Alberta High School Mathematics Competition

Report on the First Round of the 52nd Contest

Andy Liu

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Individual Results

The first part of the 52nd Alberta High School Mathematics Competition was written on November 20, 2007, by 712 students—282 girls and 430 boys. The numbers of students in Grades 8, 9, 10, 11 and 12 are respectively 2, 35, 164, 195 and 316.

The top-scoring students are listed below. Unless otherwise indicated, the student was in Grade 12 at the time of the exam.

Rank	Score	Name	School
1	85	Frank Yang	Sir Winston Churchill High School, Calgary
		Linda Zhang	Western Canada High School, Calgary
3	84	Hunter Spink	Calgary Science School, Calgary (Grade 9)
4	80	Danny Shi	Sir Winston Churchill High School, Calgary (Grade 11)
		Jarno Sun	Western Canada High School, Calgary (Grade 11)
		Chong Shen	Sir Winston Churchill High School, Calgary
7	78	Karl Qin	Queen Elizabeth Jr/Sr High School, Calgary (Grade 11)
		Lucille Lu	Western Canada High School, Calgary (Grade 11)
		Zoe Cheung	J G Diefenbaker High School, Calgary
		Alex Chen	Sir Winston Churchill High School, Calgary
		Annie Xu	Old Scona Academic High School, Edmonton
		Darren Xu	Sir Winston Churchill High School, Calgary
13	77	Yuxiang Liu	Western Canada High School, Calgary
		Wen Wang	Western Canada High School, Calgary
15	76	Jaclyn Chang	Western Canada High School, Calgary (Grade 10)
		Chen Liu	Western Canada High School, Calgary (Grade 11)
		Taylor Hudson	J G Diefenbaker High School, Calgary
		Wen Song	Sir Winston Churchill High School, Calgary
		Kevin Tan	Sir Winston Churchill High School, Calgary
		Michael Wong	Tempo School, Edmonton

Rank	Score	Name	School
21	75	Mariya Sardarli	McKernan Junior High School, Edmonton (Grade 8)
		Andrew Qi	Vernon Barford Junior High School, Edmonton (Grade 9)
		Annie Wang	Sir Winston Churchill High School, Calgary (Grade 11)
		David Yu	Sir Winston Churchill High School, Calgary (Grade 10)
		Michael Zhou	Western Canada High School, Calgary
26	74	Stephanie Bohaichuk	Harry Ainlay High School, Edmonton (Grade 10)
		Jacky Tian	Western Canada High School, Calgary (Grade 11)
		Jonathan Wong	Western Canada High School, Calgary (Grade 11)
		Jared Gordon	Western Canada High School, Calgary
		Stephanie Laflamme	Bishop Carroll High School, Calgary
		Navid Nourian	Henry Wise Wood High School, Calgary
		Ben Wang	Sir Winston Churchill High School, Calgary
		Liz Yue	Sir Winston Churchill High School, Calgary
34	73	Maninder Longowal	Tempo School, Edmonton (Grade 11)
		David Szepesvari	Harry Ainlay High School, Edmonton (Grade 11)
		James Kim	Western Canada High School, Calgary
37	72	Spencer Boone	Western Canada High School, Calgary (Grade 11)
		Jessica Jiang	Old Scona Academic High School, Edmonton (Grade 11)
		Brett Baek	Western Canada High School, Calgary
		Victor Feng	Sir Winston Churchill High School, Calgary
		Lian Tang	Jasper Place High School, Edmonton
		Glen Wang	Western Canada High School, Calgary (Grade 11)
43	71	Di Mo	Sir Winston Churchill High School, Calgary (Grade 10)
		Anna Yu	Sir Winston Churchill High School, Calgary (Grade 11)
		Douglas Cheung	Old Scona Academic High School, Edmonton
		Brenna Pickell	Archbishop Jordan High School, Sherwood Park
47	**70**	Nafisah Tyebkhan	Tempo School, Edmonton (Grade 9)
		David Gordon	Western Canada High School, Calgary (Grade 10)
		Yuri Delanghe	Harry Ainlay High School, Edmonton (Grade 10)
		Alexander Neame	Harry Ainlay High School, Edmonton (Grade 10)
		Edward Xu	Henry Wise Wood High School, Calgary (Grade 10)
		Elsie Young	Western Canada High School, Calgary (Grade 10)
		Natasha Birchall	Tempo School, Edmonton (Grade 11)
		Steven Dien	Western Canada High School, Calgary (Grade 11)
		Mandi Xu	Western Canada High School, Calgary (Grade 11)
		Yingyu Yao	Sir Winston Churchill High School, Calgary (Grade 11)
		Min Bai	Western Canada High School, Calgary
		Topher Flanagan	Tempo School, Edmonton
		Naheed Jivra	Strathcona-Tweedsmuir School, Okotoks
		Philip Morin	St Francis High School, Calgary

Team Results

The contest was written by 34 schools. There were ten schools from Zone I (Calgary) with 314 students, six schools from Zone II (southern rural Alberta) with 73 students, ten schools from Zone III (Edmonton) with 174 students and eight schools from Zone IV (northern rural Alberta) with 151 students.

The top teams are listed below.

Rank	Score	Team Members and Manager
1	245	Sir Winston Churchill High School, Calgary—Frank Yan, Danni Shi and Chong Shen, managed by Neil Hamel
2	243	Western Canada High School, Calgary—Linda Zhang, Jarno Sun and Lucille Lu, managed by Renata Delisle
3	222	J G Diefenbaker High School, Calgary—Zoe Cheung, Taylor Hudson and Zinzhu Wei, managed by Terry Loschuk
4	221	Old Scona Academic High School, Edmonton—Annie Xu, Jessica Jiang and Douglas Cheung, managed by Ihor Lytviak
5	219	Tempo School, Edmonton—Michael Wong, Maninder Longowal and Nafisah Tyebkahn, managed by Lorne Rusnell
6	217	Harry Ainlay High School, Edmonton—Stephanie Bohaichuk, David Szepesvari and Yuri Delanghe/ Alexander Neame, managed by Jacqueline Coulas
7	210	Henry Wise Wood High School, Calgary—Navid Nourian, Edward Xu and Xin Zhang, managed by Michael Retallack
8	208	Queen Elizabeth Junior/Senior High School, Calgary—Karl Qin, Raphaell Masquillier and Fay Qian, managed by Sharon Reid
9	204	Jasper Place High School, Edmonton—Liang Tang, Duhao Meng and Jingchen Ge, managed by John MacNab
10	203	St Francis High School, Calgary—Philip Morin, Nicole Veltri and Kirsten Marshall, managed by Peter Walker

Other participating schools were

- Zone I (Calgary)
  - Bishop Carroll High School—Toni Fazio, manager
  - Calgary Science School—Scot Doehlar, manager
  - Central Memorial High School—Gerald Krabbe, manager
  - William Aberhart High School—James Kotow, manager
- Zone II (Southern Rural Alberta)
  - Crowsnest Consolidated High School (Crowsnest Pass)—Jodi Peebles, manager
  - Hughenden Public School—Crystal Chudley, manager
  - Oilfields High School (Turner Valley)—Chris Hughes, manager
  - Prairie Christian Academy (Three Hills)—Robert Hill, manager
  - St Gabriel the Archangel School (Chestermere)—Adrienne Busch, manager
  - Senator Gershaw School (Bow Island)—Linda Atwood, manager
  - Strathcona-Tweedsmuir School (Okotoks)—Nola Adam, manager
- Zone III (Edmonton)
  - Archbishop MacDonald High School—John Campbell, manager
  - Holy Trinity High School—Len Bonifacio, manager
  - McKernan Junior High School—Ward Patterson, manager
  - McNally High School—Brian Pike, manager
  - Vernon Barford Junior High School—Robert Wong, manager
  - Vimy Ridge Academy—Delcy Rolheiser, manager

  Ross Sheppard High School (JeremyKlassen, manager) registered for the contest, but was unable to hold it on the day.
- Zone IV (Northern Rural Alberta)
  - Archbishop Jordan High School (Sherwood Park)—Marge Hallonquist, manager
  - Ardrossan Junior Senior High School—Rebecca Gustafson, manager
  - École Secondaire Ste Marguerite d'Youville (St Albert)—Lisa La Rose, manager
  - Father Patrick Mercredi High School (Fort McMurray)—Ted Venne, manager
  - J A Williams High School (Lac La Biche)—Matt Dyck, manager
  - Leduc High School—Corlene Balding, manager
  - Paul Kane High School (St Albert)—Percy Zalasky, manager

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1. A positive integer has 1001 digits, all of which are 1s. When this number is divided by 1001, the remainder is
   (a) 1 (b) 10 (c) 11 (d) 100 (e) none of these

2. Some cats have got into the pigeon loft because the total head count is 34 but the total leg count is 80. The number of cats among the pigeons is
   (a) 6 (b) 12 (c) 17 (d) 22 (e) 28

3. In triangle ABC, $AB \leq 1 \leq BC \leq 2 \leq CA \leq 3$. The maximum area of triangle ABC is
   (a) 1 (b) 3/2 (c) 2 (d) 5/2 (e) none of these

4. The number of ways in which five As and six Bs can be arranged in a row that reads the same backwards and forwards is
   (a) 1 (b) 5 (c) 10 (d) 15 (e) none of these

5. Among twenty consecutive integers each at least 9, the maximum number of them that can be prime is
   (a) 4 (b) 5 (c) 6 (d) 7 (e) 8

6. The non-negative numbers x and y are such that $2x + y = 5$. The sum of the maximum value of $x + y$ and the minimum value of $x + y$ is
   (a) 0 (b) 5/2 (c) 5 (d) 15/2 (e) none of these

7. We wish to choose some of the positive integers from 1 to 1000 inclusive, such that no two differ by 3 or 5. The maximum number of positive integers we can choose is
   (a) 200 (b) 300 (c) 333 (d) 500 (e) none of these

8. The number of polynomials p with integral coefficients such that $p(9) = 13$ and $p(13) = 20$ is
   (a) 0 (b) 1 (c) 2 (d) 3 (e) infinitely many

9. The number of pairs (a, b) of positive integers such that all three roots of the cubic equation $x^3 - 10x^2 + ax - b = 0$ are positive integers is
   (a) 3 (b) 8 (c) 10 (d) 66 (e) none of these

10. In the quadrilateral ABCD, AB = CD, AD = 2 and BC = 6. AD and BC are parallel lines at a distance 8 apart. The radius of the smallest circle that can cover ABCD is
    (a) $\sqrt{18}$ (b) $\sqrt{20}$ (c) $\frac{\sqrt{85}}{2}$ (d) 5 (e) none of these

11. The real numbers x and y are such that $x + 2/y = 8/3$ and $y + 2/x = 3$. The value of xy is
    (a) 3/2 (b) 4/3 (c) 2 (d) 4 (e) not uniquely determined

12. Let θ be an acute angle such that $\sec^2\theta + \tan^2\theta = 2$. The value of $\csc^2\theta + \cot^2\theta$ is
    (a) 2 (b) 3 (c) 4 (d) 5 (e) none of these

13. The diameter AC divides a circle into two semicircular arcs. B is the midpoint of one these arcs, and D is any point on the other arc. If the area of ABCD is 16 square centimetres, the distance, in centimetres, from B to AD is
    (a) 2 (b) $2\sqrt{2}$ (c) 4 (d) $4\sqrt{2}$ (e) dependent on the radius of the circle

14. Five students took part in a contest consisting of six true-or-false questions. Student #i gave the answer T to question #j if and only if $i \leq j$. The total number of incorrect answers is 8 or 9, and there are more incorrect answers of T than incorrect answers of F. The student who has both an incorrect answer of T and an incorrect answer of F is
    (a) #1 (b) #2 (c) #3 (d) #4 (e) #5

15. An integer n is randomly chosen from 10^{99} to $10^{100} - 1$ inclusive. The real number m is defined by $m = 9n/5$. Of the following five numbers, the one closest to the probability that $10^{99} \leq m \leq 10^{100} - 1$ is
    (a) 1/3 (b) 4/9 (c) 1/2 (d) 5/9 (e) 2/3

16. The smallest value of the real number k such that $(x^2 + y^2 + z^2)^2 \leq k(x^4 + y^4 + z^4)$ holds for all real numbers x, y and z is
    (a) 1 (b) 2 (c) 3 (d) 6 (e) 9

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Solution to Part I—2007

1. The number 111111 is divisible by 1001. Now $1001 = 6 \times 166 + 5$. Hence the desired remainder is the same when we divide 11111 by 1001—that is, 100. The answer is **(d)**.

2. Tell the cats to put their front legs up. Now there are still 34 heads, but only $34 \times 2 = 68$ legs on the ground. Hence $80 - 68 = 12$ legs are up in the air, and each cat puts up 2 of them. It follows that the number of cats is $12 \div 2 = 6$. The answer is **(a)**.

3. Since $AB \le 1$ and $BC \le 2$, the area of triangle ABC is at most 2. This maximum value can be attained when AB = 1 and BC = 2 and they are perpendicular to each other. Now $CA = \sqrt{5}$, and we indeed have $2 \le CA \le 3$. The answer is **(a)**.

4. The middle symbol must be an A. The first five symbols consist of two As and three Bs, and they can be arranged in all possible ways. The last five symbols, consisting also of two As and three Bs, must be in reverse order with respect to the first five symbols. In the first five symbols, we only have to find the number of ways of choosing two of the five positions for the two As. This is given by $\binom{5}{2} = 10$. The answer is **(c)**. An exhaustive analysis also works.

5. The integers from 11 to 30 include six primes, namely 11, 13, 17, 19, 23 and 29. In the ten odd numbers among any twenty consecutive integers each at least 7, at least three are multiples of 3 and exactly two are multiples of 5, but at most one can be a multiple of 15. Hence the maximum is indeed six. The answer is **(c)**.

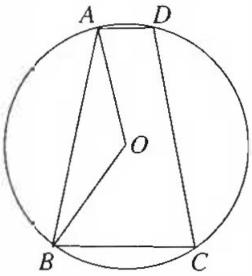
6. We have $2x + 2y = 5 + y \ge 5$, so that the minimum value of $x + y$ is 5/2, attained at $(x, y) = (5/2, 0)$. Also, $x + y = 5 - x \le 5$, so that the maximum value of $x + y$ is 5, attained at $(x, y) = (0, 5)$. The answer is **(d)**.

7. If we take all the even numbers, clearly no two will differ by 3 or 5. Hence, we can take at least 500 numbers. Now partition the integers from 1 to 1000 into blocks of 10. From each of the following five pairs, we can take at most one number: $(10n + 1, 10n + 4)$, $(10n + 2, 10n + 5)$, $(10n + 3, 10n + 8)$, $(10n + 6, 10n + 9)$ and $(10n + 7, 10n + 10)$. Hence we can take no more than 500 numbers. The answer is **(d)**.

8. Suppose there exists such a polynomial p. Since $a^n - b^n$ is divisible by $a - b$ for all positive integers a, b and n with $a \ne b$, $13 - 9$ must divide $p(13) - p(9)$. However, 4 does not divide 7, and we have a contradiction. The answer is **(a)**. A parity argument also works.

9. Let the positive integral roots be $r \le s \le t$. Then $x^3 - 10x^2 + ax + b = (x - r)(x - s)(x - t)$. Expansion yields $x^3 - (r + s + t)x^2 + (st + tr + rs)x - rst$. Hence $r + s + t = 10$. The possible partitions are (1,1,8), (1,2,7), (1,3,6), (1,4,5), (2,2,6), (2,3,5), (2,4,4) and (3,3,4). The answer is **(b)**.

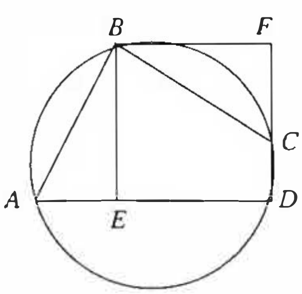
10. The centre O of the circle lies on the axis of symmetry of ABCD. Let y be its height above BC. Then $OB^2 = y^2 + 3^2$ while $OA^2 = (8 - y)^2 + 1^2$. Equating these two values yields $y = 7/2$. Hence the radius is $\sqrt{(7/2)^2 + 3^2} = \sqrt{85}/2$. The answer is **(c)**.



11. Multiplying one equation by the other, we have $xy + 4 + \frac{4}{xy} = 8$. This may be rewritten as $0 = (xy)^2 - 4xy + 4 = (xy - 2)^2$. Hence $xy = 2$. The answer is **(c)**.

12. Let $s = \sin^2\theta$ and $c = \cos^2\theta$. We have $\sec^2\theta + \tan^2\theta = \frac{1+s}{c} = 2$. Since $s + c = 1$, $2 - c = 2c$ so that $c = 2/3$. It follows that $s = 1/3$. Now $\csc^2\theta + \cot^2\theta = \frac{1+c}{s} = 5$. The answer is **(d)**.

13. Let E be the point on AD such that BE is perpendicular to AD. Complete the rectangle BEDF. Now AB = BC, $\angle AEB = 90° = \angle CFB$ and $\angle ABE = 90° - \angle CBE = \angle CBF$. Hence ABE and CBF are congruent triangles and they have equal area. It follows that BEDF is a square, and its area is also 16. Hence BE = 4. The answer is **(c)**.



14. The number of incorrect answers for each of questions 1 and 6 is 0 or 5. The number of incorrect answers for each of questions 2 and 5 is 1 or 4. The number of incorrect answers for each of questions 3 and 4 is 2 or 3. A total of 8 incorrect answers can only be made up from 0+1+3+3+1+0. However, we would have an equal number of incorrect answers of T and incorrect answers of F. Hence the total must be 9, and it can be made up from either 0+1+2+2+4+0 or 0+4+2+2+1+0. However, the latter yields more incorrect answers of F than incorrect answers of T. It follows that the correct answers for the six questions are T, F, T, F, F and F respectively. Only student #4 has both an incorrect answer of T (for question 2) and an incorrect answer of F (for question 3). The answer is (**d**).

15. We have $10^{99} \times 5/9 \leq n \leq (10^{100} - 1)5/9$. Since n is an integer, $5 \times 10^{98} < n < 5 \times 10^{99}$. However, we must eliminate those values of n where $5 \times 10^{98} < n < 10^{99}$. Thus the number of acceptable values of n is about 4.5×10^{99}. Since $10^{99} \leq n \leq 10^{100} - 1$, the desired probability is very close to ½. The answer is (**c**).

16. We can rewrite the inequality as $(k - 3)(x^4 + y^4 + z^4) + (y^2 - z^2)^2 + (z^2 - x^2)^2 + (x^2 - y^2)^2 \geq 0$, from which it is clear that $k \geq 3$. The answer is (**c**).

Editor's note: Andy Liu is a professor in the Department of Mathematical and Statistical Sciences at the University of Alberta. He enjoys working on research problems that are easy to understand but not so easy to solve.