Edmonton Junior High Math Contest 2004 Solutions

Multiple-choice Problems

1. The number of integers between $\sqrt{13}-3$ and $\sqrt{32}-2$ is

(a) 1 (b) 2 (c) 3 (d) 4

(c) We have $3<\sqrt{13}<4$ so that $0<\sqrt{13}-3<1$. Similarly, $3<\sqrt{32}-2<4$. Hence the integers which lie between $\sqrt{13}-3$ and $\sqrt{32}-2$ are 1, 2 and 3.

2. For a letter weighing up to 20 grams, the postage is 3 cents per gram. The postage for each gram beyond 20 grams but up to 40 grams is 2 cents. The postage for each gram beyond 40 grams is 1 cent. The postage for a letter weighing 45 grams is

(a) \$1.05 (b) \$1.10 (c) \$1.25 (d) \$1.35

(a) The first 20 grams cost 60 cents, the next 20 grams 40 cents and the final 5 grams cost 5 cents.

3. Eve claimed to have found n integers whose sum is 0 and whose product is n. Ace said, "I do not think there is such an n." Bea said, "Yes, there is, and n must be odd." Cec said, "Actually, n must be even." Dee said, "No, n can be either odd or even." The one who was right was

(a) Ace (b) Bea (c) Cec (d) Dee

(c) The example 2(-2)1(-1)=4 shows that Ace and Bea are both wrong. To show that Cec is right and Bea is wrong, consider any odd value of n. If there are n integers with product n, then each integer must be odd. However, the sum of an odd number of odd integers must be odd, and cannot be equal to 0.

4. The product of two numbers is equal to the product of two other numbers. Three of these four numbers are 3, 4 and 12. The number of possible values of the fourth number is

(a) 1 (b) 2 (c) 3 (d) 4

(c) We have 12x4 = 3x16, 12x3=4x9 and 4x3=12x1. Hence the fourth number can be any of 16, 9 or 1.

5. In the first round of World Cup soccer, there are four teams in each group. Every team plays each of the other three once, getting 3 points for a win, 1 point for a draw and 0 points for a loss. The top two teams in each group advance to the second round. The minimum number of points, which will guarantee advancement without getting involved in any tiebreakers, is

(a) 3 (b) 6 (c) 7 (d) 9

(c) A team with 7 points must have beaten two other teams, each of which can have at most 6 points. Thus advancement is guaranteed. We may have three teams each with 6 points, having drawn with one another and beaten the fourth team. Thus advancement is not guaranteed.

6. A merchant bought 3 pigs in a poke for x dollars each, and later bought 2 more pigs in a poke for y dollars each. When all 5 were sold at $\frac{x+y}{2}$ dollars each, the merchant lost money. The reason for this was

(a) $x<y$ (b) $x=y$ (c) $x>y$ (d) unrelated to the values of x and y

(c) The total cost was 3x+2y dollars while the total sale was $\frac{5x+5y}{2}$ dollars. Since it was a loss, $3x+2y>\frac{5x+5y}{2}$, which simplifies to $x>y$.

7. A non-zero digit is added between two adjacent digits of 2004 to obtain a multiple of 7. The number of possible values of this digit is

(a) 1 (b) 2 (c) 3 (d) 4

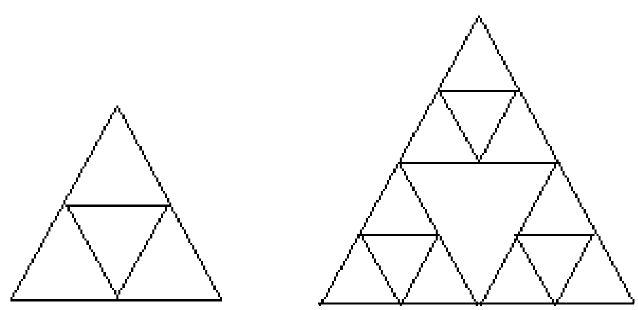
(d) Since 21 and 4004 are multiples of 7, the digit to be added between 2 and 0 must be 5. Since 203 and 504 are multiples of 7, the digit to be added between 0 and 0 can be 8, but it can also be 1. Finally, since 2002 and 14 are multiples of 7, the digit to be added between 0 and 4 must be 3.

8. The price of nuts in Walmart is $\frac{1}{3}$ that in Cocomart, and $\frac{7}{10}$ that in Doughmart. The price of nuts in Cocomart is some fraction of that in Doughmart. This fraction is

(a) $\frac{3}{7}$ (b) $\frac{10}{21}$ (c) $\frac{21}{10}$ d) $\frac{7}{3}$

(c) This fraction is $\frac{7}{10} \div \frac{1}{3} = \frac{21}{10}$.

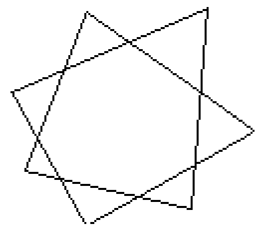
9. A sequence of diagrams is constructed as follows. The first diagram is an equilateral triangle pointing up. Each subsequent diagram is obtained from the preceding one by dividing each equilateral triangle pointing up into four congruent equilateral triangles. The second and third diagrams are shown below. There are respectively 5 and 17 triangles in them, counting overlapping ones. The total number of triangles in the fifth diagram is
(a) 122 (b) 125 (c) 134 (d) 161



(d) In each diagram, each equilateral triangle pointing up is subdivided into three equilateral triangles pointing up and one pointing down. Hence the number of triangles in the fourth diagram is $17+4\times3^2=53$ and that in the fifth diagram is $53+4\times3^3=161$.

10. The diagram below shows a star polygon with seven sides. The total measure of its seven angles is

(a) 360° (b) 540° (c) 720° (d) 900°



(b) Consider the seven triangles on the outside. The sum of their angles is 7x180° =1260°. Taking away the seven angles of the star-polygon, we have seven pairs of equal angles whose sum is twice the sum of the exterior angles of the central polygon with seven sides. Since the sum of its exterior angles is 360°, the sum of the seven angles of the star polygon is 1260°-2x360° =540°.

Answers-only Problems

1. A video tape costs $10 and sells at $16 for a 60% profit. A video disc sells for a 40% profit. If we sell one and a half times as many video tapes as video discs, the combined profit is 50%. How much does a video disc cost?

($15) Let a video disc cost 10x dollars. We may assume that 3 video tapes and 2 video discs are sold. The total cost is 30+20x dollars while the profit is 18+8x dollars. From 2(18+8x)=30+20x, we have $x=\frac{3}{2}$.

2. In the expression $\frac{k^2}{k^2-10k+50}$, a student put in k=1,2,3…,9 What is the sum of the nine numbers so obtained?

(9) Putting k=1, we have $\frac{1}{41}$. Putting k=9 , we have $\frac{81}{41}$. The sum of these two numbers is 2. Similarly, the sum of the numbers obtained when we put k=2 and k=8 is 2. The same goes for k=3 and k=7, as well as for k=4 and k=6. Putting k=5, we have 1 .

3. What is the sum of all positive integers each equal to four times the sum of its digits?

(120) Four times the sum of three digits is at most 108. For this to even reach 100, the smallest digit must be at least 7. Hence a number equal to four times the sum of it has at most two digits. In fact, it must have exactly two digits. Let such a number be 10x+y where x and y are digits. From 10x+y=4(x+y), we have y=2x. It follows that there are exactly four such numbers, namely, 12, 24, 36 and 48.

4. Two birds in hand plus one bird in the bush are worth $50. Three birds in hand plus two birds in the bush are worth $80. How much are four birds in hand plus three birds in the bush worth?

($110) Let a bird in hand be worth h dollars and a bird in the bush b dollars. From 2h+b=50 and 3h+2b=80 , we have 4h+3b=2(3h+2b)-(2h+b)=2(80)-50=110 .

5. The difference of two positive integers is 120. Their least common multiple is 105 times their greatest common divisor. What is their greatest common divisor?

(15) Let the greatest common divisor be d, and let the two numbers be da and db, so that a and b have no common divisors greater than 1. Then the least common multiple of da and db is dab, from which we have $ab=105$. The four possibilities are $(a,b)=(105,1)$, (35,3), (21,5) and (15,7). Then a-b=104, 32, 16 and 8 respectively. Since it must divide 120, we have a-b=8, so that d= $\frac{120}{a-b}$=15.

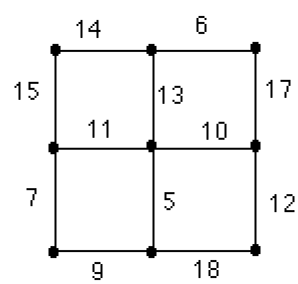
6. One hundred students in a school line up in n rows for a year-book picture, where n > 1. Several students are in the first row. Each row starting with the second contains one student more than the preceding row. There are no students left over. What is the number of possible values of n?

(2) Suppose the first row has m students. Then the subsequent rows have m+1, m+2, … , m+n-1 students. The total number is $\frac{n(2m+n-1)}{2}$=100. Note that 2m+n-1 is odd if and only if n is even, and their product is 200. Now 200 has only two odd divisors greater than 1, namely 5 and 25. Hence we may have n=5 or n= $\frac{200}{25}$=8 since n≤2m+n-1.

7. Eddy picked a two-digit number, switched the two digits to get a new number, and subtracted the smaller of the two numbers from the larger one. The difference is the cube of Eddy's age. How old is Eddy?

(3) Let $10x+y$ be the number where x and y are its digits. Then the new number is $10y+x$ and their difference $9|x-y|$ is a cube with at most two digits. Since the difference is a multiple of 9, the cube must be 27.

8. The diagram below shows nine towns connected by twelve highways. There is a robber on each highway, and the number indicates the amount of money one must pay to travel along that highway. What is the lowest amount one has to pay in highway robbery in order to go from the town at the northwest corner to the town at the southeast corner?

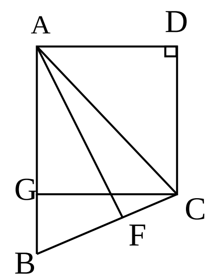


(48) We compute the minimum cost in the chart below. If we stay home in NW, the cost is 0. It costs 14 to move over to N. and another 6 to move onto NE for a total cost of 20. Similarly, it costs 15 to move down to W, and 22 to reach SW. We can arrive at C in two ways, via N or W. The former costs 14+13=27 while the latter costs 15+11=26. Obviously, we choose the lower cost. The remaining entries are obtained by comparing two options in the same manner.

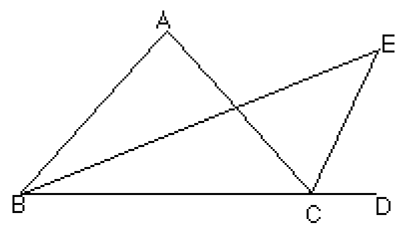
NW=0	N=14	NE=20
W=15	C=26	E=36
SW=22	S=31	SE=48

9. In the quadrilateral ABCD, both AB and CD are perpendicular to AD. AB=13, CD=8 and AD = 12. F is a point on BC such that AF is perpendicular to BC. Find the length AF.

(12) Complete the rectangle CDAG. Then CG=AD=12 and BG=AB-CD=5. By Pythagoras' Theorem, BC=$\sqrt{12^2+(13-8)^2}$ = 13. AC = $\sqrt{12^2+(8)^2}=\sqrt{208}$. Since ABCD is a trapezoid, the area is $\frac{(8+13)(12)}{2}=126$. Area of triangle ADC = $\frac{12x8}{2}=48.$ It follows that area of triangle ABC = 126-48 = 78. As well, AF is perpendicular to BC, then using $Area = 78 = \frac{13(AF)}{2}$. $AF = 12.$



10. In triangle ABC, ∠CAB=90°. D is a point on the extension of BC. The bisectors of ∠ABC and ∠ACD meet at E, as shown in the diagram below. What is the measure of ∠BEC?



(45°) We have ∠BEC=∠ECD-∠EBC=0.5(∠ACD-∠ABC)=0.5∠CAB.