Diophantine Polynomials

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What is a Diophantine Polynomial? It is a polynomial of degree 2, 3 or 4 which is factorable in the set of integers and whose derivative is factorable in the set of rational numbers. We want to discuss them to facilitate curve sketching.

The polynomials that we are about to examine can be used for both Grade 11 and calculus students, because the intercepts are easy to find and the y-values for the maxima and minima are shared among the families of curves. For example, we can ask a Grade 11 student to sketch $y = x^3 + x^2 - 16x - 16$ by finding both the x and y intercepts. We can use the very same polynomial for the Calculus 12 student who can find the intercepts easily and more readily find the x-values for both maxima and minima because the derivative is easy to factor.

My belief is that students should learn a complicated algorithm in simple progressive steps using straightforward numbers. Diophantus worked with integers and rational numbers only. Pedagogically, Diophantus was really onto something because he created methods that involved a lot of processing and sequencing while focusing on whole numbers. The distractions I refer to in curve sketching are complex and irrational numbers. It is difficult enough to learn some five to eight steps gathering enough data to accurately sketch a cubic, quartic, or quintic polynomial and/or a rational expression that may involve a diagonal asymptote without difficult-to-work-with numbers. If the student has the burden (when first learning the process) of working with irrational or complex numbers, along with concentrating on the behaviour of the curve and concavity, then he or she might simply declare "whatever" and drop the task. If the numbers are whole or integral (Diophantus), then the focus remains where it should be: on the algorithm. The task of the educator is to demonstrate algorithms in such a way that the student can master the process in sequence. The solution is to stick to the Diophantine process and to model examples that involve process and sequencing without getting tangled up with irrational numbers. To some readers this may be self-evident, but it is not as simple as it sounds to find cubics, or quartics with single integral roots whose derivatives have single rational roots. Finding them involved testing hundreds of polynomials using DERIVE (an algebra software developed by Texas Instruments), as I was determined to find easy-to-calculate polynomials, which would facilitate graphing curves like $y = x^3 + 11x^2 + 24x$ without worrying about irrational and complex numbers. There was another challenge, of course, and that was to keep the constant of the polynomial relatively small so that working without a calculator would not be arduous.

Another aspect of this approach with whole numbers is that when the student knows that the numbers are designed to work, learning of the method or algorithm remains the priority. The student also knows that there is something wrong if the numbers do not work. It is kind of a security blanket for the beginner, but it eliminates doubt, which so often takes away confidence in ability and performance. Later on, after mastering the technique, the student gains confidence through the ease of this, and therefore can tackle problems with both irrational and complex numbers.

It is my objective to propose families of cubics and quartics that are factorable in the integers and whose derivatives are factorable in the set of rational numbers. I will also propose methods using DERIVE by which you can construct your own polynomials. We'll start with the very basic table of linear and quadratic polynomials, then lead up to the cubics and quartics. I will end the paper with a brief discussion of the quintic, which should have worked but did not.

Table I contains all the various linear and quadratic forms along with the general set of cubics.

Using Table I

Take a cubic of the form $(x + a^2)(x + 2ab)(x + b^2)$ whose derivative has rational roots. Choosing any integers, $a = 1$ and $b = 3$ for example, our new polynomial is $(x+1)(x+6)(x+9)$ with roots $-1, -6$, and -9. The differential form is $3x^2 + 32x + 69$, whose roots are -3 and $-\frac{23}{3}$.

Table I

Family Function	Roots	Derivative	Roots	Conditions on coefficients and constants to have integral roots
a	none	0	none	not applicable
ax	$x=0$	a	none	not applicable
$ax+b$	$x=\frac{-b}{a}$	a	none	not applicable
$(x+a)^2$	$x=-a$	$2(x+a)$	$x=-a$	none
$x^2+ax=x(x+a)$	$x=0,-a$	$2x+a$	$x=\frac{-a}{2}$	a must be even
$x^2+x(a+b)+ab$ $=(x+a)(x+b)$	$x=-a,-b$	$2x+a+b$	$x=\frac{-a-b}{2}$	a and b are both odd or both even
$acx^2+x(ad+bc)+bd$ $=(ax+b)(cx+d)$	$x=\frac{-b}{a},\frac{-d}{c}$	$2acx+ad+bc$	$x=\frac{-ad-bc}{2ac}$	$a\neq 0, c\neq 0$ $ad+bc$ must either equal ac or be an even multiple of it
$(x+a)^3$	$x=-a$	$3(x+a)^2$	$x=-a$	none
$(x+a)^2(x+b)$	$x=-a$, $x=-b$	$2(x+a)(3x+2b+a)$	$x=-a$ $x=\frac{-2b-a}{3}$	none $2b+a$ is 3 or a multiple of 3
$x(x+a)(x+b)$	$x=0$ $x=-a$ $x=-b$	$3x^2+2x(a+b)+ab$	$x=\frac{-(a+b)\pm\sqrt{a^2-ab+b^2}}{3}$	a^2-ab+b^2 equals zero or a perfect square
$(x+a^2)(x+2ab)(x+b^2)$	$x=-a^2$ $x=-2ab, x=-b^2$	$3x^2+x(2a^2+4ab+2b^2)$ $+ab(2a^2+ab+2b^2)$	$x=-ab$ $x=\frac{-2a^2-ab-2b^2}{3}$	$2a^2+ab+2b^2$ must be 3 or a multiple of 3
$(x+1)(x-a)(x+a)$	$x=-1$ $x=a$ $x=-a$	$3x^2-2x-a^2$	$x=\frac{2\pm\sqrt{4+12a^2}}{6}$	$4+12a^2$ must be a perfect square ($a=0,1,4,16...$)

The polynomials in Table II consist of the particular numerical families with single roots. These are the ones that are ready to use in your classroom today.

As we observe the families in Table II, it is hard not to notice the pattern 8, 15, 21, 30, 35 and 36. It is a quadratic arithmetic sequence whose elements (except for a couple) all work as families of curves.

Table II

Family Function	Roots	Derivative	Roots	Transformation
$x(x+3)(x+8)$ x^3+11x^2+24x	$0,-3,-8$	$(3x+4)(x+6)$ $3x^2+22x+24$	$-\frac{4}{3},6$	$(x\pm k)(x\pm 3a\pm k)(x\pm 8a\pm k)$
$x(x+5)(x+8)$ x^3+13x^2+40x	$0,-5,-8$	$(3x+20)(x+2)$ $3x^2+26x+40$	$-\frac{20}{3},-2$	$(x\pm k)(x\pm 5a\pm k)(x\pm 8a\pm k)$
$x(x+7)(x+15)$ x^3+22x^2+105x	$0,-7,-15$	$(3x+35)(x+3)$ $3x^2+44x+105$	$-\frac{35}{3},-3$	$(x\pm k)(x\pm 7a\pm k)(x\pm 15a\pm k)$
$x(x+8)(x+15)$ x^3+23x^2+120x	$0,-8,-15$	$(3x+10)(x+12)$ $3x^2+46x+120$	$-\frac{10}{3},-12$	$(x\pm k)(x\pm 8a\pm k)(x\pm 15a\pm k)$
$x(x+5)(x+21)$ x^3+26x^2+105x	$0,-5,-21$	$(3x+7)(x+15)$ $3x^2+52x+105$	$-\frac{7}{3},-15$	$(x\pm k)(x\pm 5a\pm k)(x\pm 21a\pm k)$
$x(x+16)(x+21)$ x^3+37x^2+336x	$0,-16,-21$	$(3x+56)(x+6)$ $3x^2+74x+336$	$-\frac{56}{3},-6$	$(x\pm k)(x\pm 16a\pm k)(x\pm 21a\pm k)$
$x(x+26-a)(x+26)$ $x^3+x^2(52-a)+26(26-a)x$	$0,a-26,-26$	$3x^2+52x+26a-a^2$	not rational	
$x(x+14)(x+30)$ x^3+44x^2+420x	$0,-14,-30$	$(3x+70)(x+6)$ $3x^2+88x+420$	$-\frac{70}{3},-6$	$(x\pm k)(x\pm 14a\pm k)(x\pm 30a\pm k)$
$x(x+16)(x+30)$ x^3+46x^2+480x	$0,-16,-30$	$(3x+20)(x+24)$ $3x^2+92x+480$	$-\frac{20}{3},-24$	$(x\pm k)(x\pm 16a\pm k)(x\pm 30a\pm k)$
$x(x+33-a)(x+33)$ $x^3+x^2(66-a)+33(33-a)$	$0,a-33,-33$	$3x^2+2x(66-a)+33(33-a)$	not rational	
$x(x+11)(x+35)$ x^3+46x^2+385x	$0,-11,-35$	$(3x+72)(x+5)$ $3x^2+92x+385$	$-\frac{72}{3},-5$	$(x\pm k)(x\pm 11a\pm k)(x\pm 35a\pm k)$
$x(x+24)(x+35)$ x^3+59x^2+840x	$0,-24,-35$	$(3x+14)(x+15)$ $3x^2+118x+840$	$-\frac{14}{3},-15$	$(x\pm k)(x\pm 24a\pm k)(x\pm 35a\pm k)$
$x(x+36-a)(x+36)$ $x^3+x^2(72-a)+36(36-a)x$	$0,a-36,-36$	$3x^2+2x(72-a)+36(36-a)$	not rational	

Methods Using DERIVE

Regarding methods for single roots, let us begin by entering the form $x(x+a)(x+b)$ into DERIVE. This guarantees a factorable form. Press C for Calculus and differentiate. The resulting form is put in function form as DECLARE. Now we can either guess values and hope that our quadratic is factorable, or fix a value for a, and then guess values for b until the quadratic is factorable. The question is, do we have anything to guide our guessing? In fact, we do. Visually, the values of x for maxima and minima will occur between the first and last x-intercepts. Hence, if we were to choose 0 and 8 as two of our first and last roots, we would know that the third one must come between them. It is just a question of leaving enough room between the roots so that the critical points can occur as integers and/or rational numbers. Algebraically, we enter $x(x-a)(x-8)$ into DERIVE, and then differentiate giving $3x^2+2x(a-8)+8a$. Since we have a quadratic, the discriminant B^2-4AC must equal a perfect square in order to be factorable. Using the command DECLARE, we set $f(a)=4a^2-32a+256=4(a^2-8a+64)$ and evaluate (or use the TI83 where second function gives TABLE and we search it for perfect squares). Both 3 and 5 come up quickly, implying that both $x(x-3)(x-8)$ and $x(x-5)(x-8)$ have derivatives whose roots are rational.

I do not pretend to have all the families, but applying translations to any given family will yield many polynomials. The following is a small sample arrived at by adding a constant to all the terms.

Given family	$x(x-3)(x-8)$
Add 1	$(x+1)(x-2)(x-7)$
Add 2	$(x+2)(x-1)(x-6)$
Add 3	$(x+3)(x)(x-5)$
Add 4	$(x+4)(x+1)(x-4)$
Add 5	$(x+5)(x+2)(x-3)$
Add 6	$(x+6)(x+3)(x-2)$
Add 7	$(x+7)(x+4)(x-1)$
Add 8	$(x+8)(x+5)(x)$, etc.

Interestingly enough, the entire above shares a max height of $\frac{400}{27}=\left(\frac{5}{3}\right)\left(\frac{4}{3}\right)\left(\frac{20}{3}\right)$, and minimum low of -36, and the difference between their corresponding x-coordinates is exactly $\frac{14}{3}$. A linear relationship exists between these values and those found in the quartics. We shall explore this after exploring the quartic family of curves.

Having fully explored the cubic, the quartic family of curves presented quite a challenge because there would be three roots, other than zero, to find. Visually, I opted for a span of 7 (one less than the 8 for cubics), entered $x(x-a)(x-b)(x-7)$ into DERIVE, fixed $a=3$ (only because it had worked with the cubic), took the derivative and evaluated b from 1 to 7 hoping that some value b would work. The derived form was $4x^3+x^2(-3b-30)+x(20b+42)-21b$, and by declaring f as the function I simply tested values for b and systematically factored (pressing F). To my great delight $b=4$ worked, giving $2(x-6)(x-1)(2x-7)$. Observing 3 and 4 together, I acted on a hunch that Pythagorean Triples might work. So, following in the footsteps of Diophantus, I tried triplets beginning with odd numbers and even numbers, and they worked beautifully. An added bonus was those triplets with consecutive legs such as 20-21-29 and 119-120-169, etc, which also worked wonderfully. The patterns appear in Table III. The numerical families of the form $x(x+a)(x+b)(x+c)$ appear in Table IV. The numerical families for the form $x^2(x+a)(x+b)$ appear in Table V.

Table III

Family Function	**Roots**	**Derivative**	**Roots**	**Conditions For Integral Roots**
$(x+a)^4$	$-a$	$4(x+a)^3$	$-a$	no restrictions
$(x+a)^3(x+b)$	$-a,-b$	$(a+b)^2(4x+3b+a)$	$-a$ and $\frac{-3b-a}{4}$	$-3b-a$ must be a multiple of 4 for b=1 a=1,5,9… 4k-3 for b=2 a=2,6,10…4k-2 for b=3 a=3,7,11…4k-1 for b=4 a=4,8,12…4k
$(x+a)^2(x+b)^2$	$-a,-b$	$2(x+a)(x+b)(2x+a+b)$	$-a,-b$ and $\frac{-b-a}{2}$	a and b must be both odd or both even
$(x+a)^2(x+b)(x+c)$	$-a,-b,-c$	$(x+a)\left[4x^2+x(3b+3c+2a)+2bc+ab+ac\right]$	$\frac{-3b-3c-2a\pm\sqrt{4a^2+9b^2-4ab-4ac-14bc}}{8}$	$4a^2+9b+9c$ $-4ab-4ac-14bc$ must be perfect square
The product of (x) and $(x+2n+1)$ and $(x+2n^2+2n)$ and $(x+2n^2+4n+1)$	$-2n-1$, $-2n^2-2n$ and $-2n^2-4n-1$	$2(x+n)(x+2n^2+3n+1)(2x+2n^2+4n+1)$	$-n,-2n^2-3n-1$ and $-2n^2-4n-1$	no conditions
The product of (x) and $(x+2n+1)$ and $(x+2n^2+2n)$ and $(x+2n^2+4n+1)$	$-2n,1-n^2$, and $-n^2-2n+1$	$2(x+n-1)(x+n^2+n)(2x+n^2+2n-1)$	$1-n,-n^2-n$ and $-n^2-2n+1$	no conditions

Table IV

Family Function	**Roots**	**Derivative**	**Roots**	**Transformation**
Odd Pythagorean Triplets				
$x(x+3)(x+4)(x+7)$	$0,-3,-4,-7$	$2(x+1)(x+6)(2x+7)$	$-1,-6,\frac{-7}{2}$	$(x\pm k)(x\pm 3a\pm k)(x\pm 4a\pm k)(x\pm 7a\pm k)$
$x(x+5)(x+12)(x+17)$	$0,-5,-12,-17$	$2(x+2)(x+15)(2x+17)$	$-2,-15,\frac{-17}{2}$	$(x\pm k)(x\pm 5a\pm k)(x\pm 12a\pm k)(x\pm 17a\pm k)$
$x(x+7)(x+24)(x+31)$	$0,-7,-24,-31$	$2(x+3)(x+28)(2x+31)$	$-3,-28,\frac{-31}{2}$	$(x\pm k)(x\pm 7a\pm k)(x\pm 24a\pm k)(x\pm 31a\pm k)$
				etc
Even Pythagorean Triplets				
$x(x+4)(x+3)(x+7)$	$0,-3,-4,-7$	$2(x+1)(x+6)(2x+7)$	$-1,-6,\frac{-7}{2}$	$(x\pm k)(x\pm 4a\pm k)(x\pm 3a\pm k)(x\pm 7a\pm k)$
$x(x+6)(x+8)(x+14)$	$0,-6,-8,-14$	$4(x+2)(x+7)(x+12)$	$-2,-7,-12$	$(x\pm k)(x\pm 6a\pm k)(x\pm 8a\pm k)(x\pm 14a\pm k)$
$x(x+8)(x+15)(x+23)$	$0,-8,-15,-23$	$2(x+3)(x+20)(x+23)$	$-3,-20,\frac{-23}{2}$	$(x\pm k)(x\pm 8a\pm k)(x\pm 15a\pm k)(x\pm 23a\pm k)$
				etc
Consecutive-Leg Triplets				
$x(x+3)(x+4)(x+7)$	$0,-3,-4,-7$	$2(x+1)(x+6)(2x+7)$	$-1,-6,\frac{-7}{2}$	$(x\pm k)(x\pm 3a\pm k)(x\pm 4a\pm k)(x\pm 7a\pm k)$
$x(x+20)(x+21)(x+41)$	$0,-20,-21,-41$	$2(x+6)(x+35)(2x+41)$	$-6,-35,\frac{-41}{2}$	$(x\pm k)(x\pm 20a\pm k)(x\pm 21a\pm k)(x\pm 41a\pm k)$
$x(x+119)(x+120)(x+239)$	$0,-119,-120,-239$	$2(x+35)(x+204)(2x+239)$	$-35,-204,\frac{-239}{2}$	$(x\pm k)(x\pm 119a\pm k)(x\pm 120a\pm k)(x\pm 239a\pm k)$
				etc

Table V

Family Function	**Roots**	**Derivative**	**Roots**	**Transformation**
$x^2(x+5)(x-7)$	$0,-5,7$	$2x(x-5)(2x+7)$	$0,5,\frac{-7}{2}$	$(x^2\pm k)(x+5a\pm k)(x-7a\pm k)$
$x^2(x+5)(x+2)$	$0,-5,-2$	$x(x+4)(4x+5)$	$0,-4,\frac{-5}{4}$	$(x^2\pm k)(x+5a\pm k)(x+2a\pm k)$
$x^2(x+5)(x+9)$	$0,-5,-9$	$x(x+3)(2x+15)$	$0,-3,\frac{-15}{2}$	$(x^2\pm k)(x+5a\pm k)(x+9a\pm k)$
$x^2(x+7)(x+10)$	$0,-7,-10$	$x(x+4)(4x+35)$	$0,-4,\frac{-35}{4}$	$(x^2\pm k)(x+7a\pm k)(x+10a\pm k)$
$x^2(x+9)(x+14)$	$0,-9,-14$	$x(x+12)(4x+21)$	$0,-12,\frac{-21}{4}$	$(x^2\pm k)(x+9a\pm k)(x+14a\pm k)$

Table VI

Family Function	**Roots**	**Derivative**	**Roots**	**Conditions For Integral Roots**
$(x+a)^5$	$-a$	$5(x+a)^4$	$-a$	none
$(x+a)^4(x+b)$	$-a,-b$	$(x+a)^3(5x+a+4b)$	$-a$ and $\dfrac{-a-4b}{5}$	$a+4b$ must be 5 or a multiple of 5
$(x+a)^3(x+b)(x+c)$	$-a,-b,$ $-c$	$(x+a)^2\left(5x^2+2x(a+2b+2c)+ab+ac+3bc\right)$	$-a$ and $\dfrac{-(a+2b+2c)\pm\sqrt{a^2+4b^2-ab-ac-7bc}}{5}$	Conditions: $a^2+4b^2+4c^2$ $-ab-ac-7bc$ must be zero or a perfect square
$(x+a)^3(x+b)^2$	$-a,-b$	$(x+a)^2(x+b)(5x+2a+3b)$	$-a,-b$ and $\dfrac{-2a-3b}{5}$	$2a+3b$ must be 5 or a multiple of 5
$(x+a)^2(x+b)(x+c)$	$-a,-b,$ $-c$	$(x+b)(x+a)\left[5x^2+x(3a+3b+4c)+ab+2ac+2bc\right]$	$-a,-b,$ and $\dfrac{-(3a+3b+4c)\pm\sqrt{9a^2+9b^2+16c^2-2ab-16bc-16-c}}{10}$	Conditions: $9a^2+9b^2+16c^2$ $-2ab-16bc-16ac$ must be a perfect square
$x^2(x+a)(x+b)(x+c)$	$0,-a,$ $-b,-c$	$x\left(5x^3+x^2(4a+4b+4c)+x(3ab+3ac+3bc)+2abc\right)$	no rational roots	N/A
$x(x+a)(x+b)(x+c)(x+d)$	$0,-a,$ $-b,-c,$ $-d$	$5x^4+4x^3(a+b+c+d)+3x^2(ab+ac+ad+bc+bd+cd)$ $+2x(abc+abd+acd+bcd)+abcd$	no rational roots	N/A

The Quintic

In terms of multiple roots, the quintic lends itself nicely to easy-to-work-with numbers that are small in quantity. However, for quintics of the form $x(x-a)(x-b)(x-c)(x-d)$, the derivative has no rational roots, primarily because of Fermat's Last Theorem whereby there are no integral values for which $x^4+y^4=z^4$. Having run the computer through thousands of number combinations (just to be sure), no derivative with rational roots could be found. Our Table VI contains multiple roots only. Table VII contains the numerical families for the forms

$$x^3(x+a)(x+b) \text{ and } x^2(x+a)^2(x+b).$$

Table VII

Family Function	Roots	Derivative	Roots	Transformation
$x^3(x+3)(x+4)$ $x^5+7x^4+12x^3$	$0,-3,-4$	$x^2(x+2)(5x+18)$ $5x^4+28x^3+36x^2$	$0,-2,\frac{-18}{5}$	$(x\pm k)^3(x\pm 3a\pm k)(x\pm 4a\pm k)$
$x^3(x+3)(x+11)$ $x^5+14x^4+33x^3$	$0,-3,-11$	$x^2(x+9)(5x+18)$ $5x^4+56x^3+99x^2$	$0,-9,\frac{-18}{5}$	$(x\pm k)^3(x\pm 3a\pm k)(x\pm 11a\pm k)$
$x^3(x+4)(x+7)$ $x^5+11x^4+28x^3$	$0,-4,-7$	$x^2(x+6)(5x+14)$ $3x^4+44x^3+84x^2$	$0,-6,\frac{-14}{5}$	$(x\pm k)^3(x\pm 4a\pm k)(x\pm 7a\pm k)$
$x^3(x+5)(x+12)$ $x^5+17x^4+60x^3$	$0,-5,-12$	$x^2(x+10)(5x+18)$ $5x^4+68x^3+180x^2$	$0,-10,\frac{-18}{5}$	$(x\pm k)^3(x\pm 5a\pm k)(x\pm 12a\pm k)$
$x^2(x-3)^2(x-1)$ $x^5-7x^4+15x^3-9x^2$	$0,3,1$	$x(x-3)(x-2)(5x-3)$ $5x^4-28x^3+45x^2-18x$	$0,3,2,\frac{3}{5}$	$(x\pm k)^2(x-3a\pm k)^2(x-a\pm k)$
$x^2(x-3)^2(x-2)$ $x^5-8x^4+21x^3-18x^2$	$0,3,2$	$x(x-3)(x-1)(5x-12)$ $5x^4-32x^3+63x^2-36x$	$0,3,1,\frac{12}{5}$	$(x\pm k)^2(x-3a\pm k)^2(x-2a\pm k)$
$x^2(x-3)^2(x-7)$ $x^5-13x^4+51x^3-63x^2$	$0,3,7$	$x(x-6)(x-3)(5x-7)$ $5x^4-52x^3+153x^2-126x$	$0,6,3,\frac{7}{5}$	$(x\pm k)^2(x-3a\pm k)^2(x-7a\pm k)$
$x^2(x-3)^2(x+4)$ $x^5-2x^4-15x^3+36x^2$	$0,3,-4$	$x(x-3)(x+3)(5x-8)$ $5x^4-8x^3-45x^2+72x$	$0,3,-3,\frac{8}{5}$	$(x\pm k)^2(x-3a\pm k)^2(x+4a\pm k)$

With all the patterns that do work, it was too tempting not to try to make a linear link among the cubic, quartic, and quintic forms. Let us examine the following facts.

I. Cubic Form $x(x-5)(x-8)$	Smallest Root 0	Largest Root 8	Range 8	Sum of Roots 13
Derivative $(3x-20)(x-2)$	2	$\frac{20}{3}$	$\frac{14}{3}$	$\frac{26}{3}$
II. Quartic Form $x(x-3)(x-4)(x-7)$	Smallest Root 0	Largest Root 7	Range 7	Sum of Roots 14
Derivative $2(x-1)(2x-7)(x-6)$	1	6	$5=\frac{20}{4}$	$\frac{42}{4}$
III. Quintic Form $x(x-2)(x-3)(x-6)$	Smallest Root 0	Largest Root 6	Range 6	Sum of Roots 15
Derivative $5x^4-60x^3+240x^2-360x+144$	.616036…	5.383960…	4.767924…	$\frac{60}{5}$

We realize very quickly that we can come close to rational roots, but cannot obtain them as our constant term would have to be a multiple of 5 in order to be factorable, which is impossible in this situation.

Summary

If nothing else, the reader now has a partial list of cubic and quartic polynomials with multiple or single integral roots whose derivatives have multiple or single rational roots. The quintic avails itself to multiple but not to single roots.

I would never have attempted all this work without the user-friendly program DERIVE, as I was able to test many polynomials in seconds and quickly find derivative and corresponding factored forms. The same Diophantine process can be applied to rational forms, making life a little easier for the curve sketcher.

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