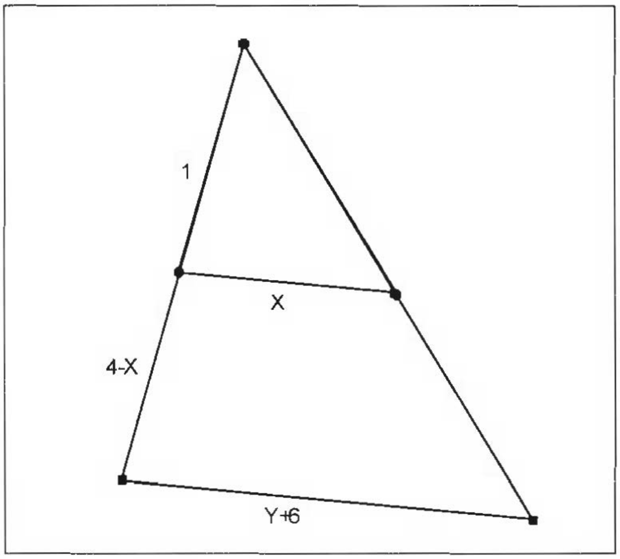
$$\frac{x+2}{3x+3} = \frac{2x+2}{6x+1}$$
$$6x^2 + 13x + 2 = 6x^2 + 12x + 6$$
$$13x + 2 = 12x + 6$$
$$x = 4$$

The general solution for this type of problem can be thought of geometrically as the intersection of two quadratic functions (as in the second line of the previous solution, considering each side to represent an independent quadratic). However, the general solution is not particularly insightful for designing questions, because it is beset by conditions representing the following requirements: all triangle sides must have positive lengths, and the problem must have a unique solution. There are cases with multiple solutions. Consider two equilateral triangles—one with all sides of length x, and the other with all sides of length $2x$. In this case, any positive value of x will suffice.

Last, similar triangles like this could be used to develop further types of questions, such as those requiring quadratic factoring with two unique positive solutions (in Figure 8, set $y = 0$ and solve for x). While the earlier questions have merit because of the connections to algebra and equation solving, casting problems of this type for factoring would be contrived. However, this type of problem, with the y in place, could be useful in higher mathematics classes to address domain and range issues. Consider the scenario in Figure 8, where the task is to determine the conditions for x and y so that the problem can be solved.

The relationship between x and y is quadratic ($y = -x^2 + 5x - 6$), and the restriction that side lengths must be positive provides restrictions for the domain and range. The domain is $0 < x < 4$, so that side lengths x and $4 - x$ are both positive. Over this domain the quadratic reaches a maximum at $x = 2.5$ and $y = 0.25$. The minimum value over the domain occurs as x approaches 0 and y approaches -6. So the range is $-6 < y < 0.25$. Note that this type of domain and range

Figure 8
Domain and Range Question



question can also be formed with linear functions, for instance by changing $4 - x$ to 4.

Concluding Remarks

This article highlights opportunities to use the concept of similar triangles for more than simply introducing trigonometry. The questions are designed for making connections between mathematical concepts that are often viewed as distinct. These connections lead to a deeper appreciation of the conceptual interconnections surrounding similar triangles.

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