Edmonton Junior High Mathematics Contest

Multiple Choice Questions:

1. How many positive integers from 1 to 2008 are there which have no common divisor with 2008 greater than 1?

   (a) 996 (b) 1000 (c) 1004 (d) none of these

   Solution:
   Note that $2008 = 2^3 \times 251$ and 251 is a prime number. Of the 2008 positive integers in consideration, there are $2008 \div 2 = 1004$ multiples of 2 which are to be eliminated. There are also $2008 \div 251 = 8$ multiples of 251 which are to be eliminated. However, the $2008 \div 2 \div 251 = 4$ multiples of 2×251 have been eliminated twice, and must be added back. Hence the number of positive integers with the desired property is $2008 - 1004 - 8 + 4 = 1000$. The answer is **(b)**.

2. Let a, b and c be three different positive integers whose product is 16. What is the maximum value of $a^b - b^c + c^a$?

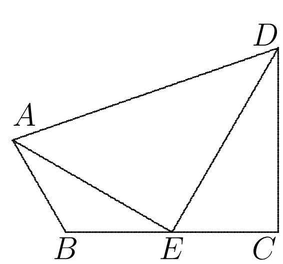
   (a) 63 (b) 65 (c) 249 (d) 263

   Solution:
   Clearly, the three numbers are 1, 2 and 8. The maximum value of c^a is 2^8. Hence the maximum value of $a^b - b^c + c^a$ is $8^1 - 1^2 + 2^8 = 263$. The answer is **(d)**.

3. In the quadrilateral $ABCD$, $AB = 1$, $BC = 2$, $CD = \sqrt{3}$, $\angle ABC = 120°$ and $\angle BCD = 90°$. E is the midpoint of BC. Find $\angle DEA$.

   (a) 60° (b) 90° (c) 120° (d) none of these

   Solution:
   Since $BE = \frac{1}{2}BC = 1 = AB$, we have $\angle BEA = \angle BAE$. From $\angle ABC = 120°$, we have $\angle BEA = \frac{1}{2}(180° - 120°) = 30°$. Since $\angle BCD = 90°$, it follows from Pythagoras' Theorem that $DE = \sqrt{CE^2 + CD^2} = \sqrt{1+3} = 2$. Hence CDE is half of an equilateral triangle, so that $\angle DEC = 60°$. Finally, $\angle AED = 180° - \angle BEA - \angle DEC = 90°$. The answer is **(b)**.



Answers Only Problems:

1. We write down the positive integers in order in a spiral pattern as follows. Start with 1 and put 2 to its left. Put 3 below 2, 4 to the right of 3 and 5 to the left of 4. Now 6 goes above 5 and 7 goes above 6. Then 8, 9 and 10 follow in that order to the left of 7 and so on. We are always making left turns, the first time at 2, the second time at 3, the third time at 5, the fourth time at 7, and so on. At which number will we be making the 20th left turn?

   Solution:
   The numbers at which the left turns are made lie on four diagonals, the fourth leading to the northeast. The 20th left turn will be the 5th one along this diagonal.

49	48	47	46	45	44	**43**
26	25	24	23	22	**21**	42
27	**10**	9	8	**7**	20	41
28	11	**2**	1	6	19	40
29	12	**3**	4	**5**	18	39
30	**13**	14	15	16	**17**	38
31	32	33	34	35	36	**37**

   The first three left turns are at 7, 21 and 43. In going from 7 to 21, we run through the numbers from 8 to 20. We can pair 8 and 9 with 23 and 24 above, 10, 11, 12 and 13 with 27, 28, 29 and 30 to the left, 14, 15 and 16 with 33, 34 and 35 below, and 17, 18, 19 and 20 with 38, 39, 40 and 41 to the right. In going from 21 to 43, we run through the numbers from 22 to 42. The eight numbers 22, 25, 26, 31, 32, 36, 37 and 42 have not been pairs. This shows that the gaps between 7, 21 and 43 widen at 8 at a time, starting at 14. It follows that the 4th left turn along this diagonal will be at the number 43+30=73 amd the 5th left turn at 73+38=111.

2. Let a, b and c be any positive integers. What is the maximum value of

$$\frac{1}{a}+\frac{1}{b}\left(1+\frac{1}{a}\right)+\frac{1}{c}\left(1+\frac{1}{a}\right)\left(1+\frac{1}{b}\right)-\left(1+\frac{1}{a}\right)\left(1+\frac{1}{b}\right)\left(1+\frac{1}{c}\right)?$$

   Solution:
   The expression is equal to

$$\begin{aligned}
& \frac{1}{a}+\left(1+\frac{1}{a}\right)\left(\frac{1}{b}+\left(1+\frac{1}{b}\right)\left(\frac{1}{c}-\left(1+\frac{1}{c}\right)\right)\right)\\
= & \frac{1}{a}+\left(1+\frac{1}{a}\right)\left(\frac{1}{b}-\left(1+\frac{1}{b}\right)\right)\\
= & \frac{1}{a}-\left(1+\frac{1}{a}\right)\\
= & -1.
\end{aligned}$$

   Since it is constant, its maximum value is -1.

3. A rectangular piece of paper $ABCD$ is such that $AB = 4$ and $BC = 8$. It is folded along the diagonal BD so that triangle BCD lands on top of triangle BAD. Let C' denote the new position of C, and let E be the point of intersection of AD and BC'. What is the area of triangle BED?

Solution:

Triangles BAD and $DC'B$ are congruent to each other. So triangles BAE and $DC'E$ are also congruent to each other. Let $AE = x$. By Pythagoras' Theorem, $BE = \sqrt{4^2 + x^2}$. We also have $BE = C'B - C'E = 8 - x$. Squaring both expressions and equating them, we have $16 + x^2 = 64 - 16x + x^2$, which simplies to $16x = 48$ and $x = 3$. Hence the area of triangle BED is $\frac{1}{2}DC \cdot DE = 2(8 - 3) = 10$.