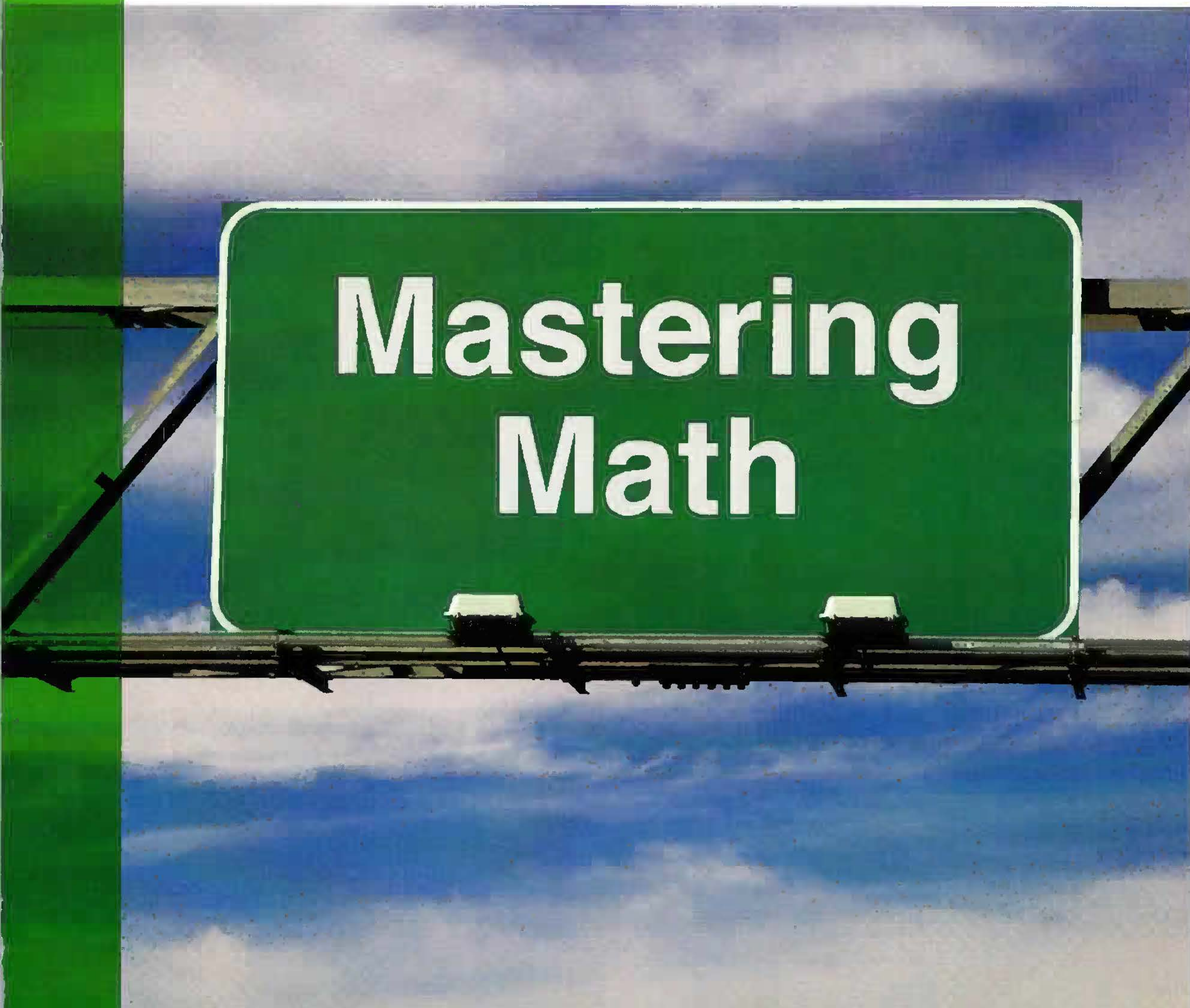
delta-k

Journal of the Mathematics Council of the Alberta Teachers' Association

Volume 53, Number 2 September 2016



Mathematical ideas = Springboard to learning

Guidelines for Manuscripts

delta-K is a professional journal for mathematics teachers in Alberta. It is published twice a year to

- promote the professional development of mathematics educators, and
- stimulate thinking, explore new ideas and offer various viewpoints.

Submissions are requested that have a classroom as well as a scholarly focus. They may include

- personal explorations of significant classroom experiences;
- descriptions of innovative classroom and school practices;
- reviews or evaluations of instructional and curricular methods, programs or materials;
- discussions of trends, issues or policies;
- a specific focus on technology in the classroom; or
- a focus on the curriculum, professional and assessment standards of the NCTM.

Suggestions for Writers

1. *delta-K* is a refereed journal. Manuscripts submitted to *delta-K* should be original material. Articles currently under consideration by other journals will not be reviewed.
2. If a manuscript is accepted for publication, its author(s) will agree to transfer copyright to the Mathematics Council of the Alberta Teachers' Association for the republication, representation and distribution of the original and derivative material.
3. Peer-reviewed articles are normally 8–10 pages in length.
4. All manuscripts should be typewritten, double-spaced and properly referenced. All pages should be numbered.
5. The author's name and full address should be provided on a separate page. If an article has more than one author, the contact author must be clearly identified. Authors should avoid all other references that may reveal their identities to the reviewers.
6. All manuscripts should be submitted electronically, using Microsoft Word format.
7. Pictures or illustrations should be clearly labelled and placed where you want them to appear in the article. A caption and photo credit should accompany each photograph.
8. References and citations should be formatted consistently using *The Chicago Manual of Style*'s author-date system or the American Psychological Association (APA) style manual.
9. If any student sample work is included, please provide a release letter from the student's parent/guardian allowing publication in the journal. The editor will provide this letter on request.
10. Letters to the editor, description of teaching practices or reviews of curriculum materials are welcome.
11. Send manuscripts and inquiries to the editor: Lorelei Boschman, c/o Medicine Hat College, Division of Arts and Education, 299 College Drive SE, Medicine Hat, AB T1A 3Y6; e-mail lboschman@mhc.ab.ca.

MCATA Mission Statement

Providing leadership to encourage the continuing enhancement of teaching, learning and understanding mathematics.

delta-K

Volume 53, Number 2 September 2016

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 delta-K is published by the ATA for the Mathematics Council (MCATA). EDITOR: Lorelei Boschman, c/o Medicine Hat College, Division of Arts and Education, 299 College Drive SE, Medicine Hat, AB T1A 3Y6; e-mail lboschman@mhc.ab.ca. EDITORIAL AND PRODUCTION SERVICES: Document Production staff, ATA. Opinions expressed herein are not necessarily those of MCATA or of the ATA. Address correspondence regarding this publication to the editor. *delta-K* is indexed in CBCA Education. ISSN 0319-8367

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From the Editor's Desk

Lorelei Boschman

One thing is certain as I travel through the landscape of mathematics teaching, learning and discourse with other educators—we have a large group of people dedicated to students' mathematical development and understandings. I do not say this lightly. Over the last 27 years of mathematics teaching and learning focus, I have met many educators with this at the forefront of their thinking. I have observed vast amounts of dedication and effort right from kindergarten to Grade 12 (and beyond) from educators who develop lessons, activities, assessments and adaptations with the sole purpose of stretching and enhancing students' mathematical understandings of the world.

It is an admirable cause. When we consider that mathematics weaves its way through so much of our life, we need to develop a fluency with it. When we understand that our students absolutely need the essential reasoning and problem-solving skills that mathematics develops, we are spurred onwards.

So what do I see as some of the emerging main focuses within mathematics education? As educators in 2016, we recognize that our students today are growing up in a somewhat different world than we grew up in. There are parts that are the same, such as work ethic, effort and practice, among others; however, we are increasingly realizing the importance of engaging our learners, and research has shown the value of this engagement with respect to learning, retention, application and attitudes. We are also furthering the discussion of assessment in the classroom, specifically formative assessment, as well as putting it into practice with tremendous results. Cooperative learning is presenting educators with the benefits of increased participation and learning practice for students. As math educators, we are increasingly analyzing our lessons and resources as well as thinking about how and why we are doing what we do, trying to make it the best learning environment for our students, all the while considering the complex world in which we live.

We do this always with our students in mind. I see teachers creating engaging lessons, activities and assessments; I see students excited about mathematics, solving problems with their inquiry and mathematical process skills; and I feel proud of who we are and what we do. The effort that mathematics educators are putting in to keep current within a changing educational landscape is admirable. Leading-edge thoughts and ideas propel us to excellent instructional ideas and methods that shape our true beneficiaries, the students.

This journal will present articles exploring much of the above. In the "Conversation Starters" section, you will enter into thinking about the process of grading as well as ideas for the engagement of students. "Research Articles" will present ways to analyze resources for effectiveness and depth as well as the "productive struggle" of problem solving and mathematical mastery that is so important to a person's mathematical development and learning. "Teaching Ideas" include using an attribute-based game idea, cooperative learning structures that would fit any level of classroom and an example of a cross-curricular application of a topic. The combined Mathematics/Science Conference 2015 highlights are also included, as well as a book review, problem-solving moments and the NCTM Illuminations website highlight.

Last, thanks to all of you who are the face of mathematics education in our landscape. Continue to have this discourse and create these opportunities for mathematical learning and to build on the expertise that so many of you exhibit daily in your teaching. I am very proud of our landscape!

How I Abolished Grading

David Martin

Here is the story of one teacher who abolished grading in a high school calculus class.

I started teaching high school calculus at my school a couple of years ago. When I started teaching the course, I used a traditional assessment strategy: I would assign homework daily, end the week with a quiz and then end the unit with a multiple choice/ written exam.

My classes would start with around 30 students, and by the end of the semester the class size would be 20. What I did was weed out the weak. One day I realized that I wasn't weeding out the weak mathematicians, but instead weeding out the weak test writers.

This year, after many talks with first-year university and college professors, administrators, teachers, students, and parents, I am proud to say that I have abolished grading. We are currently in the middle of our semester and I have not graded a single item of student work.

Before you continue, I want to remind you that this does not mean I have not assessed, but not one student in my calculus classes has received a grade at this point (other than the report card mark, which I must give).

How does it work?

First, I went through my outcomes, given to me by the government, and identified what the "rocks" are. These rocks are the outcomes that I expect the students to master above all other outcomes. I chose these particular outcomes after my discussions with others and considering what will be helpful for students to succeed in the future.

Next, these outcomes were rewritten in student-friendly language and then provided to the students on the first day of class.

My teaching schedule did not change, nor did the speed at which I have taught the course; what has changed is the speed at which the students can learn. Once I had taught two or three outcomes to a level at which I felt that the class had mastered the outcome, I administered a summative assessment. Each child wrote it as a traditional exam, but it looked drastically different from a traditional exam. Each assessment was entirely written, broken up by outcomes and tested only the basics of the outcomes. There were no trick questions, just simple questions that would assess "Can the child demonstrate this outcome, on his or her own, at a basic level of understanding?"

What has changed is the speed at which the students can learn.

When I assessed these assessments, I would write only comments on them, and either *Outcome demonstrated* or *Need to learn* for each outcome assessed (not on the overall assessment). It is very important to understand that "Outcome demonstrated" is not a

100 per cent mark, as a student could make a minor mistake and still achieve this—I am assessing understanding the outcome, not perfection.

Next, if the child received a "Need to learn," he or she had to

1. demonstrate the understanding of the questions given at a later date; this usually occurred after a lunch session, a quick conversation or multiple conversations with the child;
2. hold a conversation explaining how he or she made the mistake earlier and how the student's understanding had changed; and
3. write another assessment on the outcomes.

If, after completing these three steps, the student could demonstrate the outcomes, I would count this as "Outcome demonstrated," just as if the child had done it the first time. I do not deduct marks based on the number of tries needed.

If the child still does not demonstrate understanding (which, I have seen, is extremely unlikely), then he or she must repeat the same three steps.

After five to seven outcomes have been taught, each child is assigned an open-ended project. This project consists of each student creating a problem illustrating the math in the five to seven outcomes and solving it. The expectation is that the problem is one that is deep, relevant and for a purpose. This part is not always easy! For example, a student, to demonstrate his understanding, created a Call of Duty video and determined the rate of change of a ballistic knife falling in the video.

These projects usually range from three to five pages and must be handed in individually, but can be worked on with assistance from others and/or textbooks.

To assess these projects, I follow the same pedagogy noted above. I use comments only, and give guidance regarding any errors I see. The projects are then handed back to each student, who can go back, make corrections, and resubmit it. This process is repeated until the child achieves perfection on the project.

I have even abolished the traditional final exam. The expectation now is that the students must give me a 30- to 45-minute presentation about the rocks of the course and demonstrate their understanding of all the rocks.

How do I get a final mark percentage?

I simply take the number of outcomes and projects completed (at the end of the course) and divide by the total number of outcomes and projects. This may not be the best strategy, but it seems to work for me right now. I do weight projects twice as much—I have 20 outcomes and 5 projects, so the total is $(20 + 5 \times 2 = 30)$.

Below is my updated list of rocks.

Determine solutions to P(x)>0	Demonstrate the product, and chain, quotient rule, and implicit diff using various functions	Determine the area between curves (or the x-axis) over a given interval
Computing limits using theorems and calculator	Draw a function using derivatives	Determine the antiderivative (both definite and indefinite) of Trig and rational functions, both with and without u substitution
Explain the concept of a limit	Solve an optimization problem	Demonstrate the integral properties of sum, difference and multiplication by a constant to integrals, as well as switching the bounds
Explain continuous vs discontinuous, and sketching	Determine points of inflection, both graphically and algebraically	Determine the area between curves over a given interval
Show, with an explanation, that the slope of a tangent line is a limit	Determine the area between curves (or the x-axis) over a given interval	Calculate the mean value of a function over an interval
Determine the equation of tangent lines on a curve	Link displacement, velocity and acceleration of an object moving with nonuniform acceleration with trig, rational and poly functions	

David Martin has been a teacher for nine years. He has a master of mathematics from University of Waterloo, and is currently a division math and science lead teacher. Over the last six years, he has been part of the Math Council of the Alberta Teachers' Association. Zombies, prime numbers and scary movies are only some of his many interests. For more of David's blogs please visit http://realteachingmeansreallearning.blogspot.ca.

Math That Feels Good

George Gadanidis

This article was originally published in CMS Notes *47, no 6 (December 2015) and is reprinted with the permission of the Canadian Mathematical Society. Minor amendments have been made in accordance with ATA style.*

For many young people, mathematics feels sterile and disconnected from the world. In this edition of "Education Notes," George Gadanidis provides several ideas and resources for connecting abstract mathematics to concrete activities that children can tell stories about. These activities allow children to talk about mathematics and stay engaged.

Despite popular views to the contrary, school math can be an aesthetic experience, full of surprise, insight and beauty. What's holding us back?

Children begin their lives as eager and competent learners.

Kids these days!

One obstacle is our negative view of what young children are capable of. We remember the good old days, when kids worked hard, were polite, paid attention and knew their math facts. And, of course, in those days we walked 10 miles to school and back, uphill both ways, and we never complained. If only the new generation measured up. Maybe then we could do some cool math with them, instead of the basic skills they seem to lack.

But if we step back and look at the generations that precede ours, we realize that our parents had the same views of us, and their parents of them and so forth. Daniels (1983) documents this generational pattern as far back as ancient Sumeria. So it's not surprising that as adults we are attracted to educational theories of what children cannot do, such as Piaget's stages of cognitive development, which "absolutely dominate in education" (Egan 2002, 105).

Papert (1980), who worked with Piaget, disagrees with the linear progression of his developmental stages, suggesting that it does not exist in children's minds but in the learning culture we create for them. "Children begin their lives as eager and competent learners. They have to learn to have trouble with learning in general and mathematics in particular" (Papert 1980, 40). Dienes, in an interview with Sriraman and Lesh (2007), comments that "Children do not need to reach a certain developmental stage to experience the joy, or the thrill of thinking mathematically and experiencing the process of doing mathematics" (p 61). Egan (1997), Fernandez-Armesto (1997) and Schmittau (2005) challenge Piaget's notion that young children are not capable of abstract thinking, which Egan identifies as an integral element of language development.

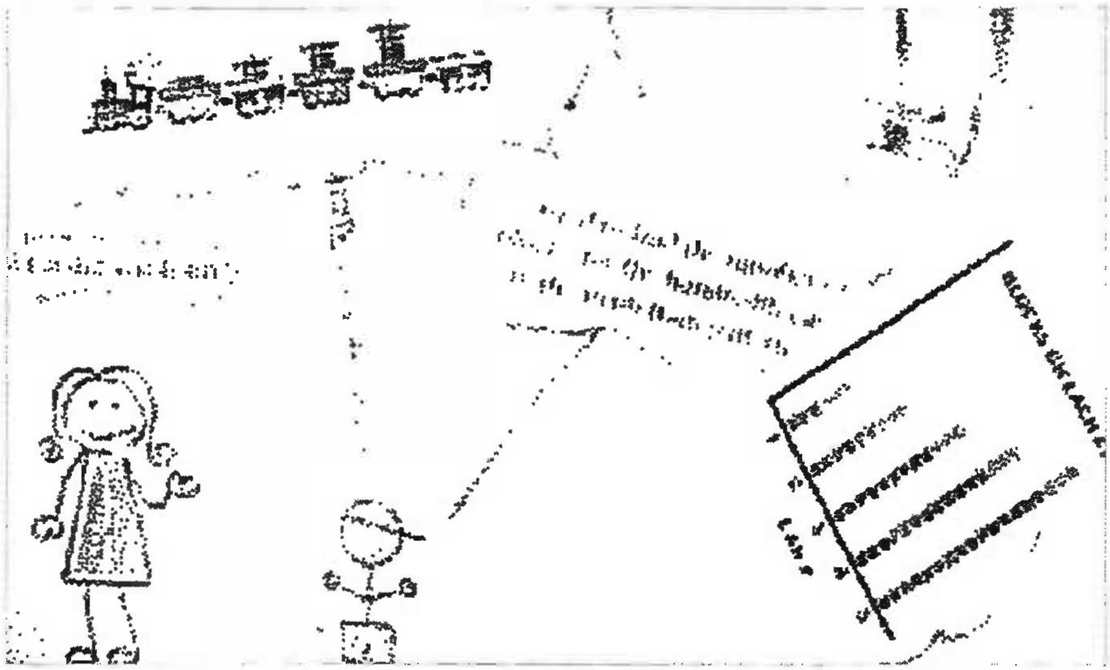
Movshovitz-Hadar (1994) notes the need for nontrivial mathematical relationships in eliciting mathematical surprise, and Gadanidis, Hughes and Cordy (2011) point to challenging mathematics as a corequisite for aesthetic mathematics experience.

Tell Me a Math Story

A second obstacle to school math as an aesthetic experience is that we have not developed a capacity for framing math ideas as stories that can be shared beyond the classroom. When I ask parents what their children say when asked "What did you do in math today?" the common responses are "Nothing," "I don't know" or the mention of a math topic, like fractions.

Story is not a frill that we can set aside just because we have developed a cultural pattern of ignoring it in mathematics.

Story is not a frill that we can set aside just because we have developed a cultural pattern of ignoring it in mathematics (Gadanidis 2012). Story is a biological necessity, an evolutionary adaptation that "train(s) us to explore possibility as well as actuality, effortlessly and even playfully, and that capacity makes all the difference" (Boyd 2009, 188). Story makes us human and adds humanity to mathematics. Boyd (2001)



Grades 1–2 students use comics to share with parents their learning about linear functions.

notes that good storytelling involves solving artistic puzzles of how to create situations where the audience experiences the pleasure of surprise and insight. Solving such artistic puzzles in mathematics pedagogy results in tremendous pleasure for students, teachers, parents and the wider community (Gadanidis 2012).

What Did You Do in Math Today?

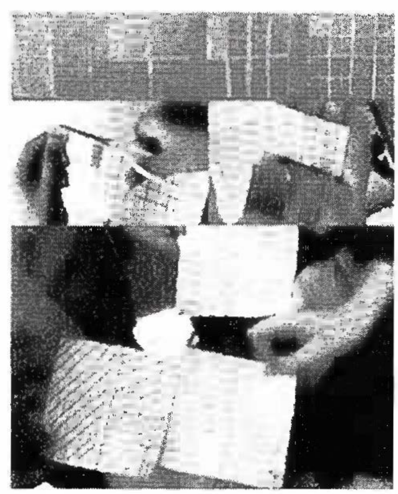
For several years I have been spending 50 to 60 days annually in elementary school classrooms, collaborating with teachers to develop aesthetic experiences for young mathematicians.

Here's how we work together:

1. We start with teacher needs. For example, when teachers asked for help with teaching area representations of fractions, we collaborated to develop an activity that covered this topic in the context of infinity and limit (see figure at right).
2. We don't change the curriculum. We simply add a richer mathematical context for teaching mandated content.
3. Our pedagogical goal is to prepare students to share their learning with family and friends in ways that offer mathematical surprise and insight, emotional engagement, and visceral sensation of mathematical beauty.

We seek to occasionally (say, once a unit) create mathematics experiences worthy of attention, worthy of conversation, worthy of children's incredible minds, which thirst for knowledge and for opportunities to explore, question, flex their imagination, discover, discuss and share their learning.

To the right are lyrics to a song that shares parent comments after Grade 3 children shared their learning of circular functions. You can view an animated music video of the Grade 3 students singing this song at http://researchideas.ca/wmt/c2d4.html.



Grades 2–3 students in Canada and Brazil discover that the infinite set of fractions 1/2, 1/4, 1/8, 1/16 and so forth fit in a single square, and share the surprise "I can hold infinity in my hand!"

Building Capacity

Toward the goal of "math that feels good," with funding from SSHRC, KNAER, the Fields Institute and Western's Teaching Support Centre, we have been developing online resources that publicly share ideas from research classrooms. Below are some examples. See more at http://researchideas.ca.

1. ***What will you do in math today?*** (www.researchideas.ca/wmt)—This resource shares mathactivities from classroom-based research. It is used as a classroom resource by teachers, for professional development and for mathematics teacher education courses. It includes lesson development, interactive content, simulations, interviews with mathematicians working on the same math tasks and classroom documentaries.
2. ***Math e-cards*** (www.researchideas.ca/randomacts)—This online tool allows you to share short videos of the math surprises in the above resource as math e-cards. Teachers can send these to parents to inform them of what their children are studying. They can also be used to share cool math ideas more widely.
3. ***Short courses for teachers*** (www.researchideas.ca/wmt/courses.html)—In collaboration with the Fields Institute, we offer short courses for teachers on number, pattern and algebra; measurement and geometry; and data and probability. The courses are freely available. Teachers can register and receive certificates of completion for a minimal fee of $30/course, or school districts can use these courses to offer their own certificates of completion.
4. ***Math + Coding***—We have been exploring the intersection of coding and mathematics education as another way to model, investigate and experience mathematical beauty The following are some resources we have made available:
   a. *Math + Coding 'Zine* (www.researchideas.ca/mc)—an online magazine offering ideas for incorporating coding in mathematics teaching and learning
   b. Math+ Coding Events (www.researchideas.ca/coding-events)—a Fields-funded project that offers support for organizing student-led math + coding community events
   c. Math+ Coding Symposium (www.researchideas.ca/coding)—videos of keynotes by Celia Hoyles, Yasmin Kafai and Richard Noss at a recent symposium funded by Fields and SSHRC
   d. Math + Coding Resources (http://researchideas.ca/mathncode)—math + coding simulations, games and more.
5. ***Math Music*** (www.researchideas.ca/jx)—Funded by the Fields Institute, we have been performing math songs from research classrooms for elementary schools across Ontario. Songs and music videos are available at this website.

Dots, Clocks and Waves

my daughter explained
how to conduct experiments
and make bar graphs
plotting the results

she was amazed
by the wave pattern
excited to explain it
to her brothers at home

a dot on a car tire
makes a wave pattern
at first I thought
it would be a spiral

the wave pattern
is still there
even if the wheels
even if they are square

it's great to see my
son excited
about school and about math
it's great to see enthusiasm
and interest in school math

my son enjoyed
testing his hypothesis
he was surprised
surprised by the result

he shared his comics
of what he learned
about math waves
on tires and clocks

the height of every hour
on a grandfather clock
plotted on a bar graph
makes a wave shape

like the height of a dot
on a rolling tire
or seasonal temperatures
or sunrise and sunset times

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References

Boyd, B. 2001. "The Origin of Stories: Horton Hears a Who." *Philosophy and Literature* 25, no 2: 197–214.

———. 2009. *On the Origin of Stories: Evolution, Cognition, and Fiction.* Cambridge, Mass: Harvard University Press.

Daniels, H. 1983. *Famous Last Words: The American Language Crisis Reconsidered.* Carbondale, Ill: Southern Illinois University Press.

Egan, K. 1997. *The Educated Mind: How Cognitive Tools Shape Our Understanding.* Chicago, Ill: University of Chicago Press.

———. 2002. *Getting It Wrong from the Beginning: Our Progressive Inheritance from Herbert Spencer, John Dewey, and Jean Piaget.* New Haven, Conn: Yale University Press.

Fernandez-Armesto, F. 1997. *Truth: A History and a Guide for the Perplexed.* London: Bartam.

Gadanidis, G. 2012. "Why Can't I Be a Mathematician?" *For the Learning of Mathematics* 32, no 2: 20–26.

Gadanidis, G, J Hughes and M Cordy. 2011. "Mathematics for Gifted Students in an Arts- and Technology-Rich Setting." *Journal for the Education of the Gifted* 34, no 3: 397–433.

Movshovitz-Hadar, N. 1994. "Mathematics Theorems: An Endless Source of Surprise." *For the Learning of Mathematics* 8, no 3: 34–40.

Papert, S. 1980. *Mindstorms: Children, Computers, and Powerful Ideas.* New York: Basic Books.

Schmittau, J. 2005. "The Development of Algebraic Thinking – A Vygotskian Perspective." *ZDM* 37, no 1: 16–22.

Sriraman, B, and R Lesh. 2007. "A Conversation with Zoltan P. Dienes." *Mathematical Thinking and Learning* 9, no 1: 57–75.

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Comments from Your Executive

The article by George Gadanidis brings up an interesting point about how we view "kids these days." I believe that it is important that teachers maintain a positive outlook on the students sitting in their classroom. A teacher cannot write off a room full of kids and their mathematical abilities because of a widespread belief that they are not hard working, or that they are not able to tackle hard problems because of a lack of ability. This article shows that students at any age can be introduced to complicated ideas like infinity through the use of fractions and a simple piece of paper. It is these types of topics that are going to inspire curiosity, which will lead to more mathematical exploration and a sense of excitement. In my opinion, this excitement is t"g through some of the resources listed at the end of the article. As a computer science major, I am constantly looking for ways to introduce coding into the math class, so I was really excited to see a host of resources related to this topic. To me, coding and math go hand in hand. The type of thinking that goes into coming up with a successful algorithm is the same thought process as solving a complicated math problem. Having students involved with this type of problem will have them excited to get to math class. If more students came home from school excited to talk about what they did in math class, I am sure we would see an impressive improvement in students' mathematical abilities.

Matthew McDonald is a recent graduate of the Werklund School of Education at the University of Calgary. He received a bachelor of education with a concentration in secondary mathematics. Outside of education, Matthew has a passion for musical theatre and competitive curling. Recently he received his first temporary contract and has been teaching Grade 10 mathematics in Calgary.

Internet Sensations

There have been a few viral math Internet sensation questions in the past year. Try your problem-solving skills here with a couple of them. (Answers are provided at the websites listed in the footnotes.)

1. The first problem is from Britain's 2015 General Certificate of Secondary Education (GCSE) exam, which is given to 16-year-olds.[1]

   There are n sweets in a bag.

   Six of the sweets are orange.

   The rest of the sweets are yellow.

   Hannah takes at random a sweet from the bag.

   She eats the sweet.

   Hannah then takes at random another sweet from the bag.

   She eats the sweet.

   The probability that Hannah eats two orange sweets is 1/3.

   (a) Show that n^2 - n - 90 = 0.

   (b) Solve n^2 - n - 90 = 0 to find the value of n.

1. www.bbc.com/news/blogs-trending-33029606

2. The second problem is from Singapore and is said to be given to high-school-aged math students.[2]

   Albert and Bernard just met Cheryl. "When's your birthday?" Albert asked Cheryl.

   Cheryl thought a second and said, "I'm not going to tell you, but I'll give you some clues." She wrote down a list of 10 dates:

   May 15, May 16, May 19

   June 17, June 18

   July 14, July 16

   August 14, August 15, August 17

   "My birthday is one of these," she said.

   Then Cheryl whispered in Albert's ear the month—and only the month—of her birthday. To Bernard, she whispered the day, and only the day.

   "Can you figure it out now?" she asked Albert.

   Albert said, "I don't know when your birthday is, but I know Bernard doesn't know, either."

   Bernard said, "I didn't know originally, but now I do."

   Albert said, "Well, now I know, too!"

   When is Cheryl's birthday?

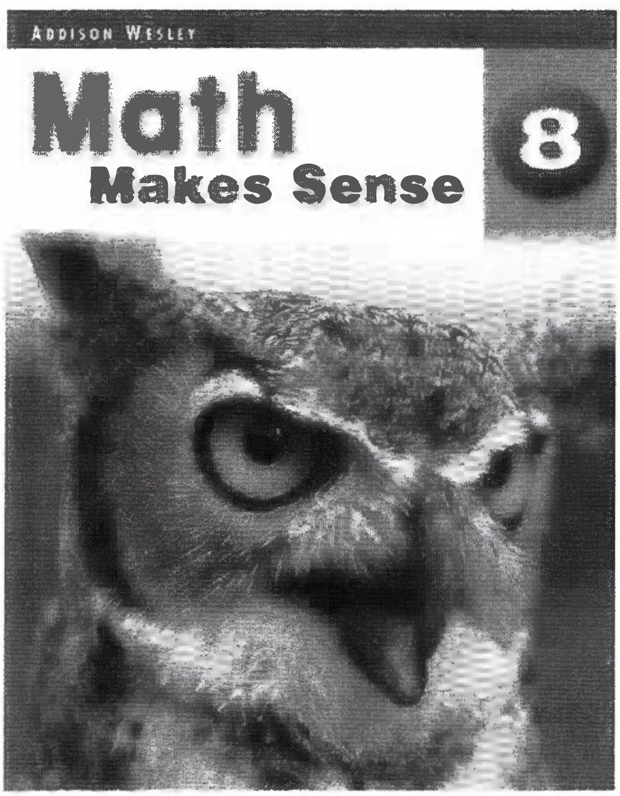
2. www.bdcwire.com/singapore-math-problem-solution

Evaluation of the Tasks from *Math Makes Sense 8*: Focusing on Equation Solving

Xiong Wang

Abstract

Mathematics tasks have been regarded as important for promoting students' understanding in classroom teaching. However, there has not been much research that has closely examined tasks from textbooks or supporting resources that teachers use in daily teaching. This paper aims to evaluate the tasks from the math textbook *Math Makes Sense 8* (Baron et al 2008), which has been adopted in Alberta, Canada. The tasks in question are selected from two lessons, Solving Equations Using Models and Solving Equations Using Algebra, in *Math Makes Sense 8* (6.1 and 6.2), and the corresponding sections in the practice and homework book and the ProGuide for teachers, and are analyzed with three kinds of analysis framework. In order to understand the knowledge level involved in the tasks, the standards for scoring assignments have been used for reference and modified into a knowledge-level framework. In addition, the levels of cognitive demands have been employed as an examiner of the cognitive demands required in the tasks. Finally, the factors associated with the maintenance of high-level cognitive demands have been applied to verify the questions and strategies provided in the ProGuide. The results show that the tasks are eventually coded as submedium knowledge level, medium-level cognitive demands and higher-level knowledge communication, and that scaffolding students' thinking and reasoning is a major factor in the supportive strategies provided for teachers.



Engaging students in mathematical thinking and reasoning has been recognized and held in esteem by many researchers.

Background

Mathematics tasks have been regarded as important vehicles for promoting students' understanding in classroom teaching; for example, Hiebert et al (1997) identify tasks as the core component of classroom teaching. As for the role of mathematics tasks, engaging students in mathematical thinking and reasoning has been recognized and held in esteem by many researchers (Cai and Lester 2005; Stein et al 2000). In fact, in daily teaching practice, teachers rely heavily on textbooks (Pepin and Haggarty 2001) and supporting materials such as students' workbooks and teachers' manuals to manage their teaching. However, analysis of mathematics tasks is too often conducted in the classroom, as Shimizu et al (2010) have done. Very little research has been done focusing on the tasks in textbooks and the role of supporting materials related to the tasks, particularly on the tasks in textbooks used in Alberta. This paper concentrates on the analysis of certain tasks from the Grade 8 *Math Makes Sense* textbook and the corresponding students' workbook and teachers' manual, and on the examination of their features and roles in supporting students' mathematics learning by adopting three kinds of analysis framework: knowledge level, cognitive demands and supportive factors.

Introduction of Solving Equations Using Models and Algebra

The tasks to be analyzed have been selected from two lessons, Solving Equations Using Models (6.1) and Solving Equations Using Algebra (6.2), in the textbook *Math Makes Sense 8* (Baron et al 2008, 318–32). With a view to examining their supports for the students' learning, other relevant resources have been taken into account, such as *Math Makes Sense 8: Practice and Homework Book* (6.1 and 6.2) (Berglind et al 2009, 138–43) and *Math Makes Sense 8: ProGuide* (6.1 and 6.2) (Appel et al 2007, 4–18).

Each resource comprises several parts. Typically, the two lessons selected from the textbook have such similar constructions as Investigate, Connect and Practice. The two lessons from the practice and homework book have similar constructions, such as Quick Review and Practice. The ProGuide book for teachers provides guiding questions and strategies at three stages: before (Get Started), during (Investigate) and after (Connect).

The two lessons are targeted to lead students to solve equations by using algebra tiles, balance scales and algebra. At the end of the two lessons, the students are expected to write an equation to represent a problem.

Methods

Data

Data is taken from *Math Makes Sense 8* (textbook, 6.1 and 6.2), *Math Makes Sense 8 Practice and Homework Book* (6.1 and 6.2) and *Math Makes Sense 8 ProGuide* (6.1 and 6.2) (see Table 1).

Table 1: *Content by resource*

Textbook (T)	Practice and homework book (PH)	ProGuide (PG)
T1 Investigate (T1-I) Connect Example 1 (T1-CE1) Example 2 (T1-CE2) Example 3 (T1-CE3) Practice Check (T1-PC) Apply (T1-PA) Assessment focus (T1-PAF)	PH1 Quick Review (PH1-QR) Practice (1-7) (PH1-P)	PG1 Before (Get Started) (PG1-BGS) During (Investigate) (PG1-DI) After (Connect) (PG1-AC)
T2 Investigate Connect Example 1(T2-CE1) Example 2(T2-CE1) Practice Check (T2-PC) Apply (T2-PA) Assessment focus (T2-PAF)	Quick Review (PH2-QR) Practice (1-5) (PH2-QR)	Before (Get Started) (PG2-BGS) During (Investigate) (PG2-DI) After (Connect) (PG2-AC)

Analysis Frameworks

Three kinds of analysis framework have been adopted to analyze the tasks from the textbook, the practice and homework book and the ProGuide book for teachers. In order to understand the knowledge level involved in the tasks, the standards for scoring assignments (Koh and Lee 2004) have been employed and modified into a knowledge-level framework to analyze the tasks from the textbook and the practice and homework book.

Specifically, three standards (standard 1, standard 2 and standard 3) selected from the standards of scoring assignments from Koh and Lee (2004) have been modified into dimension 1, dimension 2 and dimension 3, respectively, in the knowledge-level framework. Moreover, for the purpose of demonstrating a feature of tasks in the textbook *Math Makes Sense 8*, encouraging students' communication, an extra knowledge level, dimension 4—knowledge communication—is added to the knowledge-level framework. Each category from the four dimensions is coded to the tasks from Solving Equations Using Models (6.1 in the textbook), and its content is modified to inosculate the tasks content. Based on those modifications, the framework of a new knowledge level (see Table 2) is built and capitalized to code all the tasks from the textbook and the practice and homework book.

In addition, the levels of cognitive demands are applied to check the cognitive demands included in these tasks. Finally, the factors associated with the maintenance of high-level cognitive demands are adopted to examine the questions and strategies provided in the ProGuide book.

Table 2: *Knowledge level of tasks*

	Level 1	Level 2	Level 3	Level 4
Dimension 1 Depth of knowledge	**Factual knowledge** Possible indicators are tasks that require students to describe routine computational procedures and perform routine equation operations.	**Procedural knowledge** Possible indicators are tasks that require students to know how to carry out a set of steps to solve equations using models and algebra; to use a variety of computational procedures and tools; and to manipulate the written symbols of algebra.	**Advanced knowledge** Possible indicators are tasks that require students to make connections to other mathematical concepts and procedures; to explain one or more mathematical relations; and to understand how a mathematical topic relates to real-world situations.	
Dimension 2 Knowledge criticism	**Presentation of knowledge as truth or given** Possible indicators are tasks that require students to accept or present ideas or solutions as truth or a fixed body of truths; to perform a well-developed equation; and to perform clear steps.	**Comparing and contrasting information or knowledge** Possible indicators are tasks that require students to compare different methods of solving equations.	**Critiquing information or knowledge** Possible indicators are tasks that require students to comment on different equation solutions; to discuss and evaluate approaches to equation-based problems; and to make mathematical arguments, and pose and formulate equation problems.	
Dimension 3 Knowledge manipulation	**Reproduction** Possible indicators are tasks that require students to reproduce procedures; to recognize equality, to manipulate equation expressions containing symbols and formulae in standard form; to carry out computations; to apply routine mathematical procedures and technical skills, and to apply equality concepts and procedures to the solution of routine equations.	**Organization, interpretation, analysis or evaluation** Possible indicators are tasks that require students to write and interpret equations and to consider alternative solutions or strategies.	**Application or problem solving** Possible indicators are tasks that require students to apply equation concepts to create a problem; and to apply equations to the solution of the problem.	**Generation or construction of knowledge new to students** Possible indicators are tasks that require students to generalize strategies and solutions to new problem situations and to apply modelling to new contexts.
Dimension 4 Knowledge communication	Guides students to represent their thinking in pictures, numbers or symbols.	Provides a balance of oral and written communication opportunities	Reflects different models to solve an equation or different equation types using algebra	

The levels of cognitive demands of tasks (Table 3) are cited directly from Stein et al (2000, 16).

Table 3: *Level of cognitive demand of tasks*

Level 1: Memorization	
• Involve either reproducing previously learned facts, rules, formulas or definitions, or committing facts, rules, formulas or definitions to memory. • Cannot be solved using procedures because a procedure does not exist or because the time frame in which the task is being completed is too short to use a procedure.	• Are not ambiguous. Such tasks involve the exact reproduction of previously seen material, and what is to be reproduced is clearly and directly stated. • Have no connection to the concepts or meaning that underlie the facts, rules, formulas or definitions being learned or reproduced.
Level 2: Procedures without connections	
• Are algorithmic. Use of the procedure either is specifically called for or is evident from prior instruction, experience or placement of the task. • Require limited cognitive demand for successful completion. Little ambiguity exists about what needs to be done and how to do it. • Have no connection to the concepts or meaning that underlie the procedure being used.	• Are focused on producing correct answers instead of on developing mathematical understanding. • Require no explanations, or explanations that focus solely on describing the procedure that was used.
Level 3: Procedures with connections	
• Focus students' attention on the use of procedures for the purpose of developing deeper levels of understanding of mathematical concepts and ideas. • Suggest explicitly or implicitly pathways to follow that are broad general procedures that have close connections to underlying conceptual ideas as opposed to narrow algorithms that are opaque with respect to underlying concepts.	• Usually are represented in multiple ways, such as visual diagrams, manipulatives, symbols and problem situations. Making connections among multiple representations helps develop meaning. • Require some degree of cognitive effort. Although general procedures may be followed, they cannot be followed mindlessly. Students need to engage with conceptual ideas that underlie the procedures to complete the task successfully and that develop understanding.
Level 4: Doing mathematics	
• Require complex and non-algorithmic thinking—a predictable, well-rehearsed approach or pathway is not explicitly suggested by the task, task instructions or a worked-out example. • Require students to explore and understand the nature of mathematical concepts, processes or relationships. • Demand self-monitoring or self-regulation of one's own cognitive processes.	• Require students to access relevant knowledge and experiences and make appropriate use of them in working through the task. • Require students to analyze the task and actively examine task constraints that may limit possible solution strategies and solutions. • Require considerable cognitive effort and may involve some level of anxiety for the student because of the unpredictable nature of the solution process required.

Six factors associated with the maintenance of high-level cognitive demands (Table 4) are adopted from Stein et al (2000) to examine the supports for students' cognitive processing from the questions and strategies provided in the ProGuide book.

Table 4: *Supportive factors associated with students' cognitive processing (Stein et al 2000)*

Factors
F1: Scaffolding student thinking and reasoning
F2: Offering students the means of monitoring their own progress
F3: Modelling alternative performance
F4: Emphasizing justifications and explanations through questioning
F5: Using students' prior knowledge
F6: Drawing conceptual connections

Code Up Technique

A "code up" technique (Garrison, Anderson and Archer 2001) is employed in the coding process in terms of multiple questions within a task, such as three questions in T1-PC. The highest levels of knowledge and cognitive demands encoded within a task are used as the final code for the entire task.

Coding Reliability

The use of multiple researchers is adopted to confirm the conformability of the data (Ertmer, Sadaf and Ertmer 2011). Specifically, a graduate student from my department is invited to participate in the coding process. The graduate student and I first code the data individually and then collaboratively develop a consensus on the coding results for all the tasks and questions.

Results

The coding results are shown in tables 5, 6 and 7. The sections that follow outline features of tasks that we discovered and make suggestions about future tasks for textbooks and supportive resources.

Submedium Level of Knowledge Type, Criticism and Manipulation

Table 5 panoramically reveals that the knowledge levels of tasks from the textbook and the practice and homework book are low or submedium, with the exception of knowledge communication (dimension 4).

Dimension 1, depth of knowledge, demonstrates that, except for T1-CE3, all the tasks focus mainly on the procedural knowledge (medium level) of equation solving, using either models or algebra. For example, writing an equation and using tiles to solve the equation are required in T1-I.

Dimension 2, knowledge criticism, outlines that 13 out of 17 tasks are at level 1, presentation of knowledge as truth or given. For example, algebra is used in T2-CE2 to solve the equation: $16t-69 = -13$, verify the solution and present the solving equation knowledge. Generally, tasks under the dimension-knowledge criticism stay at a lower level. Furthermore, dimension 3 illustrates that 16 of 17 tasks are at L1 (reproduction) and L2 (organization, interpretation, analysis or evaluation), suggesting a lower level of knowledge manipulation. For example, in PH1-P, a model is used in question 7 to solve the problem "one less than three times a number is eleven," verify the solution and write a concluding statement. Using the model to solve the typical quantitative relationship is a reproduction of the knowledge of using the model to solve an equation. Verifying the solution and writing the conclusion statement represent interpretation and analysis of knowledge manipulation.

Higher Level of Knowledge Communication in the Textbook

Table 5 indicates that all 13 tasks from *Math Makes Sense 8* (6.1 and 6.2) at L2 and L3 possess higher levels of knowledge communication. Those tasks have clear guidance to encourage students to communicate with their pair partner or reflect their own ideas. For example, there are distinct statements guiding students to reflect and share in T2-I: "Compare the equation you wrote with that of another pair of classmates; if the equations are different, is each equation correct ..." (Baron et al 2008, 327).

However, tasks from the practice and homework book mainly require students to represent equations in tiles, numbers or symbols and provide their solutions, thus failing to encourage students to communicate or reflect their own ideas. For example, question 3 of PH2-P is designed to "Use algebra to solve each equation. Verify the solution. (a) $6m+5=7$; (b) $3c-2=2$; (c) $2+5y=2$; (d) $4-3x=-5$" (Berglind et al 2009, 143). There is no prompt in the task, to encourage students to communicate or reflect their own ideas. In fact, it is possible to use such prompts to trigger students to reflect on the process of solving equations and recognize the significance and meaning of solving equations using algebra rather than mainly

Table 5: *The knowledge level of tasks*

Tasks	Knowledge Level of Tasks												
	Dimension 1			Dimension 2			Dimension 3				Dimension 4		
	L1	L2	L3	L1	L2	L3	L1	L2	L3	L4	L1	L2	L3
T1-I		●		●				●				●	
T1-CE1		●		●				●				●	
T1-CE2		●		●				●				●	
T1-CE3	●			●			●					●	
T1-PC		●		●			●						●
T1-PA		●		●				●					●
T1-PAF		●				●		●					●
T2-I		●		●				●				●	
T2-CE1		●		●				●				●	
T2-CE2		●		●			●					●	
T2-PC		●				●	●						●
T2-PA		●		●				●					●
T2-PAF		●				●			●				●
PH1-QR		●		●			●				●		
PH1-P		●		●				●			●		
PH2-QR		●		●			●				●		
PH2-P		●				●	●				●		

focus on solving procedures; for example, Baron et al (2008) suggest "Which types of equations do you prefer to solve using algebra? Explain why you may not want to use algebra tiles or a balance-scales model" (p 332).

Medium Level of Cognitive Demands

Table 6 shows that the levels of cognitive demand of tasks generally remain at medium. Specifically, 15 out of 17 tasks at L2 and L3 focus on procedure without connections or with connections. One example of question 3 of PH2-P requires the use of a well-established procedure without connections to meaning for finding the solution of equations. Another example of T1-I, "Marie's bonds," focuses attention on the procedures for finding and solving an equation in a meaningful context.

Only 2 out of the 17 tasks use the Assessment Focus of Practice (PAF) from the textbook. For example, question 14 of T2-PAF requires writing a problem solvable by applying an equation and by adding such information as "Boat rental $300" and "Fishing rod rental $20." This task has no suggestion of any pathway; instead, its focus is on searching for the underlying mathematical equation, which requires complex thinking.

Table 6: *Levels of cognitive demands of tasks*

Tasks	Levels			
	L1	L2	L3	L4
T1-I			•	
T1-CE1			•	
T1-CE2			•	
T1-CE3			•	
T1-PC		•		
T1-PA			•	
T1-PAF				•
T2-I			•	
T2-CE1			•	
T2-CE2		•		
T2-PC		•		
T2-PA			•	
T2-PAF				•
PH1-QR		•		
PH1-P		•		
PH2-QR		•		
PH2-P		•		

Major Factor: Scaffolding Students' Thinking and Reasoning

The ProGuide book indicates that F1 in Table 7 is the major factor, which is broadly demonstrated through a large quantity of suggested questions and instructional strategies (Burnett 2007). For example, in PG1-DI, 12 questions are recommended to observe and understand students' thinking, such as "What problem-solving strategies could you use to help you with this problem?"(p 4) and "How can you check that your expression is correct?" (p 5). Similarly, in PG2-BGS, 11 questions are provided to promote students' thinking, such as "How would you use algebra to solve the equation?" (p 14). In addition to the above questions, alternative instructional strategies are also offered to enhance students' thinking and reasoning. For example, in PG1-AC, an alternative solution is suggested to scaffold students who "may have difficulty using a model to verify the solution to an equation that has negative variable tiles" (p 10). Also, in PG2-AC, an extra strategy is provided to facilitate students' thinking in question 5: "For students who need extra help to complete this question, refer them to Example 1" (p 17).

Table 7: *Factors associated with the maintenance of high-level cognitive demands*

Tasks	Factors: F1, F2, F3, F4, F5, F6
PG1-BGS	F1, F5
PG1-DI	F1, F6
PG1-AC	F1, F4, F5
PG2-BGS	F1
PG2-DI	F1
PG2-AC	F1, F4

Minor Factors: Justifications, Usage of Students' Prior Knowledge, and Conceptual Connections

Several other factors are also employed in the ProGuide book (Burnett 2007) to support teachers' classroom teaching. For example, in PG1-AC, certain questions are suggested to help students make justifications or explanations (F4), such as "Why did you add a unit tile to each side?" (p 6) and "Could we have used a different variable? Justify your answer" (p 8). PG2-AC also offered similar questions, such as "When using algebra to solve the equation, why did you start by subtracting 5 from each side rather than dividing each side by 2?" (p 14).

In addition, some suggestions are presented to remind students of connecting with their prior knowledge (F5). For instance, PG1-BGS suggests that teachers review how to use red and yellow unit tiles to represent positive and negative numbers so that students could recognize different tiles representing different variables. Using those tiles is very helpful for students to solve the equations in the whole lesson. In PG1-AC, the suggestion is made to "remind students to define a variable before they use it in an equation" (p 10). Meanwhile, some questions are brought forward to stimulate students' conceptual connections (F6). For instance, in PG1-DI, such questions as "How did you use tiles to represent the equation?"(p 5) are recommended to facilitate students' understanding of the relationship between tiles representation and equation.

Conclusions

Lower Level of Knowledge and Cognitive Demands of Tasks

Generally speaking, the evaluated tasks are at the lower levels of knowledge type, knowledge criticism and knowledge manipulation. Only a few tasks possess higher-level cognitive demands. However, the results did not imply that all the evaluated tasks under each dimension should reach the high level. My concern was that the lower knowledge level and cognitive demands of the evaluated tasks might lead to students' learning becoming "surface learning" (Davis and Renert 2014, 30), such as memorizing procedures by rote, or "rote learning" (Mayer 2002, 227). Certainly, the efforts of setting certain tasks at the higher level were made to achieve "deep learning" (Davis and Renert 2014, 30) or "meaningful learning" (Mayer 2002, 227). However, the big gap between the major tasks at the lower level and the minor ones at the higher level might "press students toward a more mechanical attitude" (Davis and Renert 2014, 30).

The efforts of setting certain tasks at the higher level were made to achieve "deep learning."

Higher Level of Knowledge Communication in Textbook

Fortunately, an appealing feature emerging from our coding results is that, highlighting the higher levels of knowledge communication, the tasks in *Math Makes Sense 8* (6.1 and 6.2) have very clear statements to encourage students to share their work with their pair partners as well as to reflect their own thinking.

Relative Monotone of Factors Associated with Cognitive Supporting

Among the factors of supporting students' cognitive processing, the major factor in the supportive strategies provided for teachers is scaffolding students' thinking and reasoning. However, very few factors connect with monitoring students' progress, alternative demonstration, justifications, prior knowledge and conceptual understanding. Thus, the ProGuide has some limitations regarding supports for teachers to facilitate students' cognitive processing.

Implications

Generally speaking, the tasks from *Math Makes Sense 8* (6.1 and 6.2), and the practice and homework book (6.1 and 6.2) are procedural in nature. The knowledge communication highlighted in the textbook demonstrates a core idea of *Math Makes Sense*—creating a math community in the classroom (Burnett 2007, 3–11).

The balanced instruction advocated in *Math Makes Sense 8* includes four key components: problem solving, understanding concepts, application of procedures and communication (Burnett 2007, 8). There is still much to be desired in problem solving and understanding concepts. For instance, "Investigate" in each lesson is designed for doing mathematics. However, the tasks involved lower levels of knowledge and their cognitive demands failed to achieve the curriculum goal. Therefore, this study highly recommends that the levels of knowledge and cognitive demands of the tasks be enhanced. I endeavoured to acquire from literature an answer as to how many enhancing degrees are regarded as reasonable, but in vain. After referring to my own teaching and research experience, I believe that balancing the tasks at different levels might be a practical way to select or design the learning tasks so that the lower-level tasks are necessary for advanced-level ones (Brean 2014).

Balancing the tasks at different levels might be a practical way to select or design the learning tasks.

In addition, the tasks within a session do not seem to be designed with cognitive hierarchies, resulting in the failure to nourish higher-level cognitive processing. It is possible to design such tasks by using variation theory (Marton and Booth 1997) to lead students to the access to the level of problem solving. Finally, it is suggested that questions or instructional strategies to support students' cognitive processing pay attention to the conceptual connections in order to promote students' understanding of concepts.

This research not only indicated the knowledge level and cognitive demand level of tasks from the *Math Makes Sense 8* textbook and the supportive factors in the teacher guide book but also presented a way of analyzing tasks applied in teaching practice. The results and the applied method could be used by teachers to help them use the tasks in an analytical way in their classrooms.

References

Appel, R, F Glanfield, G Pallen, S Ball, L J Hantelman, J Pusic, S Beaudin et al. 2007. *Math Makes Sense 8: ProGuide*. Toronto: Pearson Education Canada.

Baron, L, T Brown, G Davis, S Jeroski, S Ludwig, E Milne, K Neel et al. 2008. *Math Makes Sense 8*. Toronto: Pearson Education Canada.

Berglind, R, J Menie, E Milne, S Ludwig and D Sufrin. 2009. *Math Makes Sense 8: Practice and Homework Book*. Toronto: Pearson Education Canada.

Brean, J. 2014. "Math Wars: Rote Memorization Plays Crucial Role in Teaching Students How to Solve Complex Calculations, Study Says." *National Post*, August 21. Available at http://news.nationalpost.com/news/canada/math-wars-rote-memorization-plays-crucial-role-in-teaching-students-how-to-solve-complex-calculations-study-says or http://tinyurl.com/nhml6x5 (accessed July 7, 2016).

Cai, J, and F Lester Jr. 2005. "Solution Representations and Pedagogical Representations in Chinese and U.S. Classrooms." *Journal of Mathematical Behavior* 24, no 3–4: 221–37.

Davis, B, and M Renert. 2014. *The Math Teachers Know: Profound Understanding of Emergent Mathematics*. New York: Taylor & Francis.

Ertmer, P, A Sadaf and D Ertmer. 2011. "Student-Content Interactions in Online Courses: The Role of Question Prompts in Facilitating Higher-Level Engagement with Course Content." *Journal of Computing Higher Education* 23: 157–86.

Garrison, D R, T Anderson and W Archer. 2001. "Critical Thinking, Cognitive Presence, and Computer Conferencing in Distance Education." *American Journal of Distance Education* 15, no 1: 7–23.

Hiebert, J, T P Carpenter, E Fennema, K Fuson, D Wearne, H Murray, A Oliver and P Human. 1997. *Making Sense: Teaching and Learning Mathematics with Understanding*. Portsmouth, NH: Heinemann.

Koh, K H, and A N Lee. 2004. *Technical Report: Manual for Scoring Teacher Assignments and Assessment Tasks*. Singapore: National Institute of Education, Centre for Research in Pedagogy and Practice.

Marton, F, and S Booth. 1997. *Learning and Awareness*. Mahwah, NJ: Lawrence Erlbaum.

Mayer, R E. 2002. "Rote Versus Meaningful Learning." *Theory into Practice* 41, no 4: 226–32.

Pepin, B, and L Haggarty. 2001. "Mathematics Textbooks and Their Use in English, French and German Classrooms: A Way to Understand Teaching and Learning Cultures." *Zentralblatt für Didaktik der Mathematik* 33, no 5: 158–75.

Shimizu, Y, B Kaur, R Huang and D J Clarke. 2010. *Mathematical Tasks in Classrooms Around the World*. Rotterdam: Sense.

Stein, M K, M S Smith, M A Henningsen and E A Silver. 2000. *Implementing Standards-Based Mathematics Instruction: A Casebook for Professional Development*. New York: Teachers College Press.

Acknowledgement

The research is financed by Humanities and Social Science Research Project of the Ministry of Education, People's Republic of China, no 13YJC880075.

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Mathworks 12: A Critical Analysis of Text

Ellen Watson

Concerned was an understatement; thanks to the incoming mathematics curriculum, I had to plan and execute at least four new courses, all with different mathematical mandates. "Don't worry," my principal assured me, "The new textbooks are foolproof. They will make teaching math easy." Mathematics education today relies on the textbook. According to Skovsmose and Penteado (2012), "the teacher can follow the textbook, and do so with good conscience, as the textbook is assumed to provide a recipe for good teaching" (p 74). Due to its integral role in the mathematics classroom, we must analyze the textbook and its purpose, pedagogy and power.

We must analyze the textbook and its purpose, pedagogy and power.

Guiding pedagogical and curricular decisions, the textbook is not meant to solely control the curriculum. Textbooks are not written to replace a teacher—they are to be utilized and mediated by a teacher (Love and Pimm 1996). In fact, textbooks "provide a framework for thinking about what will be taught, to whom, when, and how," (Nicol and Crespo 2006, 331). The curriculum document and teacher should collaboratively develop this framework prior to filling in the frame with learner needs and the textbook. If the textbook is central to the construction of the classroom, teachers should make every attempt to analyze its reasoning and purpose before implementation.

MATH *ON THE JOB*

Lori Ann, Stefan, and René Regnier are members of the team at Blue Lagoon Organics that grows certified organic fruits, herbs, and vegetables on its farm near St. François Xavier, Manitoba. Stefan and René grew up in St. François Xavier, while Lori Ann was raised in Winnipeg.

The Blue Lagoon Organics team participates in a community-supported agriculture (CSA) system. Fifty-seven community members pay a yearly fee to receive regular deliveries of the farm's produce. To equally distribute the produce it grows, the farm team weighs it and divides it among the members. Lori Ann, René, and Stefan also calculate prices, create invoices, and work out ratios.

When distributing equal portions of produce to the community members, why is it not important that the weigh scale gives a correct weight?

The farm also sells produce to the general community on a price-per-kilogram basis. Give two reasons why it is important that the scale records the correct weight.

Lori Ann uses an electronic scale to weigh pumpkins before selling them to a customer. Weigh scales used to sell products on a price-per-weight basis must be regularly checked for accuracy.

Figure 1: *Math on the job (*Mathworks 12, *76)*

The Text to Be Considered

Throughout this paper I intend to analyze the purpose, pedagogy and power of the textbook *Mathworks 12* (Angst et al 2012a).[1] A specific focus will be made on the unit Limits to Measurement in *Mathworks 12*. Unique to the workplace mathematics pathway (also known as "dash 3") in the Western and Northern Canadian Protocol (WNCP) mathematics curriculum; this unit is not covered in the precalculus ("dash 1") or foundations ("dash 2") streams. The impact, ideology and authority of *Mathworks 12* all play a role in this textbook's curricular orientation.

Purpose

Producing a Good Worker

Mathworks 12 has a distinct mandate: to educate students in real-world mathematics required to succeed in the workforce. Since its students are destined to pursue trades, certified occupations or direct entry into the workforce, *Mathworks 12* addresses the curriculum through real-world contexts (Angst et al 2012b). For example, "Math on the Job" sections give students a sense of occupations requiring the mathematics they are about to learn, complete with a sample problem. The Limits to Measurement unit's job is an organic farm worker (Figure 1). Every effort

Mathworks 12 addresses the curriculum through real-world contexts.

is taken to ensure that students recognize the real-world basis of this job, including the location of the job, the place of origin of the workers and a specific explanation of what the work entails. Finally, students are asked to consider why they must record correct measurements with a direct relation to the price per kilogram—a good employee measures correctly. Another example to consider is a question on page 97, "What could happen if the tolerances for the door height or the location of the hinges were not followed?" At first these questions appear to offer mathematical reflection, but the "good worker" vantage point reflects the undertones of reinforcing proper workplace habits. If we cut the door outside the tolerance it will not close, and thus we produce unsatisfactory work. Both of these examples ask students to consider the impact of accuracy and tolerance in the workplace; substandard work is not affordable, nor is it acceptable. Anyon (1980) described working-class education as a means to retaining a good job. *Mathworks 12* supports this recognition that good workers receive good jobs, successfully placing its learners within the working class. Students not only need to be able to *do* mathematics but also recognize how it affects employability.

An apparent goal of this textbook is to develop good workers, despite the claims of many theorists that education is not to be used for job training. For example, Dewey (1929) said "education … is a process of living and not a preparation for future living," (p 36). His reading of the situation is supported in literature such as Adler (2009) and his call to educate all students, not simply train the working class for jobs and educate the elite. Teaching workplace students to become "good workers" has implications far beyond textbook considerations and into societal concerns. Success in mathematics gives students the ability to climb our societal structure (Gates and Vistro-Yu 2003). *Mathworks 12* students are given the chance to succeed in mathematics, but not to the extent of the other pathways; thus, workplace (or dash 3) mathematics students are taught to be successful in their paradigm, as good workers.

Common Sense?

Mathworks 12 attempts to connect students' "common sense" to mathematics. Consistent use of concrete, real-world problems, such as the measurements of a stool (p 98), the time of a race (p 86) or the placement of patio blocks (p 97) clearly shows this intention. According to Polya (1985), accessing and unlocking students' common sense promotes mathematical exploration. However, many students struggle with the "common-sense" problems identified in *Mathworks 12*, particularly those who have not taken many practical and applied arts courses. According to Williams (2005), there is no such thing as common sense; everyone's sense comes from personal experiences. *Mathworks 12* assumes that students have a basic understanding of many real-world pro-

1. This work will be cited as *Mathworks 12* throughout this article to avoid redundancy.

cesses, most of which only some learners have encountered. Since the textbook is a single resource to be used in instruction, it can attempt inclusion through varying examples, but its reach is limited. The unit on limits and measurement relies heavily on an understanding of machinery, building or baking; a student who does not have these skills will inevitably struggle, unable to connect to the concrete example meant to ground the abstract mathematics.

Pedagogy

Investigative or Prescriptive?

Mathworks 12 could appear as investigative; it contains many activities intended to move beyond traditional drill-and-practise methods. However, these activities consist of specific steps for students to follow and data to collect (complete with sample tables). According to Smith and Stein (1998), higher-level mathematics processes include reflection on mathematics, require cognitive effort and suggest pathways to follow. While the tasks included in *Mathworks 12* offer students reflection questions, rarely do they require cognitive effort. For example, on page 83, students are asked a conclusion question: "State the precision and uncertainty of your measured times." A straightforward and direct application of the information being covered, this question does not move students to consider higher-level mathematics.

Mathworks 12 rarely asks cognition-provoking questions due to the method of activity execution: specific directions. Rarely, *Mathworks 12* students may be asked to develop a procedure; typically, students are consistently provided with specific directions on how to complete an activity. Directions are written in a clear and authoritative manner, ensuring an extreme amount of specificity so that students cannot deviate. According to Polya (1985), mathematics problems require four things: understanding what is required, identifying connections to make a plan, carrying out this plan and, finally, reflection. *Mathworks 12* students are not being asked mathematics problems, as they rarely have to develop a plan or reflect; rather, they are focused on reproducing prescriptive mathematics procedures.

The focus on prescriptive mathematics procedure could be because these students have been deemed "unsuccessful" in mathematics. In my experience, students in this pathway typically struggled with the "normal" mathematics classroom, but not necessarily the mathematics itself. Thus, these students are not part of the successful (or powerful) mathematics group. Students outside of the culture of power require specific instructions to succeed (Delpit 1988). If workplace mathematics students are not part of the power culture in mathematics, the author could possibly have interpreted this as a reason to provide specific direc-

b) To find the nominal value, add the maximum and minimum values and divide by 2.

$$\text{nominal value} = \frac{\text{maximum} + \text{minimum}}{2}$$

$$\text{nominal value} = \frac{0.250'' + 0.230''}{2}$$

$$\text{nominal value} = \frac{0.480''}{2}$$

$$\text{nominal value} = 0.240''$$

To find the tolerance, subtract the minimum value from the maximum value.

$$\text{tolerance} = \text{maximum} - \text{minimum}$$

$$\text{tolerance} = 0.250'' - 0.230''$$

$$\text{tolerance} = 0.020''$$

Calculate half the tolerance.

$$\frac{1}{2}(\text{tolerance}) = 0.020'' \div 2$$

$$\frac{1}{2}(\text{tolerance}) = 0.010''$$

The measurement can be written as $0.240'' \pm 0.010''$.

Figure 2: *Problem solution (*Mathworks 12, *94)*

tions to complete mathematical activities. Activities and projects in this textbook rarely reach into the investigative realm; thus, students are rarely taught to think, but are instead taught to follow instructions.

Order and Pacing

Linear in its framework, *Mathworks 12* follows a pattern common to most mathematics textbooks. Information is organized numerically (eg, unit 2 lesson 1 is titled 2.1) and presented in a sequential order. While teachers can choose to break this sequence, many new teachers often follow the textbook sequence provided (Nicol and Crespo 2006). In my teaching experience, students are frustrated by jumping around within the textbook content. Linearity in a textbook is a way of controlling time and sequence in the classroom; very few textual materials break linearity, and then typically only in the form of reference, such as answers in the back or using the margins to emphasize (Love and Pimm 1996). This textbook has linearity; in fact, the only writing within the margins is superfluous information or images one

PUZZLE IT OUT

MIXING CEMENT

Anil needs to add 4 gallons of water to a volume of cement that he is mixing. He has three buckets: an 8-gal one filled with water, a 3-gal bucket, and a 5-gal bucket. The buckets have only a "full" marking at their capacity, so he can only completely empty or completely fill a bucket to measure a volume. How can he divide the water into two equal 4-gallon portions?

HINT

The solution takes seven steps!

Figure 3: *Puzzle It Out question (*Mathworks 12, *87)*

could use to vaguely support the topic. This order and power is welcomed by students, as they prefer a familiar structure in their mathematics classroom. Again, conforming to expected work habits and rigid structure is required, further conditioning students into workers by limiting their mathematical thought and exploration.

Rigour

This textbook appears to be written for "lower-ability" mathematics students. According to Dowling (1991), "'working class' is part of what it means to be 'less able'" (p 148). As aforementioned, *Mathworks 12* students are equipped to enter the trades and/or workforce; these students are destined to become working class. Workplace-oriented mathematics textbooks are seen to be antiacademic and reinforce the importance of manual work (Dowling 1991). *Mathworks 12* focuses on developing good workers, seemingly due to their perceived lack of mathematical ability. Examples within this textbook always offer specific solutions; work is neat and down the page (see Figure 2). There is a plan shown in the steps (Figure 2), but the plan is only revealed as each subsequent step is completed, indicating that students should be planning their math work; it is more important that students learn to focus on completing proper procedures as the steps advance from correctly calculated values.

Workplace mathematics students are rarely given problems with more than one or two steps, but *Mathworks 12* does offer a few chances to move into higher-level tasks. The most interesting activities for students have varying solutions, such as the "Puzzle It Out" questions. For example, in the Limits to Measurement chapter, students are asked to split eight litres of water into two sets of four litres each (Figure 3). Finally, students can move into Smith and Stein's (1998) higher-level demands of planning, representation and cognition-activation. Students are asked not only to solve the problem but, more important, also to share *how* the problem can be solved. However, the authors anticipate that students will have difficulty with this open problem and thus offer the helpful hint bolded in the margin. Immediately, learners may feel limited to these seven steps. Workplace mathematics students are rarely offered the opportunity to independently undertake rigorous mathematics, presumably because of their alleged ability or lack thereof.

Power

Authority in the Classroom

According to Love and Pimm (1996), teachers and students reorient themselves according to the ideas presented in the textbook. Consequently, the textbook is given an unprecedented amount of authority; this is strongly utilized in *Mathworks 12*. For example, in an attempt to elicit knowledge, *Mathworks 12* questions students "What assumptions do you make?" (p 78); unfortunately, solutions are typically printed immediately following these questions. Students' learning (and thus thinking) is directed onto a specific path—the correct path as defined by the textbook. Ensuring that students follow specific, laid-out steps throughout activities (such as in Figure 4, overleaf) to find the correct solution further reinforces the power of the textbook. Many mathematics texts are "closed texts": there is a determined and fixed way to complete each problem and readers' steps are confined to this process (Love and Pimm 1996). Closed correctness gives the textbook an air of authority—it alone contains accurate solutions. Textbooks hold the correct solutions and thus determine the norm in learning mathematics (Delpit 1988). Defining the normative system forces students and teachers to reorient and submit to the power of the textbook.

What to do:

1. Examine a toy block. Decide which dimensions of the block should be reviewed for quality control.

2. Make a sketch of the block showing the front, side, top, and bottom views. Label the critical dimensions with letters.

3. Make a quality control table with a column for each critical dimension and enough rows to measure at least 24 blocks. Your table should include length, width, height, outer nib diameter, inner hole diameter, the position of the nubs and holes with respect to each other, and any other dimensions you think are important to the quality of the product.

4. For each of the 24 blocks, make and record the dimension measurements.

5. Find the maximum and minimum values of each dimension. Based on your investigation, what is the tolerance for each dimension? Write each critical dimension in the form nominal value $\pm \frac{1}{2}$(tolerance).

Figure 4: *Specific directions (*Mathworks 12, *99)*

Language

On page 76 of *Mathworks 12*, students are told, "You will explore the concepts of accuracy, precision, uncertainty, and tolerance." The use of *you will* is prevalent in *Mathworks 12.* "Addressing the reader as *you* may indicate a claim to a relatively close relationship between author and reader or between reader and subject matter" (Morgan 1996, 6). The appearance of a personal relationship lessens the necessity to submit to the textbook, yet still indicates the necessity to surrender one's authority. The use of the indicative *you will*, again, gives authority to the textbook; the textbook gives direct instructions to students of what learning is to be achieved, and students are required to yield to these demands.

The textbook uses *you will* in an attempt to imply a personal relationship; yet when undertaking mathematical instruction, it uses authoritative and aloof language. There is no relation to the author or the student, merely a relationship with mathematics. The solution to the example in Appendix A uses phrases such as "the measurement is shown …," "the uncertainty of the measurement is …" and "in this case, the measurement precision and uncertainty are not clear." According to Morgan (1996), the use of this reserved, neutral voice creates a formal relationship. The author has decided to shift between an informal and formal relationship with the reader, but the important aspect is *when* this shift occurs. The reader is addressed formally when the reader is invited to solve mathematical questions; yet, when students are being instructed, informal language is used. Thus, the textbook becomes the expert for the student and for mathematics as the ultimate authority.

Concluding Thoughts

As illustrated by Nicol and Crespo (2006), Love and Pimm (1996) and Skovsmose and Penteado (2012), the mathematics textbook is central to instruction. Thus, mathematics educators and educational researchers are called to analyze the textbooks used. This article has analyzed and criticized the purpose, pedagogy and power of *Mathworks 12*. The purpose of *Mathworks 12* appears to be to produce good workers, successfully driving social reification. Commonsense math is also a focus of this textbook, but the common sense addressed is not always common; this assumption of commonality could alienate students further from mathematics. Pedagogically, this textbook is prescriptive: students are given specific steps for doing mathematics. The linear pacing of this textbook contributes to a restrictive learning of mathematics, most likely due to the perceived lower ability of these students, resulting in a textbook with little rigour in instruction. Finally, *Mathworks 12*, through the use of both formal and informal language, strongly employs authority over its readers; the authors appear to have designed power structures of "textbook over student" and "mathematics over all." Teachers must be roused to the impacts of mathematics education on our world (Kumashiro 2009). Textbook analysis can bring this awakening, as shown by the investigation into *Mathworks 12*; after analysis, teachers may be able to truly consider the impact that texts have on their students.

References

Adler, M. 2009. "The Paideia Proposal." *In The Curriculum Studies Reader*, 3rd ed, ed D Flinders and S Thornton, 176–79. New York: Routledge.

Angst, C, K Borgen, B Dominik, M Healy and A Kaisser. 2012a. *Mathworks 12*. Vancouver, BC: Pacific Educational.

———. 2012b. *Mathworks 12: Teacher Resource*. Vancouver, BC: Pacific Educational.

Anyon, J. 1980. "Social Class and the Hidden Curriculum of Work." *Journal of Education* 162, no 1: 67–92.

Delpit, L D. 1988. "The Silenced Dialogue: Power and Pedagogy in Educating Other People's Children." *Harvard Educational Review* 58, no 3: 280–98.

Dewey, J. 1929. "My Pedagogic Creed." In *The Curriculum Studies Reader*, 3rd ed, ed D Flinders and S Thornton, 34–41. New York: Routledge, 2009.

Dowling, P. 1991. "A Touch of Class: Ability, Social Class and Intertext in SMP 11-16." In *Teaching and Learning School Mathematics, a Reader*, ed D Y E Love and E Pimm, 137–52. London, UK: Hodder & Stoughton.

Gates, P. and C P Vistro-Yu. 2003. "Is Mathematics for All?" In *Second International Handbook of Mathematics Education*, vol 10 of *Springer International Handbooks of Education*, ed A J Bishop, M A Clements, C Keitel, J Kilpatrick and F K S Leung, 31–73. Dordrecht, Neth: Kluwer.

Kumashiro, K. 2009. *Against Common Sense: Teaching and Learning Toward Social Justice*, rev ed. New York: Routledge.

Love, E, and D Pimm. 1996. "'This Is So': A Text on Texts." In *International Handbook of Mathematics Education*, vol 4 of *Springer International Handbooks of Education*, ed A J Bishop, M A Clements, C Keitel, J Kilpatrick and C Laborde, 371–409. Dordrecht, Neth: Kluwer.

Morgan, C. 1996. "'The Language of Mathematics': Towards a Critical Analysis of Mathematics Texts." *For the Learning of Mathematics* 16, no 3: 2–10.

Nicol, C, and S Crespo. 2006. "Learning to Teach with Mathematics Textbooks: How Preservice Teachers Interpret and Use Curriculum Materials." *Educational Studies in Mathematics* 62, no 3: 331–55.

Polya, G. 1985. "In the Classroom." In *How to Solve It: A New Aspect of Mathematical Method*, 2nd ed, 1–32. Princeton, NJ: Princeton University Press.

Skovsmose, O, and M G Penteado. 2012. "Concerns of a Critical Mathematics Education: Challenges for Teacher Education." In *Mathematics Teacher Education in the Public Interest: Equity and Social Justice*, ed L J Jacobsen, J Mistele and B Sriraman, 65–79. Charlotte, NC: Information Age.

Smith, M S, and M Stein. 1998. "Selecting and Creating Mathematical Tasks: From Research to Practice." *Mathematics Teaching in Middle School* 3, no 5: 344–50.

Williams, J. 2005. *Understanding Poststructuralism*. Durham, UK: Acumen.

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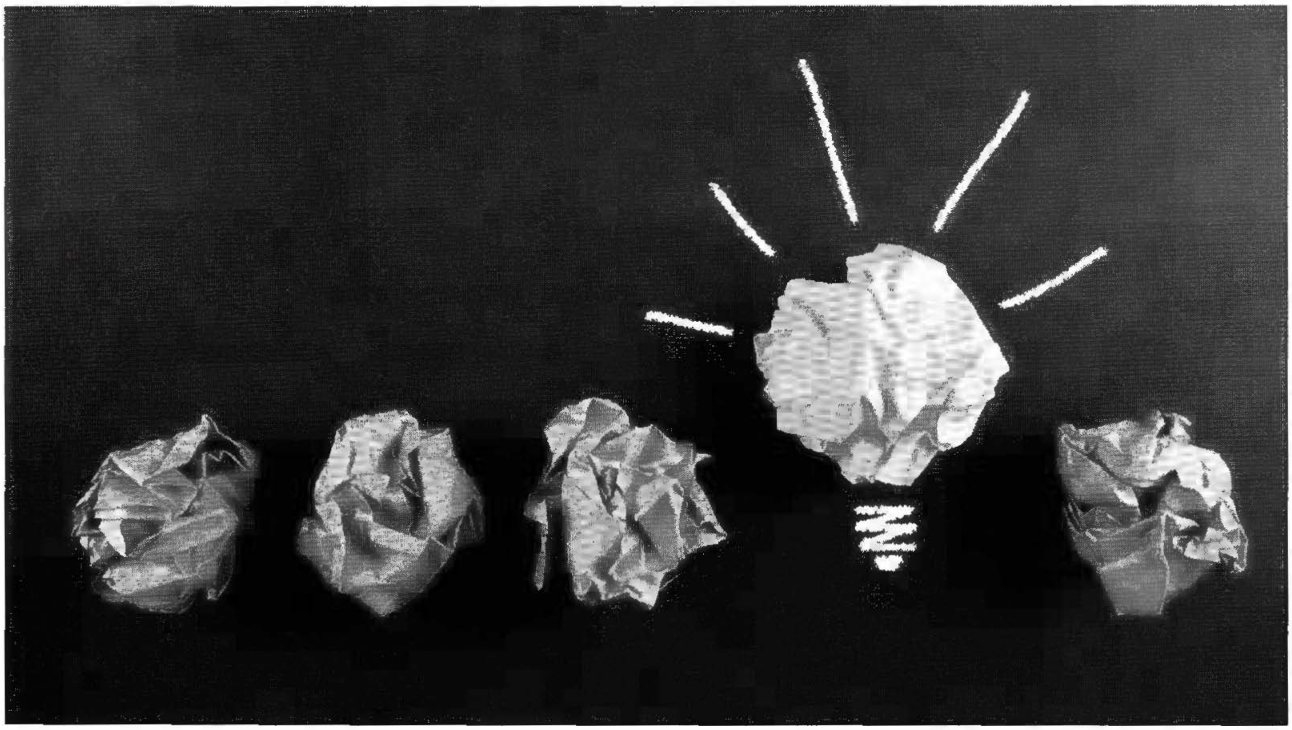
Making Sense of Problem Solving and Productive Struggle

Paul Betts and Sari Rosenberg



Introduction

How do we help children develop their abilities to solve problems? Can this problem-solving ability be developed so that it is also available in nonmathematical situations? These two questions inspired us (a group of teachers and a university professor) to embark on a professional learning journey concerning the teaching and learning of problem solving. In this paper, *problem solving* should be taken to mean the heuristics and metacognitive regulation available to a problem solver while navigating an initially unknown scenario (Schoenfeld 1992). We started with linear conceptions of the nature of problem solving and minimal conceptions of how it should be taught.

Can this problem-solving ability be developed so that it is also available in nonmathematical situations?

By watching children try to solve problems, and by adjusting our activities based on these observations, we have developed a rich understanding of problem-solving ability and how it can be developed. In this paper, we focus on productive struggle, which we believe is a fundamental component of problem-solving ability.

Context for a Professional Journey

Our professional learning journey was structured on principles of lesson study (Fernandez 2002), with school division support of teacher release time. We regularly developed and refined problem-solving activities by coplanning, coteaching and codebriefing, always focusing on our collective observations of children during these activities. As teachers, we deliberately decided to focus on teaching problem solving, embedded within our own practice and teaching concerns. All of us were concerned with how to help all children to successfully solve problems because we experienced minimal success in the past.

We wanted every child to be engaged and successful. We also knew that mathematics education organizations such as the National Council of Teachers of Mathematics recommend shifting instruction toward inquiry and scaffolding the learning of children, rather than only tightly directing the learning of children through modelling and practice (eg, NCTM 2014). But we didn't know how to accomplish these goals. As it turned out, our observations of children helped us to broaden our conceptions of problem-solving ability and to achieve these goals.

In particular, we developed the notion of *productive struggle*, which is the core idea unveiled throughout this paper. The idea starts with our initial belief that math teaching is successful when it makes learning simple. And yet, mathematics is not simple. We have also observed the debilitating effect of math anxiety on learning and students' lack of interest in mathematics. In the past, trying a word problem resulted in frustration for many children. Hence, we started our journey with an overall skepticism of problem-solving tasks because we feared that they were too difficult for the children. In the past, we tended to adopt easier word problems, if we tried them at all. "Keep it simple to ensure sucess for all" was our mantra, even though some children still became frustrated.

Productive struggle is a significant part of our problem-solving pedagogy because it is a fundamental component of problem-solving ability.

We faced our belief in the need to reject difficult problems very early in our journey. We decided to try the Neighbouring Numbers problem (see above) in a Grade 1/2 class, despite our concern that the problem was too difficult. Our planning session focused on what to do with specific students when they quickly gave up on the problem. We were surprised when our predictions of excessive frustration did not come to fruition. This started our journey to uncover why we needed to change our beliefs about problem-solving ability. We have learned to accept struggle and have developed our professional fluency in scaffolding struggle so that it is a space for learning about problem solving. Productive struggle is a significant part of our problem-solving pedagogy because it is a fundamental component of problem-solving ability.

Snapshots of a Journey to Productive Struggle

In what follows, we use two problem-solving activities to illustrate the idea of productive struggle, both as a fundamental component of problem-solving ability and as a pedagogic technique. The first problem, Neighbouring Numbers, involves arranging the digits 1 to 8 into a network so that no two neighbouring numbers are consecutive. We launched the problem with a short story about an apartment block (symbolized by the diagram in Figure 1) where the numbers 1 to 8 live. The premise of the story was that, when the numbers get home from work, they want a break from the order required at work and choose to live so that no neighbours are consecutive. Figure 1 is partially completed and illustrates two neighbours that are consecutive (3 and 4), which is not allowed. The students worked on the problem with a partner (see Figures 2 and 3—both arrangements are incorrect). We consolidated the activity by having students describe to the class their strategies to solve the problem.

When planning for the Neighbouring Numbers problem, many of us struggled to find a solution, hence our concern that it would be too difficult and cause frustration. So we developed the following scaffolds in anticipation of struggling students:

1. Based on their current work, we could ask students what would happen if a certain number was placed in a certain location, in order to illustrate allowed and not-allowed arrangements of numbers.
2. If students became frustrated to the point where they might give up, then we could suggest placing a 1 or an 8, or both, in one of the centre locations.

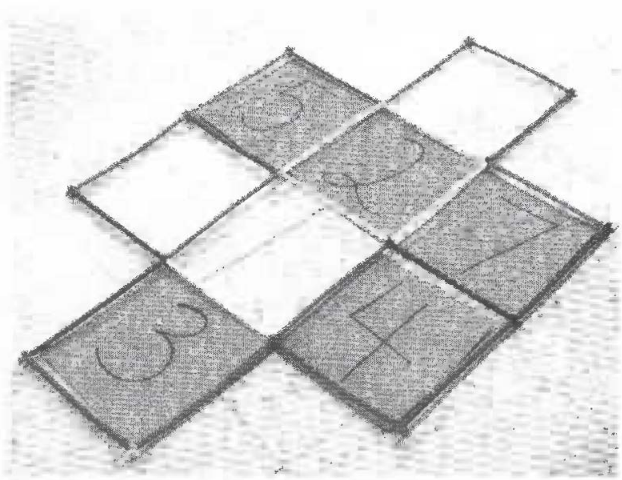


Figure 1: *Introducing the Neighbouring Numbers problem*

(The 1 and the 8 are located in the centre for any correct solution.)

We also anticipated that some students might quickly find a correct arrangement, and we could challenge them by suggesting they find a different solution. Finally, we concerned ourselves with engaging the children because we had noticed that students sometimes resist even starting to try a problem. We felt that when children find a problem interesting, they are naturally motivated to try to start solving the problem, which we hoped would be accomplished by the apartment-block story.

Despite our concerns about the problem's difficulty, the story motivated all the children, and we observed all the Grade 1/2 children sustain their interest in the problem for about 30 minutes. All children immediately started working on the problem, trying to find a correct arrangement of the numbers. They would excitedly raise their hand when they thought they had found a solution; in most cases, there was a problem with the solution and we would provide an "Are you sure?" scaffold. To our surprise, the children happily kept trying to find a solution. We observed some students struggling, but encouragement was enough to sustain their engagement. While consolidating the activity with the class, children shared strategies such as "fixing problems" and "placing consecutive numbers far apart." During this discussion, we emphasized their problem-solving strategies and praised their effort and willingness to keep trying.

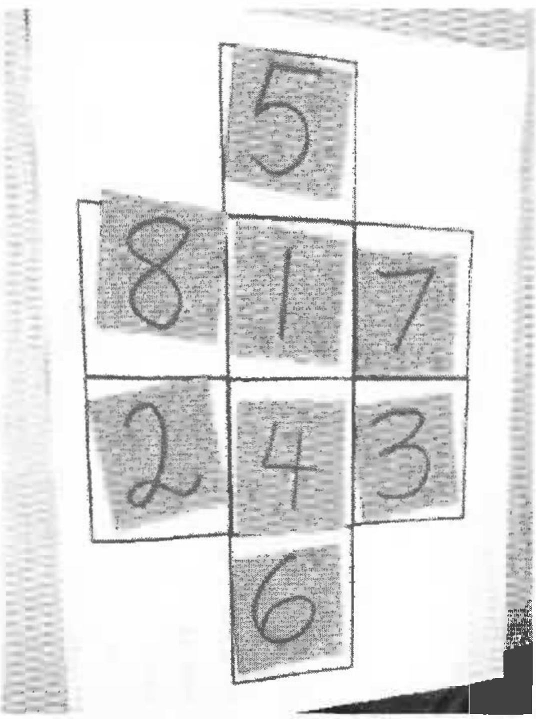


Figure 2: *The arrangement is not allowed because 1 and 2 are consecutive neighbours.*

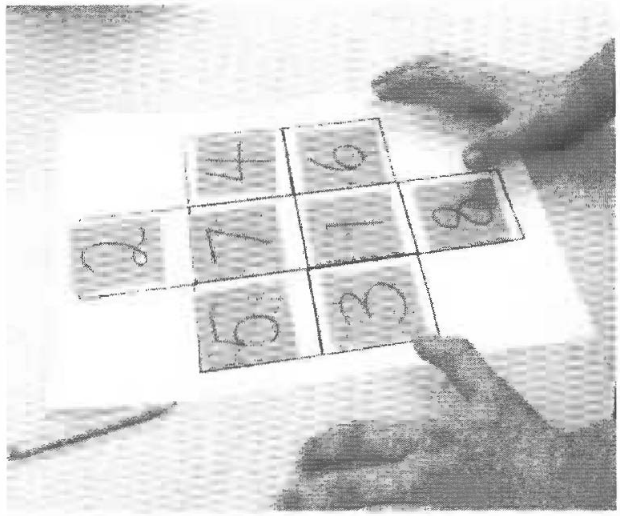


Figure 3: *The arrangement is not allowed because 6 and 7 are consecutive neighbours*

Even though many children struggled to find a solution, and some did not find a solution, the activity was a success for all. We were surprised, so our debriefing session as teachers focused on why a problem that seemed too hard was still a success. We noticed that all children naturally used trial and error as a problem-solving strategy. In part, the problem was successful because the children were engaged and the children quickly made progress by simply trying different arrangements of numbers.

We also noticed that children were willing to struggle. This contradicted our "make it easy" assumption. We had privileged protecting children from failure to such an extent that we couldn't see the benefit of temporary failure. At this moment, we began to realize the danger of overprotecting children from failure. Regulating frustration is an important goal in education, so some frustration was a reasonable event in school problem solving. Our challenge as teachers was now to develop our ability to support children through their frustration.

We praised their effort and willingness to keep trying.

We tried the same problem again in a Grade 1 class, staying conscious of supporting frustration while scaffolding the students. Supporting frustration allowed us to notice how different students used the trial-and-error problem-solving strategy. Some tended to adjust the whole arrangement, which tended to make it more difficult to use the previous trial to inform their thinking. Other students focused on fixing an error to a given arrangement, which would usually generate an unexpected new problem with the arrangement. A few students tried to plan ahead by asking questions such as "If I fix this problem by switching these numbers, do I create a new problem?" These variations illustrate some of the nuance of problem-solving ability—in this case, how there can

Resist telling students what to do.

be differing degrees of sophistication in the use of an incorrect arrangement to support thinking. Our class consolidation focused on the different ways that children used trial and error. We reinforced how every child was problem solving, and also encouraged the students' willingness to keep trying.

When we debriefed as teachers after the activity, our observations of the differences between children's use of trial and error reiterated for us a significant problem of teaching problem solving: How much [should] we help children when they struggle? Understanding what children could do to solve the problem helps us to develop scaffolds for the next time. For example, a teacher could encourage a student to look more closely at how to fix a specific problem in an arrangement, without actually finding and fixing a problem for the student. It is in this scaffold that we realized an important pedagogic disposition: *Resist telling students what to do*. When we tell students what to do, we have minimized the potential for children to learn about problem solving. When we tell students what to do, we eliminate the potential for children to be frustrated but still succeed, and these kinds of experiences are a life skill. We decided that our response to frustration should be to provide the minimal amount of support needed for children to keep trying to solve the problem.

During our journey, we had been developing our professional ability to provide a minimal amount of support, in both planning an activity and in-the-moment teacher decision making. We are convinced of the need to resist telling students what to do, in favour of a minimal amount of support. The second problem we use to illustrate productive struggle was tried later in our journey. Our work with this problem, called Handshakes at a Party, illustrates our deepening understanding of scaffolding productive struggle as a fundamental problem-solving ability.

The Handshakes at a Party problem considers how many handshakes occur if everyone at a party shakes hands with everyone else exactly once. We launch the problem by acting out the problem with five students. While five students perform the handshakes, the rest of the class counts how many handshakes occurred. After several tries acting out the problem, the class agrees that there are 10 handshakes. We then ask the students to determine the number of handshakes for a larger number of people. In a Grade 2 class, we asked the students to try 10 people, with an extra challenge to try 20 people. In higher grades, we ask the students to find the number of handshakes if 20 people are at the party, and challenge students to determine a general method regardless of the number of people at the party. In what follows, we described what happened when we tried the problem for the first time, which was with a Grade 2/3 class.

Many students immediately guessed that there would be 20 handshakes among 10 people because it is double of the situation with 5 people. This is wrong, but we have learned to resist telling students what to do, so we used an "Are you sure?" scaffold. This immediately caused confusion for the students. Students often expect teachers to respond with "Right" or "Wrong" when a student gives an answer. We did neither, and the students struggled. For some students we suggested they try to produce a convincing argument without relying on their "doubling"

Rather than tell the students they were wrong, we asked why.

observation, whereas for others we suggested they model or act out 10 handshakes to see if they were correct. With a few students who continued to struggle, we suggested they try to make a drawing or try the case of handshakes among 6 people. With patience, all students used blocks or a diagram to model the problem in order to count the number of handshakes. Some students used their model to recognize a pattern for adding up the number of handshakes: with 5 people, the number of handshakes is 4+3+2+1; and with 10 people, the number of handshakes is 9+8+ … +2+1.

When we debriefed as teachers after the activity, we saw a pattern in our scaffolding. The minimal amount of support depends on our knowledge of the children and on what type of progress they have made on the problem. Rather than tell the students they were wrong, we asked why. When struggle appeared to be at the point of excessive frustration, we provided further minimal guidance, to tell more without doing the problem for the student. We are looking for just the right amount of guidance, so that frustration is a learning opportunity and not interpreted by the student as complete failure. Our final analysis suggested that our scaffolding decision making is grounded in our professional relationships with each student, and with ensuring that the student experiences at least some problem solving.

Problems and struggling are intimately woven together.

Describing Productive Struggle as a Problem-Solving Ability and Pedagogy

Productive struggle occurs when a child learns something about problem solving when he or she cannot immediately solve a problem; productive struggle cannot happen when a child does not problem solve (Warshauer 2015). Struggling to make sense of mathematics is a necessary condition of learning mathematics (Hiebert and Grouws 2007). We concur: if a child does not struggle, then we believe that the experience was not problem solving for the student. Problems and struggling are intimately woven together. On the other hand, too much struggle, to the point of frustration, is not productive struggle because the process of problem solving is stopped. Negative emotions will mitigate against a child learning something about problem solving.

Given the foundational role of productive struggle as a problem-solving ability, we have developed a productive struggle pedagogy. Although we believe that this pedagogy is grounded in our professional decision making emerging from the context of a problem solving activity, there are still three recommendations we can make for other teachers.

First, cultivate an open-minded disposition for what students can do, and resist the desire to tell them what to do. If we are closed minded, we cannot see what students can do. If we are open minded, we are able to notice and tell students what they did. This is a metacognitive turn: when we label for students what they did as problem solvers, they become aware of what they can do, consistent with Schoenfeld's (1992) recommendations. Problem solving is no longer an "I can" or "I cannot" experience. Students can realize the problem-solving strategies they can use, even if they ultimately did not solve a problem.

Second, teachers should focus on problem-solving processes. If the focus is on the answer, then those children who do not find an answer are labelled as failing. On the other hand, if problem-solving processes are the goal, then children's problem-solving strategies and abilities can be developed. Given an open-minded disposition, teachers can notice and provide positive feedback for the strategies used by children. In our experience, all children are able to use at least one problem-solving strategy, although this ability differs based on the problem and the child. We are always able to find something in the activity of a child that can be labelled as problem solving. Here, finding the right scaffold is paramount. Just enough help is given so that a child is still challenged but not overly challenged. Our professional relationships with children guide our decision making to find the right balance. Every child is different, so even though the class works on the same problem, our sense of what problem-solving ability we could observe and the type of scaffolds needed varies for each child.

Finally, we plan for productive struggle by developing problem-solving activities that account for the diversity of learning needs found in a classroom. The core problem should always be rich in possibilities in how it could be solved, how far a problem could be explored and possible solutions. We always develop an engaging launch to the problem. We try to phrase the problem in an open-ended way, which allows for adaptations as children make progress on the problem. An open-ended and complex problem is a necessary condition for creating conditions in which productive struggle can occur.

While planning, we consider closely how much help to provide the whole class initially by asking two questions. Do we tell the students to model the problem using a manipulative? If we tell students which manipulative to use, the focus could switch from problem solving to the manipulative, and it is important for children to learn how to model a problem. Hence, it is valuable to provide opportunities for students to model the problem in a manner that they devise and are comfortable with. Further, do we provide the students with a preorganized recording sheet? Organization is a problem-solving strategy, and we may notice that some children lack an ability to organize their thinking and data while solving a problem. If we believe a problem has natural opportunities for students to recognize the need to be organized, then we do not provide a recording sheet. On the other hand, if we feel the problem has other problem-solving possibilities, and organization would be an extra factor that could cause too much frustration, then we do provide an organized recording sheet. There are no absolute rules for making these decisions, as it often depends on the observed abilities of the child and more specific goals for the activity.

Although planning for productive struggle is not a well-defined process, the process itself prepares us for in-the-moment decisions that effectively provide the minimum amount of support needed by each child. For example, when planning the Handshakes at a Party problem for a Grade 2 class, we wondered about both organization and modelling the problem. Recognizing an organized procedure for shaking hands is tantamount to counting the number of handshakes accurately. We decided to begin the launch of the problem, with five people, with a disorganized approach that made it difficult to track the handshakes Students were then motivated to come up with a better way to organize the handshakes. This led to also modelling an organized approach: person 1 shakes hands with persons 2, 3, 4, and 5; person 2 shakes hands with persons 3, 4, and 5; and so on. Each subsequent person shakes hands only with the remaining persons, and the last person will have nobody to shake hands with because this would repeat a handshake. During the launch, we tried to guide the class to this approach, but were more than willing to just tell the class by modelling this procedure. On the other hand, we decided that creating a model for solving the problem was rich in possibilities: students could use a diagram or manipulative to track people and handshakes, or they could act out the problem with smaller numbers of people and generate a pattern. Thus, we did not provide any help to the students by modelling the problem. One pair of students used a patterning approach with one to five persons at the party to start the pattern; the rest of the students used blocks or a diagram. We also noticed some students restart their model to help them keep track of counting the handshakes, which was an unexpected opportunity to label organization as a problem-solving strategy. Importantly for us, most students developed a model with little or no help from us. When students struggled and wanted help, we used prompts such as "Can you draw a picture?" rather than risking too much help by showing them how to draw a picture.

Our planning helped us decide how much help to provide when students did struggle. Our focus on an open-minded, open-ended and process disposition, use of planned and in-the-moment minimal-help scaffolds, and deliberate consideration of possible problem solving strategies and goals leads us to notice problem-solving ability that we would not have noticed otherwise. Our conversations with children while solving a problem and while consolidating the activity focus on the problem-solving processes and strategies we have noticed, so that every student can recognize the problem solving they did and feel success as a problem solver.

Conclusion

Fundamentally, with an engaging, well-structured yet open-ended problem-solving activity, we always observe children getting stuck and trying again. We are able to label this perseverance as a problem-solving strategy—students are problem solving. Productive struggle has transformed our sense of the nature of problem solving. It is a foundational component of problem-solving ability. It is a core principle of our problem-solving pedagogy.

References

Fernandez, C. 2002. "Learning from Japanese Approaches to Professional Development: The Case of Lesson Study." *Journal of Teacher Education* 53, no 5: 393–405.

Hiebert, J, and D A Grouws. 2007. 'The Effects of Classroom Mathematics Teaching on Students' Learning." *In Second Handbook of Research on Mathematics Teaching and Learning*, ed Frank K Lester Jr, 371–404. Charlotte, NC: Information Age.

National Council of Teachers of Mathematics (NCTM). 2014. *Principles To Actions: Ensuring Mathematical Success for All.* Reston, Va: NCTM.

Schoenfeld, A H. 1992. "Learning to Think Mathematically: Problem Solving, Metacognition, and Sense Making in Mathematics." In *Handbook of Research on Mathematics Teaching and Learning*, ed D A Grouws, 334–70. New York: Macmillan.

Warshauer, H K. 2015. "Strategies to Support Productive Struggle." *Mathematics Teaching in the Middle School* 20, no 7: 390–93.

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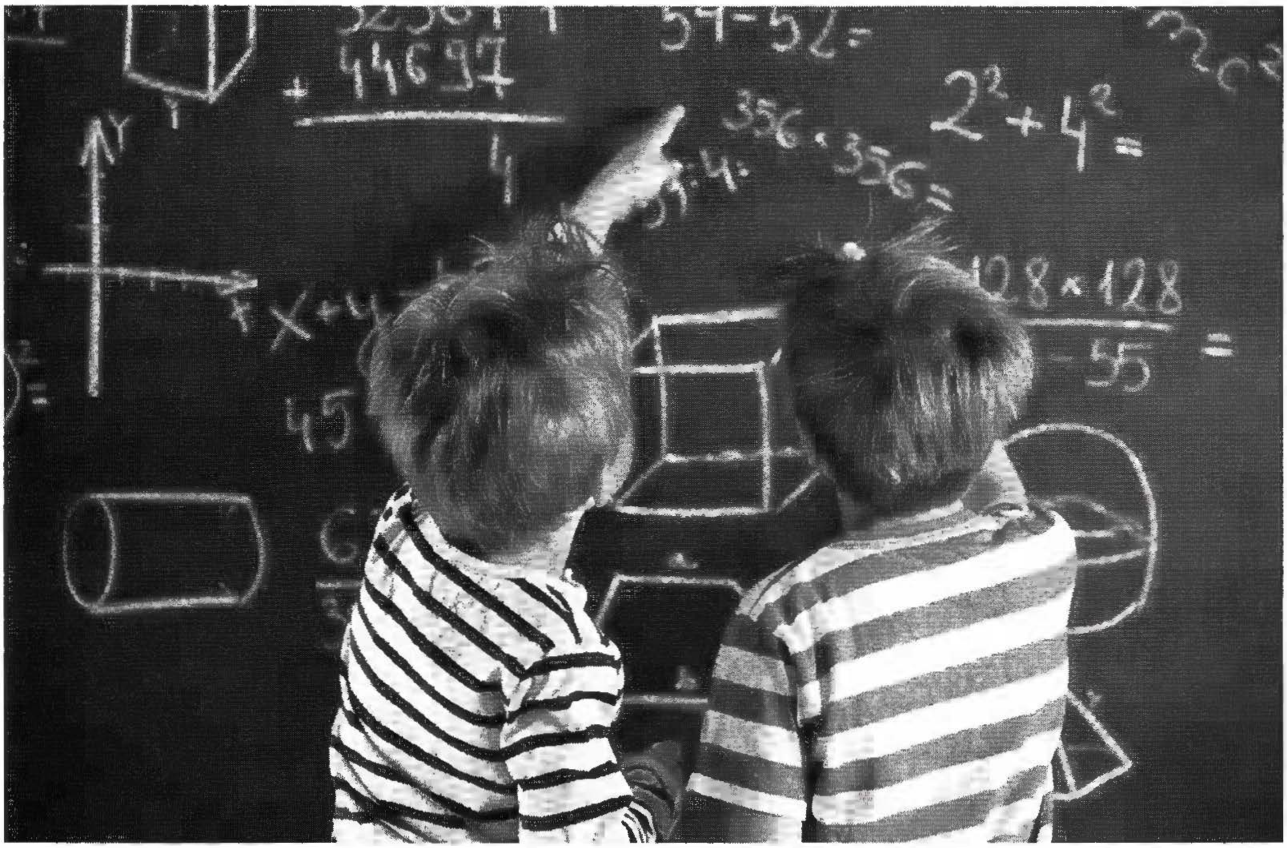
The authors thank the teachers of R F Morrison School who, with their students, assisted in this project.

Automaticity: Building the Foundation for Mathematical Mastery

Ronald Vanden Pol



Abstract

Automaticity is the ability of a student to automatically recall basic math facts without the use of strategies or manipulatives. It is an important component in the education of students and essential for student success inside and outside of the math classroom when it comes to complex problem solving. Using comparisons to reading instruction, this paper will investigate the importance of teachers helping students automate basic facts and the steps required for students to reach the mastery level of numeracy. The research for this paper was gathered through an extensive search of relevant scholarly literature using the University of Calgary's electronic library, along with the electronic databases Education Research Complete, Taylor Francis Online, ERIC, ProQuest and Google. After reviewing the information, it was found that students who were unable to automate in younger grades had difficulty with their math education as they continued in school. The automaticity of basic math facts is a foundational part of number sense and essential to helping scaffold students to higher levels of math numeracy.

Automaticity: Building the Foundation for Mathematical Mastery

One of my strongest school memories happened in a junior high math class. The teacher was working on an example math problem on the board for the rest of the class. He quickly pointed to one of the kids while working at the board and asked, "What is eight times four?" The boy didn't say anything and the teacher stopped what he was doing and asked again.

The boy began counting on his fingers and, after running out of fingers, just took a guess. The teacher then asked him again more sternly, "What is eight times four?" The student shrugged and gave a look which seemed to ask the rest of the class, how am I supposed to know that? Our math teacher then said with disappointment, "How can a student be in junior high and still not know their multiplication tables?"

Most teachers who have taught math during their career have come across students who struggle with basic math facts.

Most teachers who have taught math during their career have come across students who struggle with basic math facts. They struggle counting to 10 or they do not understand how to carry while adding. They look at their fingers while multiplying and always require two or three guesses before finally being told that was the right guess, all while hoping they were successful. They just seem to lack the basics of number sense.

The success of any student during his or her time in school is based upon many factors. In math class, one of these important factors is a student's capacity to recall basic facts quickly and accurately. The ability of a student to automatically recall facts without the use of any strategy is known as *automaticity* and is "fundamental to success in many areas of higher mathematics" (Woodward 2006, 1). By automating their basic math facts, students can have more cognitive resources free for understanding and performing increasingly complex mathematical chores (Caron 2007; Poncy, Skinner and Jaspers 2006; Ramos-Christian, Schleser and Varn 2008; Woodward 2006). It is important for teachers to ensure that students confidently know their basic math facts, to unpack the stages required to get them to the point of mastery and to fit memorizing basic math into their regular maintenance routines at all grade levels to provide students with the best chance to succeed.

The Stages of Mathematical Mastery and the Importance of Automaticity

Over the years, various researchers have agreed on three stages of understanding math facts and strategies that would help increase fluency in students (Ando and Ikeda 1971; Ashlock 1971; Carnine and Stein 1981; Garnett and Fleischner 1983; Isaacs and Carroll 1999). The first stage involves students getting a concrete or pictorial understanding of numbers. This can be seen as counting on fingers, numbering stuffed animals, or asking dad for two more scoops of ice cream. In literacy this can be understood as identifying letters and the phonemes attached to those letters. These activities are the building blocks for numeracy and literacy.

The second stage requires students to learn or figure out different methods or strategies to remember information and connect it to facts they already know or understand. To again use literacy as a comparison, this is similar to students sounding out longer words or using words which they have read in the past to help decipher the current word. Students cannot blindly go into math or reading by randomly guessing in order to get the answers, but must start at the beginning and use basic strategies in order to solve for the correct answer. Some of the most well-known strategies are "carrying the 1" or "borrowing" when it comes to addition and subtraction. When these strategies are used properly and with understanding, then students can move on to the final stage.

The final stage is, of course, mastery. Alberta Education explains that mastery occurs "when students understand and recall facts. This allows students to apply their knowledge to different and more complex computations and to be flexible in their thinking" (Alberta Education 2014a, 1). Students are expected to understand basic facts and apply that knowledge to solve problems. Students are also expected to recall facts. While recall expects students only to use a strategy to efficiently find an answer, automaticity requires students to instantly give a response. Students are not required in Alberta to automate basic math facts, but I would argue that it is very important in both math and reading.

By automating their basic math facts, students can have more cognitive resources free for understanding and perfoming increasingly complex mathematical chores.

Students do not require time and energy to sound out common words when reading, but rather have these words automated to the point that they do not even have to think about them. How many times during this article have you stopped to sound out a word? You just know the average word after many years of regular practice. This is also true for math. There comes a point in time when recognizing 145 or adding 12 + 5 does not take any brain power at all, but is just an automatic reaction to seeing a specific combination of numbers.

Here is another example of automaticity, from Samuels and Flor as they write about the need to improve automaticity in students for reading.

> There are numerous examples of skills developed to automaticity with which readers will be familiar. Driving an automobile effortlessly through traffic is one example of automaticity. However, it was not always so easy. Recall your first ventures behind the wheel, when considerable attentional energy was consciously applied to mechanical aspects of driving such as avoiding accidents and shifting gears. For the beginning driver, so much attention is focused on the mechanical aspects of driving that holding a conversation with a passenger while driving is impossible. But with practice, the mechanical aspects of driving become less demanding, and the skilled driver can simultaneously listen to the radio, hold a conversation, and appreciate the scenery. Skills practiced and learned to the point where they are considered "automatic" demand less cognitive and attentional energy; thus the person with expertise is capable of performing multiple complex tasks at the same time. (Samuels and Flor 1997, 108).

As you can read in this example, automaticity does not come instantly. It needs to be worked on and practised over time in order to ensure that the students are learning the basics. In math this is true as well, and it is key that students progress through the first two stages before attempting automaticity in order to help achieve mastery. Students who are struggling are often asked to do the impossible and skip ahead to stages they are not prepared for. If you have a struggling student, ask yourself what stage the student is in and then work on how to best assist that student at his or her level. Sometimes the best way to move forward is to take a step back.

First Stage: Counting Strategies

Number sense begins long before a child enters kindergarten. It starts with how many more bites they need to finish on their dinner plate. It comes out with how many more books they want their parents to read for them. It progresses when they notice that siblings have more than they have. These quantitative concepts begin long before a child steps into a classroom and can greatly affect how children will do in the classroom (Gersten and Chard 1999). Gersten and Chard found, based on home visits, that the lack of knowledge in young students "reflected a lack of experiences with adults or siblings that would facilitate the association of quantity and numbers and would lead to the development of an abstract numerical understanding" (Gersten and Chard 1999, 23). This lack of experience leads to a poor number sense for those students, and it is only when they have a concrete idea of number that it is possible to move onto the next stage. Early educators and Division I teachers need to be especially conscious of this lack of concrete understanding because some students are stuck in this stage for a long time. Providing students with counting and mathematical situations is crucial to support their growth with numeracy.

Number sense begins long before a child enters kindergarten.

Second Stage: Reasoning Strategies

Once students have basic numeracy skills it is possible to teach them strategies for comparing and relating numbers or solving larger problems. These strategies are used to improve the accuracy of student answers. Some examples for addition and subtraction given by Isaacs and Carroll (1999) are

1. basic concepts of addition; direct modelling and counting all for addition (number lines);
2. the 0 and 1 addition facts; counting on; adding 2;
3. doubles (6 + 6, 8 + 8 and so on);
4. complements of 10 (9 + 1, 8 + 2 and so on);
5. basic concepts of subtraction; direct modelling for subtraction;
6. easy subtraction facts (-0, -1, and -2 facts); counting back to subtract;
7. harder addition facts; derived-fact strategies for addition (near doubles, over-10 facts);
8. counting up to subtract; and
9. harder subtraction facts; derived-fact strategies for subtraction (using addition facts, over-10 facts). (p 511)

These strategies are similar to the approach that the Government of Alberta has put into the Alberta program of studies in order to encourage mental mathematics—the ability of a student to use different math strategies reliably and accurately (Alberta Education 2014b, 5). These are not limited to just addition and subtraction, but exist throughout the math curriculum in all units. Learning different strategies is a key component of the second stage of mastery because it allows for students to find ways to accurately solve problems using reliable steps.

The most important thing to remember is that students learn numeracy skills through strategies that help them, as individuals, solve problems more fluently (Baroody 2006). For past generations, teachers relied on the strategies they were taught to teach their own students, which worked well for the majority of learners. The problem is that they did not work for every student. Most people reading this can think of a kid in their class growing up that did not seem to "get" math. The teacher tried teaching the strategy over and over again, but it did not stick. For some students, solving this issue may have required going back to the first stage and working on building counting skills, but for others, it just required adapting strategies to fit the needs of a particular student. There can be different strategies for different problems, but students need to find what works for them.

Third Stage: Mastery

The final step in mathematical understanding is mastery. Students are able to solve problems using their abilities to recall and understand number facts. Alberta Education explains to teachers that "Mastery is a progression of learning. For example, students work towards mastery of multiplication and related division facts, beginning in Grade 3" (Alberta Education 2015, 1). However, it is important to know that this happens at different times for different students. Students need to have spent an appropriate amount of time working on their counting and reasoning strategies before they can be expected to fully understand and use their basic fact knowledge. Practising automaticity while working on strategies is one way to help them achieve mastery.

The Importance of Automaticity

"Automaticity is an immediate and unconscious retrieval of answers, which suggests a rate faster than one answer per second" (DeMaioribus 2011, 6). Automaticity is different than just being fluent with numbers, because there is an expectation that this process happens without thought. Why is counting on fingers not good enough and why must a student be able to do these basic math calculations automatically? "Although correct answers can be obtained using procedural knowledge, these procedures are effortful and slow, and they appear to interfere with learning and understanding higher-order concepts" (Hasselbring, Goin and Bransford 1988, 2).

If a child is reading a novel and is struggling to pronounce and decipher every word, do you think he or she would have much luck comprehending what the novel is actually about? By automating certain words and reading without conscious thought, our brains are then more open to comprehending the text. If the words are confusing to the students, how do we expect them to understand that sentence or paragraph? If students are struggling with the basic calculations in a multiple-step problem, should we expect them to be able to successfully figure it out? In fact, "without this seemingly simple set of knowledge, by eighth grade, students are virtually denied anything but minimal growth in any serious use of mathematics or related subjects for the rest of their school years, and most likely, the rest of their lives" (Caron 2007, 279). Time spent practising basic math facts is as important as time spent practising reading. The goal is to improve automaticity and allow students to focus on the problem instead of just the numbers.

The problem is that they did not work for every student.

"Researchers explored the devastating effects of the lack of automaticity in several ways. Essentially they argued that the human mind has a limited capacity to process information, and if too much energy goes into figuring out what 9 plus 8 equals, little is left over to understand the concepts underlying multidigit subtraction, long division, or complex multiplication" (Gersten and Chard 1999, 21). Automaticity is so important to students' numeracy skills because without it students' skills are extremely limited. We need students to automate as many of these basic facts as possible, so that as much attention as possible is freed up to deal with the most complex part of the problem.

Helping Students with Automaticity

In my own Grade 6 class we commit a few classes a month to improving student automaticity with basic facts. While it is not a part of the program of studies for Grade 6, Alberta Education (2014b) writes that teachers should work with students over time to "refine their strategies to increase their accuracy and efficiency" (p 9). I try to take time to go over basic addition strategies to help struggling students see success and already-successful students see even more growth. This past month we worked on adding doubles. We first went through all of the doubles that did not require regrouping and did a few math minutes using just those problems. We then went over doubles that did require regrouping and practised equations related to those numbers. Because what we were practising required remembering only small amounts of basic facts, most students had very little trouble with the problems. Those that did have some issues spent extra time practising only the equations they struggled with. Very quickly, all students were able to see success working with adding doubles.

The next time we worked on basic facts, we practised doubles + 1 without regrouping (4 + 5, 2 + 3). After students became proficient in those, we moved on to those that required regrouping. After that, doubles + 2 and then making 10s. Once most of the strategies had been shared and practised I provided students a chance to play adding games, like war with cards; adding dice; and rock, paper, numbers (two players stick out fingers like rock, paper, scissors and the first person to figure out the answer wins).

My class also has a set of Chromebooks. I, and researchers like Nelson et al (2013), have found success in using computer programs that can give students who struggle instant feedback as a way to improve fluency without the use of any mnemonic strategies, in the same way a parent reading with a child can give instant feedback and correction. For example, xtramath.org is a website that will assess a student's basic math facts abilities and then give the student questions on whatever needs the most help.

If a child is reading a novel and is struggling to pronounce and decipher every word, do you think he or she would have much luck comprehending what the novel is actually about?

At my school, a colleague uses a program that helps Grade 3 students scaffold their learning through different reasoning strategies. The class works on these strategies and practises automaticity through flashcards and math minutes. Students progress on a continuum moving from simple equations, such as adding 1, to more difficult facts once they have achieved a satisfactory level of recall and understanding. This practice is done daily for about 10 minutes per class. They focus only on addition before moving to subtraction and then multiplication and division facts. The program is modelled like a karate dojo and uses belts as a theme to encourage students, make it fun and keep them motivated. Daily practice gives students a chance to commit facts to memory and improve automaticity, which helps their problem-solving abilities in other outcomes and mathematical concepts.

While the practices mentioned above may seem like a lot of work for higher grades, it is a review of strategies they already know and requires just a little bit of time to practise. A few classes a month for older students is more than enough to maintain their current abilities and improve their speed and accuracy. Students in Grades 3 to 5 require more regular practice to learn strategies and also to play games to keep students engaged in working toward automaticity. The important thing to remember is that success in automaticity is often found in small, manageable bits. A little bit at a time is much better than everything all at once. For most students in younger grades (K–2), automaticity may be a bit early and it is probably best for them to stick to counting and simple reasoning strategies.

Conclusion

In Alberta, the program of studies clearly lays out the expectations for teaching the basic math facts from Grade 1 up to Grade 6. It is [important] for teachers during these years to ensure that they are not only creating a concrete understanding of numbers through manipulatives and strategies, but also promoting and practising automaticity to support students getting to mastery. If students are not ready to move on to the next stage, more and more practice may not be the right answer. However, when a stage is successfully realized, regular practice and feedback is a vital part of the process. The importance of automaticity is not just in helping students get the correct answer, but also about helping them achieve the correct answer while using less mental strain and allowing them to freely think about larger problems. Automaticity is about building a solid foundation on which other teachers are able to support students in understanding more and more complex problems. It is crucial that students have the automaticity needed to work through the large, complicated, real-world, multistep problems that they will encounter in daily life. And it is up to us as professionals to ensure that students are reaching this goal.

References

Alberta Education. 2014a. *Clarification of Expectations Regarding Basic Number Facts and Strategies*. Available at https://education.alberta.ca/media/481800/clarification_of_expectations_regarding_basic_num_facts_and_strategies.pdf or http://tinyurl.com/gr3632a (accessed June 22, 2016).

———. 2014b. *Program of Study: Mathematics Kindergarten to Grade 9*. Available at https://education.alberta.ca/media/464621/k_to_9_math_pos.pdf or http://tinyurl.com/jgqgoyt (accessed June 22, 2016).

———. 2015. *Bulletin for Teachers*. "Helping Parents Understand the Alberta *Mathematics Kindergarten to Grade 9 Program of Studies*." Available at https://education.alberta.ca/media/481796/teacher_bulletin.pdf or http://tinyurl.com/joqedle (accessed June 22, 2016).

Ando, M, and H Ikeda. 1971. "Learning Multiplication Facts—More Than a Drill." *Arithmetic Teacher* 18, no 6: 366–69.

Ashlock, R B. 1971. "Teaching the Basic Facts: Three Classes of Activities." *Arithmetic Teacher* 18, no 6: 359–64.

Baroody, Arthur. 2006. "Why Children Have Difficulties Mastering the Basic Number Combinations and How to Help Them." *Teaching Children Mathematics* 13, no 1: 22–31.

Carnine, D W, and M Stein. 1981. "Organizational Strategies and Practice Procedures for Teaching Basic Facts." *Journal for Research in Mathematics Education* 12, no 1: 65–69.

Caron, T A. 2007. "Learning Multiplication: The Easy Way." *Clearing House* 80, no 6: 278–82.

DeMaioribus, C E. 2011. "Automaticity of Basic Math Facts: The Key to Math Success?" Master's thesis. University of Minnesota. Available at https://d-commons.d.umn.edu/bitstream/10792/274/1/DeMaioribus,%20Carmel.pdf or http://tinyurl.com/zf5tgr5 (accessed June 24, 2016).

Garnett, K, and J E Fleischner. 1983. "Automatization and Basic Fact Performance of Normal and Learning Disabled Children." *Learning Disability Quarterly* 6, no 2: 223–30.

Gersten, R, and D Chard. 1999. "Number Sense: Rethinking Arithmetic Instruction for Students with Mathematical Disabilities." *Journal of Special Education* 33, no 1: 18–28.

Hasselbring, T S, L T Goin, and J T Bransford. 1988. "Developing Math Automaticity in Learning Handicapped Children: The Role of Computerized Drill and Practice." *Focus on Exceptional Children* 20, no 6: 1–7.

Isaacs, A C, and W M Carroll. 1999. "Strategies for Basic-Facts Instruction." *Teaching Children Mathematics* 5, no 9: 508–15.

Nelson, P M, M K Burns, R Kanive and J E Ysseldyke. 2013. "Comparison of a Math Fact Rehearsal and a Mnemonic Strategy Approach for Improving Math Fact Fluency." *Journal of School Psychology* 51, no 6: 659–67.

Poncy, B C, C H Skinner and K E Jaspers. 2006. "Evaluating and Comparing Interventions Designed to Enhance Math Fact Accuracy and Fluency: Cover, Copy, and Compare Versus Taped Problems." *Journal of Behavioral Education* 16, no 1: 27–37.

Ramos-Christian, V, R Schleser and M Varn. 2008. "Math Fluency: Accuracy Versus Speed in Preoperational and Concrete Operational First and Second Grade Children." *Early Childhood Education Journal* 35, no 6: 543–49.

Samuels, S J, and R F Flor. 2006. "The Importance of Automaticity for Developing Expertise in Reading." *Reading & Writing Quarterly: Overcoming Learning Difficulties* 13, no 2: 107–21.

Woodward, J. 2006. "Developing Automaticity in Multiplication Facts: Integrating Strategy Instruction with Timed Practice Drills." *Learning Disability Quarterly* 29, no 4: 269–89.

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Ready, SET, Play: Learning Elementary School Mathematics Through an Attribute-Based Game

Janelle McFeetors and Kaitlyn Ireland

The use of engaging activities such as games in elementary mathematics classrooms can help children form a positive disposition towards mathematics and provide meaningful learning opportunities. Games have been recommended as a way for students to develop an understanding of mathematical ideas before they move toward abstractions (Diénès 1971). Ernest (1986) identified three educational uses of games in mathematics class: gaining skill-based fluency, developing conceptual understanding and refining problem-solving approaches. Analysis of students' learning through varied game contexts has identified benefits such as (1) the context of mathematics class allows for children to readily mathematize authentic contexts originating outside the classroom (Linchevski and Williams 1999); (2) small-group settings in games, with peers and a teacher, support mathematical conversations in which learning occurs (Polaki 2002); and (3) children demonstrate improvement in attitude and motivation for learning mathematics (Lopez-Morteo and Lopez 2007).

Commercial card and board games have an important presence in many homes of the students in our classrooms. We identify games as "commercial" when they are marketed to the general public and are easily purchased through bookstores or toy stores. While often these games are used for enjoyment, we realized that many games have an educational element that can be employed in elementary mathematics classes. Different from instructional games—games created to intentionally learn or practice specific mathematics skills—commercial games can be seen as authentic contexts in which to experience mathematical ideas. Through our interactions with elementary school students in several classrooms, we explored the mathematical experiences available through commercial games.

Games have been recommended as a way for students to develop an understanding of mathematical ideas before they move toward abstractions.

In this article, we offer a sense of the richness of mathematical learning possible through incorporating a card game. We use the game SET as a specific example, first describing the rules and explaining the context of our work. Students' mathematical thinking embedded in their game play offers a strong case for incorporating SET as an opportunity to learn in mathematics class. We end by identifying curricular connections and offering some ideas for differentiating to engage all students in experiencing mathematical learning through games.

Try Making a Set!

We explored the use of the game SET in an elementary school mathematics club. The commercial card game is a visual discrimination game that students can learn quickly and that allows students to explore identifying attributes and sorting in a captivating manner. SET is a game for one or more players, and takes approximately 15 minutes to play. Players aim to make a set that consists of three cards. Twelve cards are placed face up in a rectangular arrangement on a surface for all players to see, as depicted in Figure 1.

Cards are made up of figures with four varying attributes: shape, colour, quantity and shading.

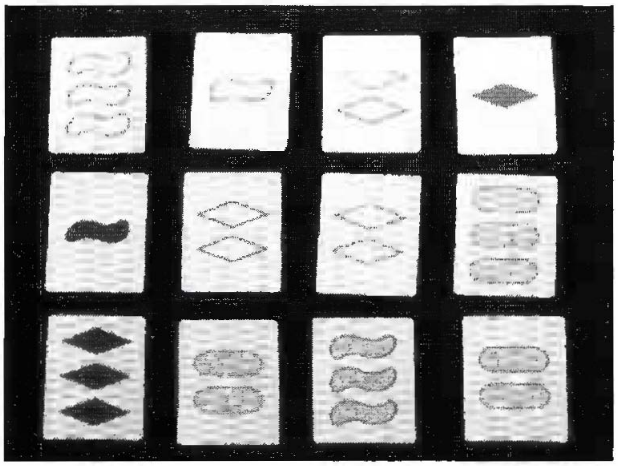

Figure 1: *The game SET*

In order to make a set, a player must identify 3 cards on which the individual attributes are either all the same or all different across the 3 cards. Once a player identifies a set, the player must say "set" out loud and remove the cards from the table; the other players verify that the set is correct. If the player forms a valid set, the player keeps the cards; otherwise, the cards are returned to the table. Figure 2 shows examples of sets. During game play, cards are replaced from the deck so that 12 cards remain at all times on the table until the cards run out. Game play is over once all cards have been used. Players count the number of sets they have collected during the game. Each set counts as a point and the player with the most points wins.

SET can be purchased inexpensively at many stores, making the game easily obtainable for teachers to add to their classroom materials to be used as part of a mathematics lesson or station. We found that students appreciated having a "real" game to play in class. Within limited budgets, it is possible to construct a homemade version of the game because it relies only on cards as playing materials. As a result of searching online we also discovered that the game is available for free through several Internet sites, including

- www.setgame.com/set/daily_puzzle,
- http://smart-games.org/en/set/start/ and
- www.lsrhs.net/faculty/seth/Puzzles/set/set.html.

SET could also be used in the classroom by app for iPad and iPod touch. We invite readers to give one of these online versions a try before continuing.

Context

To explore students' mathematical thinking within the game of SET we attended an elementary school math club over a three-month period. The math club met weekly during lunch hour as an extracurricular activity. Grades 4 to 6 students were invited to participate, with an average of more than 30 students attending each week. In the small elementary school that was the setting for the research project, almost half the Grades 4 to 6 students attended at times over the three months.

In math club, students would eat their lunch while a teacher introduced a new game. Often instructional videos were used to highlight game rules, such as this one for the game SET: www.youtube.com/watch?v=bMhJmrJVP4Q. If no new game was introduced that week, students would be asked to share some strategies they developed from the previous week. After the introduction the students selected who they wanted to play with and which game they wanted to play out of seven possible games: Farkle, SET, Othello, Gobblet Gobblers, CirKis, Equilibrio and Quartex. Students had the option to play in pairs or individually against their opponent(s). After the students had chosen their game, the rest of math club time was given to open exploration and playing the selected game.

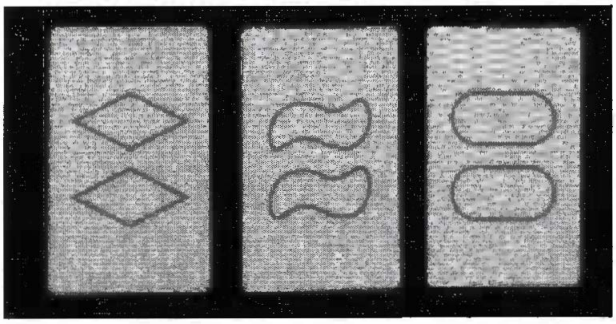

Figure 2a: *Example 1 of a set*

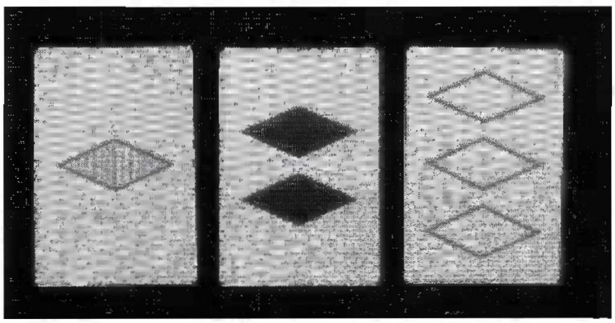

Figure 2b: *Example 2 of a set*

Two teachers, two parent volunteers and a researcher were present to play with the students and to pose questions to prompt mathematical thinking. Several times the students were invited to fill in record sheets to help them identify some of their mathematical thinking and to share some of their personal strategies. A couple of examples of reflective prompts include "How do you look for sets?" and "When you can't see a set right away, what do you do?" The students who took part in our research project also participated in informal interviews in pairs to further demonstrate their mathematical thinking and share their thoughts on the games in math club. The examples of students' thinking in this article come from both their interviews and their record sheets.

Students' Mathematical Thinking Through SET

We highlight students' learning through three particular Grades 4 and 5 students: Nicole, Zahra and Rimira (pseudonyms are used for all students). We selected these students because they saw themselves as experts and had developed facility with SET. In general, the students in math club felt they were a game expert if "you have a good chance of winning and a lot of strategies," as one student explained to the group. It became apparent that to only have one strategy to rely on in playing wasn't seen as effective by students, but a variety of strategies allowed their play to respond to each game situation, often leading to a win.

The students who played SET, however, identified that there is more to being an expert than simply winning the game. Zahra, who was very quick to identify sets, attributed her exceptionality at SET to having "good looking." In this brief explanation, Zahra demonstrates a capability to notice her own mathematical

They grew as mathematical learners.

thinking, indicating that visualizing was an important process for her as she built sets systematically.

As we observed students playing SET, explored their record sheets and interviewed the students, we were pleased with the mathematical thinking and learning occurring. We are excited to share some of the students' thinking as examples of the possibility for rich mathematical learning embedded in an engaging commercial game like SET. In particular, students developed a range of strategies as they tried to improve, they engaged in important mathematical processes and refined their use, and they grew as mathematical learners.

Learning to Sort by Using Personal Strategies

When we asked students about what made the games used in math club mathematical in nature, a common response was similar to Nicole's—the games "teach us lots of new strategies." While personal strategies are often used in the program of studies in conjunction with arithmetic computations (Alberta Education 2014), the development of strategies is

Their personal strategies relied on discriminating among attributes, classifying groups of cards, and developing and applying sorting rules.

important more generally in all mathematical learning because it indicates that students are making procedures and approaches that are meaningful to them. Related to "relational understanding" (Skemp 2006)—where students not only know what to do to craft a solution to a mathematical question, but they also know the reasons for why they carried out a particular procedure—students develop the *why* in creating their own personal strategies.

Many personal strategies emerged as students learned the game and moved toward being expert players. Their personal strategies relied on discriminating among attributes, classifying groups of cards, and developing and applying sorting rules. When students were prompted to explain how they searched for a set, students demonstrated a range of ways to proceed systematically that worked best for them.

Some students used the cards as physical objects to identify sets. They would sort cards that had potential in forming a set and narrow down related attributes. For example, Zahra preferred to physically pick up or point at cards to help her keep track of the cards she was sorting to make a set. Figure 3 shows Zahra beginning a set by selecting two cards with the same shape (ovals), same shading (solid), different colours (green and red), and different quantities (one and two). The importance of using manipulatives in mathematics class has been well documented previously (eg, Boggan, Harper and Whitmire 2010; Moyer 2001; Sowell 1989). By using the SET cards in a concrete fashion, students like Zahra were able to keep track of their possible sets.

Figure 3: *Zahra manipulating the cards*

While some students who are more competitive might not want to point to cards to reveal possible starting places for making a set, we noticed that the physical grouping of cards did not inhibit Zahra from winning many of her matches. In fact, Bonnie (the supervising teacher for math club) exclaimed that Zahra "was amazing at SET!" She followed up with Zahra by posing the question, "How do you see these [sets] so fast?" For us, this was an important question because it pointed to the process of visualizing that students began to apply as they moved beyond using the cards as manipulatives.

Another common strategy students developed was to begin with a partial set. Seen as an act of "specializing" (Mason, Burton and Stacey 2010), this refers to using specific examples or narrowing the problem into a smaller problem to be solved. For students playing SET, that often meant quickly identifying two cards that constituted part of a set. In the picture above, Zahra portrays this approach. Rimira expressed this as her most frequent strategy. She would "look for two that are pairs and then I try to see if there is something that can go with that and if not I just move on to a different two." Students using this strategy would move across the table with a pair of cards as a point of comparison, having established a sorting rule that could then be applied to the other cards on the table. Generating the sorting rules created an efficient way of discriminating, supporting growth in pattern noticing.

In another form of specializing, students would also systematically explore the 12 cards laid out on the table. A powerful approach to thinking mathematically is to be systematic in exploring the situation, where "success is more likely if the specializing is done systematically" (Mason, Burton and Stacey 2010, 5). The most systematic approach the students demonstrated was to isolate one attribute on the SET cards at a time.

During math club, Nicole explained her strategy to us as "I look for one colour, and then I look at another colour and then the other colour." Beginning with the attribute of colour, Nicole used the attribute to direct how she looked at the rest of the cards on the table. Rather than sorting by four attributes, specializing allowed Nicole to narrow down the possibilities and systematically keep track of the leading attribute. If this did not result in forming a set, Nicole would be able to justify why and move to isolating another attribute confidently.

Bonnie also described Zahra's systematic approach as "she would pick a characteristic and look for that particular one. But that would lead her to another one. ... It's like she was following a trail of them." When we asked Zahra in an interview about her expertise, she also confirmed that she liked to focus on "the detail on things" to help her make a match aiding her development of focusing on attributes. Nicole and Zahra's personal strategies established a very systematic way to play the game, which allowed them to develop a logical argument for quickly sorting cards by attributes.

Through the development and use of personal strategies, students began to make statements about broader game play. For instance, Nicole found that it was easiest to find sets where all of the cards are all the same in each of the individual attributes. This built on her personal strategy of attending to one attribute at a time. One generalization Zahra made was that the hardest sets to find were ones where the attributes were all different. In this case, she would have to look at every card rather than applying sorting rules created from pairs of cards. The process of generalizing is so important that it has been described as "the life-blood

> *We noticed growth in the students' mathematical processes of communication, visualization and reasoning.*

of mathematics" (Mason, Burton and Stacey 2010, 8). Rather than specific strategies, the generalizations that students formed articulated a global approach to sorting situations grounded in noticing patterns across many instances of playing SET.

Developing Mathematical Processes

The mathematical processes that appear in the program of studies (Alberta Education 2014), such as communication, visualization, reasoning and problem solving, are also important elements in the mathematics classroom that are integrated across all learning opportunities for students. As the students collaborated during game play, they were continually engaged in developing the mathematical processes as "shared mathematics promotes the development of problem solving, reasoning and communication skills" (Alberta Education 2010, 217). In particular, we noticed growth in the students' mathematical processes of communication, visualization and reasoning.

Communication

Communication in an elementary mathematics classroom can be characterized as "a way of clarifying students' thinking and understanding" and "a way of revealing their thinking, their reasoning, and what they know and do not know" (Greenes and Schulman 1996, 160). Students were talking with their peers frequently while they were playing SET. As they identified a set, they defended the composition by verbalizing to their opponents the validity of the set. Peers were appropriately skeptical so that no player would claim an undeserved set. Platz (2004) emphasizes that "there is a need for children to not only sort and classify objects but for them to communicate their thinking as to how they sorted or classify the set of objects provided to them" (p 90).

As students were learning SET, they often discussed the attributes of the shapes on the cards with their peers in order to assess whether they had identified a set. This helped establish a common way of grouping attributes—for instance, the patterns filling the shapes were referred to as "shading," "pattern" or "detail"—providing a rationale for developing mathematical terminology for mathematics class. As students became experts, they communicated their thinking very clearly to others.

Through game play and the rule that a player must show the other players the identified set, the students became fluent at communicating the components of their set. For example, Nicole justified why three cards were a set by explaining, "Same shape, same colour and same detail. One, two, three." Although brief, Nicole's statements were complete enough to explain her rationale and satisfy her opponent. In addition to the specificity of identifying a set, we have already demonstrated above that students communicated their strategies on how to locate a set that moves toward expressing generalizations valued in mathematics.

Visualization

Visualization is a mathematical process that is rooted in action—the act of visualizing, which grows out of manipulating objects and creating (pictorial) representations. As students learn mathematical ideas, we can understand the early development as "image making" and "image having" to develop mental images of mathematical ideas (Pirie and Kieren 1994). Visualizations are recognized as

> physical objects (i.e., illustrations, computer-generated displays); mental objects pictured in the mind (i.e. mental schemes, mental imagery, mental constructions, mental representations); or cognitive processes (i.e., cognitive functions in visual perception, manipulation and transformation of visual representation by the mind, concrete to abstract modes of thinking, and picturing facts). (Macnab, Phillips and Norris 2012, 104)

In the case of SET, students were using the cards (illustrations on physical objects) to scaffold the ability to see possible groups of cards mentally as they transformed the arrangement of cards.

SET is advertised as a game of "visual perception" (www.setgame.com), which could occasion moments for students to improve in visualizing. Zahra mentioned on several occasions that she depended on her visualization to aid her in finding a set. For instance, when asked why SET belongs in math class Zahra acknowledged that the game "gives your eye a little workout." By saying this, Zahra recognizes the importance of visualizing in the game and its place in mathematics class. To her, visualizing during a mathematics activity is a fundamental component of becoming mathematically literate. Students used the mathematical process of visualization to sort and organize objects and data, recognize same and different, form mental images, and focus on attributes. Students carry with them the mental images they create to learn related mathematical ideas in meaningful ways.

Reasoning

Mathematical reasoning is viewed as one of the most important mathematical processes and "involves exploring the mathematics at hand; generating, implementing, and evaluating conjectures; as well as justifying our thinking and actions as we engage in mathematics" (Thom 2011, 234). Students playing SET were exploring the mathematical actions of discriminating among attributes, sorting by attributes and generating rules for classification. They created conjectures as to which cards would belong in a set and identified a possible set to be assessed by opponents. The need to justify was inherent in the game and valued by students as they played. Even the students noticed that the game "gets your mind going" and "gives your brain a workout," becoming aware that they needed to be cognitively active to succeed in a mathematical game like SET.

Deductive reasoning structures began to emerge as students made if–then statements as they were playing. If students had difficulty finding a set, rather than getting frustrated they persevered and talked among their opponents, making statements such as, "If there was this one here [gesturing], then the set could have gone through." Rather than imposing this important mathematical structure, students used if–then statements in an authentic and meaningful way. If–then statements also mark in the students' reasoning that they are developing relationships among the attributes and classifications, and "students who understand such relationships are reasoning at higher levels" (Fox 2000, 573). We were excited to hear this way of generating conjectures because of the foundational experiences in reasoning that can lead to later facility with constructing proofs. As students worked together during game play, they were continually engaged in refining the mathematical processes integral to *doing* mathematics.

They were continually engaged in refining the mathematical processes integral to doing mathematics.

Growing a Positive Disposition

Each week, students eagerly picked games at the beginning of math club time. They showed enthusiasm as they developed winning strategies and defended their plays to their peers. In describing the development of mathematical proficiency, researchers have identified a productive disposition as one of five components necessary for students' success, defining it as a "habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy" (National Research Council 2001, 116). We see a willingness to engage in mathematical tasks, such as thinking mathematically during SET, as an indication of a productive disposition. It incorporates and moves beyond a positive attitude so that students "appreciate and value mathematics" (Alberta Education 2014)—a goal within the Alberta mathematics curriculum. We noticed that the students who played SET were engaged, exhibited perseverance and were confident.

In describing the development of mathematical proficiency, researchers have identified a productive disposition as one of five components necessary for students' success.

We asked the students why they like playing SET, as they often returned to the game week after week. Nicole explained that the game is "simple and fun." While it may not be the primary reason to implement the game into your mathematics class, Nicole's statement is worth considering. Nicole demonstrated in her play that she was often successful at finding sets, and connected her efficacy with the straightforward nature of the play. Rimira echoed Nicole's sentiment when she identified that the game could be used "for people who have trouble at math, they can have an easier way to learn math." Both students had confidence in their ability to play a mathematical game. Students can feel the most successful when they believe that they can accomplish something and can be more engaged and motivated to learn when the activity they are doing is enjoyable.

Out of a positive disposition often comes a willingness to persevere in problem-solving situations. We found this to be the case in playing SET on a number of occasions. Zahra identified several times where "there is no matches or anything," making it impossible to make a set within a group of 12 cards laid out on the table. When this occurred, she eagerly explained how no set could be formed. This required perseverance by systematically eliminating all possibilities. Additionally, notice one of Zahra's final reflections on her learning through the game, shown in Figure 4 overleaf. Like Zahra, students often found the last round most challenging as cards on the table diminished, but also took great pride in identifying the last set of the game, regardless of how long it took. Rather than causing frustration, coming across challenging groups of cards provided opportunities for

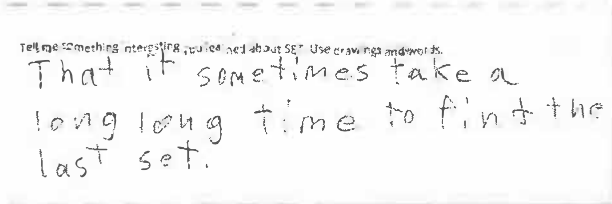

Figure 4: *Zahra's final reflection*

students to develop and use different strategies while discussing with their peers whether a set existed.

We also noticed students' confidence as they sustained engagement in playing SET, seeing themselves as growing mathematical learners. While the game was relatively quick to learn how to play, students enjoyed the variation of each group of 12 cards being different and the competition of being first to identify a set. Nicole explained that she would "just play [the game] a lot" and chose it frequently during math club. Her choice to play often provided the opportunity to develop a range of strategies, refining older strategies and testing newer ones. Even outside of math club, Rimira mentioned that she could "play SET on the Internet." While Rimira did not consider mathematics to be her favourite subject in school, Rimira's choice to play a mathematically oriented game outside of mathematics class speaks to her evolving sense of her capability in mathematical tasks. These three students demonstrate, on behalf of their peers, how incorporating commercial games with mathematical ideas can foster a shift in perspective about mathematics.

Curriculum Connections

The qualities of mathematical thinking and engagement that the students demonstrated are important reasons to incorporate SET in elementary school mathematics classrooms. We are mindful that in addition to the broad goals for students and mathematical processes that are to be incorporated in all aspects of learning mathematics in school, it is beneficial to connect mathematical tasks with specific learning outcomes.

Students in the early grades of elementary school focus on identifying attributes in objects, sorting by a predetermined rule (comparing), identifying a rule used to sort objects (pattern noticing), and creating and expressing rules for sorting (generalizing). The Alberta program of studies (Alberta Education 2014) includes the following specific learning outcomes:

- Kindergarten: Sort a set of objects based on a single attribute, and explain the sorting rule.
- Grade 1: Sort objects, using one attribute, and explain the sorting rule.
- Grade 2: Sort a set of objects, using two attributes, and explain the sorting rule.
- Grade 3: Sort objects or numbers, using one or more than one attribute. (p 60)

While SET consists of more attributes than listed above, we believe that extending the trajectory in this group of learning outcomes into later elementary grades is beneficial for encouraging growth of complexity in distinguishing attributes and patterning.

Experiences of thinking mathematically in these ways supports students' emerging understanding of complex mathematical ideas in later grades.

Building on these foundational experiences, "sorting, classifying, and ordering facilitate work with patterns, geometric shapes, and data" (National Council of Teachers of Mathematics 2000, 91). In other words, experiences of thinking mathematically in these ways supports students' emerging understanding of complex mathematical ideas in later grades. Additionally, while students classify, sort and categorize information they are also learning about relationships between the objects that they are organizing. Through further analysis of the Alberta program of studies (Alberta Education 2014), we noticed this learning trajectory through multiple content substrands in elementary school.

- Patterns: Students' understanding of *same* and *different* through identifying attributes supports pattern building and pattern noticing, while creating rules leads directly to making generalizations in pattern expressing. These are important algebraic skills.
- 3-D Objects and 2-D Shapes: Students' abilities to notice attributes to describe characteristics of shapes and classify to construct properties of geometric shapes lead to developing geometric relationships and engaging in deductive reasoning.
- Data Analysis: Students' experiences in classifying inform the way students analyze data through categorizing based on attributes; sorting in different ways highlights the interpretive nature of working with data, which in turn shapes their representation of data.

Identifying attributes, sorting into groups and classifying with rules becomes important in learning how to generalize, work with geometrical shapes, analyze

Figure 5: *Matching Madness*

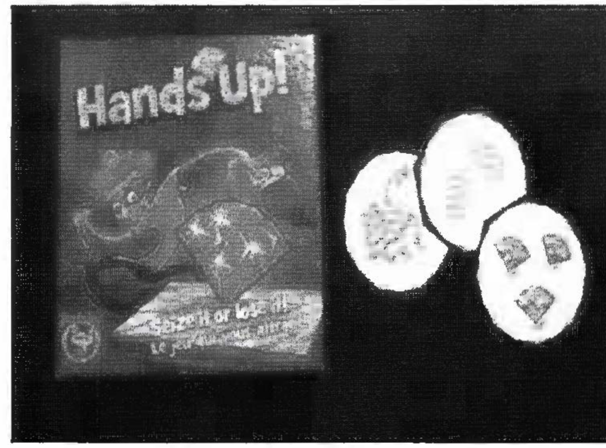

Figure 6: *Hands Up!*

data, classify information, use deductive and inductive reasoning and think systematically during problem solving (Maherally 2014). The breadth of mathematical learning that relies on the experiences students have while playing SET could support their subsequent success in learning mathematics.

Differentiating with Attribute Games

One of the encouraging aspects of incorporating commercial games into mathematical learning is that students are engaged and excited. For us, the success of *all* students is important and so we also explored ways to differentiate with attribute games. Differentiating within the game of SET is possible: for instance, instead of racing to be the first to find a set, students could be encouraged to take turns. SET has also been used in high school and college contexts to support mathematical reasoning (Quinn, Koca Jr and Weening 1999).

Beyond differentiating within the game, we located several different attribute-based games that could be used within the same classroom to address students' differing needs or across grade levels within a school. Below, we offer an explanation of how these games differ from the original SET game with some images. The different options are presented from least difficult to most challenging.

SET Junior is a double-sided board game variation of SET, in which the figures on tiles use only three attributes (shading is excluded). On one side of the board, players are limited to direct matching tiles in their hand to the preprinted SET figures on the board, akin to direct correspondence. Game play is similar to SET on the second side of the board, with some features making the game easier: (1) only 10 tiles are displayed, limiting the number of comparisons to be made; and (2) examples of sets are depicted around the board, supporting players in identifying the range of sets allowed. Both versions take about 10 minutes to play.

Matching Madness is a frog-themed game with four different attributes and a shorter playing time of about 10 minutes (Figure 5). Three differences make the game easier for children to play: (1) a greater variation in the formation of the shapes makes it easier to spot similarities; (2) players look for similarities only, rather than similarities and differences; (3) matches are made only between the top card in the discard pile and the cards in a player's hand, limiting the number of comparisons; and (4) a die directs the attention of players to focus on only one attribute in each round.

Hands Up! captures the imagination of children in its robbers-and-jewels theme (Figure 6). The game is limited to three attributes, but maintains a challenging aspect in identifying similarities or differences across each attribute. As a result of cards being constantly added to the table from players' hands, the game becomes more challenging than Matching Madness. Hands Up! adds some special cards that modify rules when played, making sets easier to make by narrowing down the choice of sets to scaffold game play. This provides momentary breaks in the intensity to find sets and allows for a more balanced play among opponents.

SET Cubed is a Scrabble-inspired game composed of dice that are rolled and then placed on a board resembling that of Scrabble (Figure 7, overleaf). This version is the most sophisticated of the attribute games because it challenges players not only to make sets with three dice but also to build new sets with

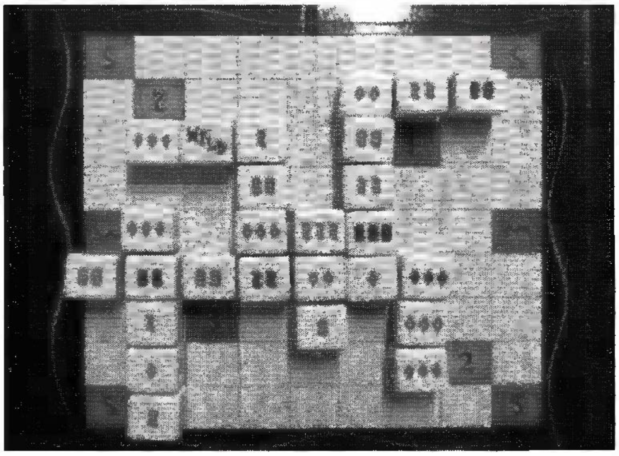

Figure 7: *SET Cubed*

the dice that are placed on the board throughout the game. Similar to SET Junior, SET Cubed does not include the shading attribute in the game; therefore, sets are made using only three attributes. The scoring is a bit more complex in SET Cubed, making it important for players to play strategically while placing their dice on the playing board.

Extending the Inquiry

One phenomenon we did not explore in this article was how students extended their exploration of the game beyond what was structured by the games. For instance, some students began to analyze the quantities and types of cards that comprised the deck. In incorporating SET into a classroom, we invite readers to attend to how their students' curiosity and interest provokes further inquiry. We imagine many productive moments of mathematical thinking!

We hope that sharing the students' engagement and mathematical thinking through the game of SET encourages teachers, parents and other educators to give the game a try. What other personal strategies are possible for children to develop? Are there other mathematical processes that arise as important in the game play? In what ways are children willing to re-engage in learning mathematics and share this with their parents? We invite you to consider how playing SET with your students may provide rich, foundational experiences for meaningful learning in mathematics class.

References

Alberta Education. 2010. *Making a Difference: Meeting Diverse Learning Needs with Differentiated Instruction.* Edmonton, Alta: Alberta Education.

———. 2014. *Mathematics Kindergarten to Grade 9.* Edmonton, Alta: Alberta Education.

Boggan, M, S Harper and A Whitmire. 2010. "Using Manipulatives to Teach Elementary Mathematics." *Journal of Instructional Pedagogies* 3: 1–6. Available at http://www.aabri.com/manuscripts/10451.pdf (accessed July 4, 2016).

Diénès, Z P. 1971. "An Example of the Passage from the Concrete to the Manipulation of a Formal System." *Educational Studies in Mathematics* 3, 3–4: 337–52.

Ernest, P. 1986. "Games. A Rationale for Their Use in the Teaching of Mathematics in School." *Mathematics in School* 15, no 1: 2–5.

Fox, T B. 2000. "Implications of Research on Children's Understanding of Geometry." *Teaching Children Mathematics* 6, no 9: 572–76.

Greenes, C, and L Schulman. 1996. "Communication Processes in Mathematical Explorations and Investigations." In *Communication in Mathematics, K-12 and Beyond*, ed P C Elliott and M J Kenney, 159–69. Reston, Va: National Council of Teachers of Mathematics.

Linchevski, L, and J Williams. 1999. "Using Intuition from Everyday Life in 'Filling' the Gap in Children's Extension of Their Number Concept to Include the Negative Numbers." *Educational Studies in Mathematics* 39, 1: 131–47.

Lopez-Morteo, G, and G Lopez. 2007. "Computer Support for Learning Mathematics: A Learning Environment Based on Recreational Learning Objects." *Computers & Education* 48, no 4: 618–41.

Macnab, J S, L M Phillips and S P Norris. 2012. "Visualizations and Visualization in Mathematics Education." In *Reading for Evidence and Interpreting Visualizations in Mathematics and Science Education*, ed S P Norris, 103–22. Rotterdam: Sense.

Maherally, M. 2014. "The Development and Validation of the Algebra Curriculum Based Measure: A Measure of Preschool Children's Sorting and Classifying Skills." EdD dissertation, University of Cincinnati, Cincinnati, Ohio. Available at https://etd.ohiolink.edu/!etd.send_file?accession=ucin1396530891&disposition=inline or http://tinyurl.com/hgqxrn6 (accessed July 4, 2016).

Mason, J, L Burton and K Stacey. 2010. *Thinking Mathematically.* 2nd ed. Harlow, England: Pearson Education.

Moyer, P S. 2001. "Are We Having Fun Yet? How Teachers Use Manipulatives to Teach Mathematics." *Educational Studies in Mathematics* 47, no 2: 175–97.

National Council of Teachers of Mathematics. 2000. *Principles and Standards for School Mathematics.* Reston, Va: National Council of Teachers of Mathematics.

National Research Council. 2001. *Adding It Up: Helping Children Learning Mathematics.* Washington, DC: National Academies.

Pirie, S, and T Kieren. 1994. "Growth in Mathematical Understanding: How Can We Characterize It and How Can We Represent It?" *Educational Studies in Mathematics* 26, no 2: 165–90.

Platz, D L. 2004. "Challenging Young Children Through Simple Sorting and Classifying: A Developmental Approach." *Education* 125, no 1: 88–96.
Polaki, M V. 2002. "Using Instruction to Identify Mathematical Practices Associated with Basotho Elementary Students' Growth in Probabilistic Thinking." *Canadian Journal for Science, Mathematics and Technology Education* 2, no 3: 357–70.
Quinn, A L, R M Koca Jr. and F Weening. 1999. "Developing Mathematical Reasoning Using Attribute Games." *Mathematics Teacher* 92, no 9: 768–75.
Skemp, R R. 2006. "Relational Understanding and Instrumental Understanding." *Mathematics Teaching in the Middle School* 12, no 2: 88–95.
Sowell, E J. 1989. "Effects of Manipulative Materials in Mathematics Instruction." *Journal for Research in Mathematics Education* 20, no 5: 498–505.
Thom, J S. 2011. "Nurturing Mathematical Reasoning." *Teaching Children Mathematics* 18, no 4: 234–43.

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The authors collaborated on a research project exploring students' rich mathematical thinking through commercial games. They are grateful to the teachers who opened their classrooms and the students who shared their mathematical thinking.

All photographs courtesy of Janelle McFeetors.

Today, You Are the Math Coach!

Michelle Hilton, Shelley Strobel and Terry Freeman

The coach calls "Time!" A whistle blows. You are given a time out. This scenario is well known to many. But can a similar approach be used in math? The answer is a simple "Yes."

One of the big concerns today in education is engaging students in their learning. In addition to this focus on engagement, teachers are trying to provide timely formative assessment and make assessment *as* learning a priority. Four years ago, teachers in Medicine Hat School Division no 76 indicated a strong desire to explore cooperative learning to address this engagement of students. All teachers in the division were instructed in the Tribes Learning Process. The use of Tribes provides safety and community in a classroom; safety allows for students to be comfortable in working with others as well as sharing answers and learning together. This spurred the interest in additional learning strategies, which led to Kagan Cooperative Structures. Over half of the school division's teachers underwent training in this area immediately. From this learning, the middle school and high school math teachers underwent training designed for secondary math in which the focus was on cooperative structures that work best in the math classroom.

Cooperative learning is student-to-student interaction over subject matter as an integral part of the learning process.

What is cooperative learning? Spencer Kagan describes it as "student-to-student interaction over subject matter as an integral part of the learning process" (Kagan 2009). Cooperative learning is students doing the work and realizing that success comes from one another. Students have equal participation, participate frequently and are held accountable for their learning. Students cannot hide in this learning environment; they are required to participate. There are many reasons cooperative learning has been proven to work. It provides immediate and frequent feedback to students, it increases their on-task time and it provides frequent practice recalling and verbalizing math processes. In his online article, Kagan (2014) summarizes the work of State University of New York (SUNY). The chart below shows that the effect size of Kagan structures, in this case the Numbered Heads cooperative structure, is clearly positive.

Kagan believes that when teachers consciously design learning situations with cooperation in mind, a wide range of positive outcomes is the result. That is the beauty of these cooperative structures. They have been carefully crafted, tried and applied. During teacher training, the structures are modelled and practised with real curricular content. Teachers are taught to view structures through the PIES filter (see figure at right). A structure can be called *cooperative* only if it meets these strict criteria.

What does this look like in the classroom? Michelle Hilton, a middle school math and science teacher with Medicine Hat School Division no 76

Effect of Kagan Structures on Achievement

Study	Effect Size	Percentile Gain
1 Numbered Heads vs Whole Class Question & Answer[16]	95	33.0
2 Numbered Heads + I vs Whole Class Question & Answer[17]	98	33.5
3 Numbered Heads vs Whole Class Question & Answer[18]	78	28.2
4 Numbered Heads – I vs Whole Class Question & Answer[19]	96	33.2
5 Response Cards vs Whole Class Question & Answer[20]	90	31.5
6 Numbered Heads vs Whole Class Question & Answer[21]	95	33.0
7 Numbered Heads vs Whole Class Question & Answer[22]	89	31.2
Average	.92	31.9

Kagan 2014

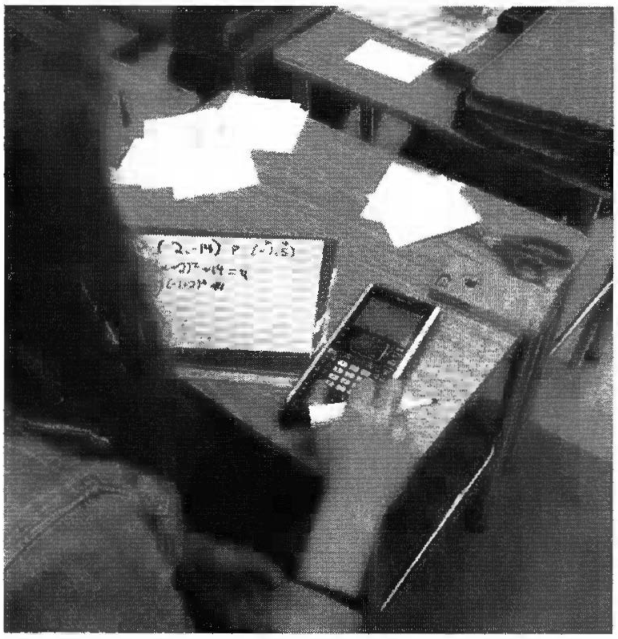
since 2004, and Shelley Strobel, a senior high math teacher with Medicine Hat School Division no 76 since 1994, reflected on their experiences in the classroom using cooperative learning structures and pursued additional training. Out of their Kagan training they petitioned Medicine Hat School Division no.76 to have Kagan Secondary Math brought to Medicine Hat for the teachers to have the same training they experienced. They both describe cooperative structures as the game changer in their classrooms. Student engagement and assessment results have improved after the use of these cooperative structures in their classrooms.

They summarize a few of these cooperative structures that have been successful in their math classrooms below:

> An excellent and easy Kagan structure is called Mix-Pair-Share. This structure promotes movement, which is good for the brain. Social interaction occurs because students are required to pair up with a new partner for each question. Class building is happening because students are out of their desks interacting with other students. This structure easily meets the criteria for PIES in the cooperative learning situation. In Mix-Pair-Share, students mix around the classroom until the teacher calls to pair up. Students pair up with the person closest to them (students who haven't found a partner raise their hands to find each other). The teacher asks a question and gives students time to think. Students then share their answer to the question with their partner using the Kagan structure called Timed Pair Share (students are given a specified time to respond to the question and switch when time is up) or they use the Kagan structure RallyRobin (students take turns responding to the question), depending on whether you want students to have equal sharing time. This structure can be used for class building or content.
>
> The math classroom of the past would ask students to practise concepts by tackling textbook questions individually and reviewing them the next day before a worksheet was assigned for individual completion. This method has the teacher running ragged around the class, attempting to coach students that were having difficulty and monitoring students' behaviour. If students are engaged and taught how to be coaches themselves, the achievement gap will be lessened over time. A cooperative math classroom shifts the responsibility of learning away from the teacher and onto the students. Learning shifts from being a spectator sport to one where students are consistently active participants. In a traditional class the teacher asks a question and calls on one student for the answer, effectively having possibly only one student actively on task at that moment. In a cooperative class the teacher would ask a question, provide think time and then have the students share with their shoulder or elbow partner. Now 50 to 100 per cent of the class has had the opportunity to share ideas.
>
> A cooperative lesson may follow this pattern: teacher-directed notes with examples, but after each example a structure could be inserted for students to practise the newly taught concept. For example, after teaching a lesson on the sine law the teacher could use the Kagan structure RallyCoach to give students an opportunity to verbalize the steps of solving a triangle using the sine law.

They both describe cooperative structures as the game changer in their classrooms. Student engagement and assessment results have improved after the use of these cooperative structures in their classrooms.

- RallyCoach has students working with their shoulder partners. Person A is given a question to solve by talking out the process while Person B watches and listens, checks and coaches if necessary. For the second question the pairs switch roles. The key to making a difference in math learning is this verbalized piece. "Verbalization increases internalization" (Jeff Dane, Kagan instructor).



Preparatory work for the Kagan structure called Showdown.

- RallyCoach can also be used for completing the assigned practice questions. Students work in pairs, coaching each other along the way to mastering the math concept being taught. The setup consists of the teacher creating a two-column worksheet, with questions on each side that require a similar process to solve. The worksheet is folded in half so only one column can be seen at a time. The coach will hold the pencil and the other partner will be required to verbally explain the process that he or she thinks would be best to solve the question at hand. It is important that the student explain the steps to the coach step by step. If the coach agrees with the process he or she will pass the pencil over so that the partner can write the answer down. If the coach does not agree, he or she will give the partner tips and coach the partner to the right process. Once both agree that the question is completed correctly, the coach gives the partner a specific praise. Then the paper is flipped over and roles are switched. Not only is content being mastered in an engaging, meaningful way, but students are also practising social skills that they will take with them outside of the school walls.

Engagement recognizes the need for students to be participants in their learning.

- The structure Pairs Check is similar to RallyCoach but adds another level of accountability. In pairs, students take turns solving problems as described above. After two problems, students check their answers and celebrate with another pair. This pattern continues until the worksheet is complete.

Engagement recognizes the need for students to be participants in their learning. Cooperative structures provide the vehicle to increase student engagement. Another excellent Kagan structure that focuses on verbalizing the steps and can be used daily is Sage-n-Scribe.

- After a concept with examples has been taught, the class is given two "Your Turn" questions. Student A would be the sage first and have to explain how to solve the problem. Student B, the scribe, would record the sage's work only if he or she agrees with what is being said. This gives student B the opportunity to coach if student A misses a step; it also gives student B a chance to ask questions about the process. For the second "Your Turn" question, students switch roles. A couple of things are happening here. By talking it out or having to coach, students retain more of what they learned than if they used a traditional method of trying the example on one's own. As well, the engagement is well approaching 100 per cent in the classroom. This frees up the teacher to help specific individuals or partner groups in the classroom.

Another example of using Kagan cooperative structures in the senior high math class is the day before a unit exam. A structure like Quiz-n-Show works well.

- For Quiz-n-Show, students are each given their own whiteboard, pen and eraser. The teacher presents a problem on the board and allows think time, and students solve the problem individually. Think time is about three to five seconds. This is crucial—it levels the playing field because it makes students pause before rushing to answer the question. It allows the slower processor to have thinking time. When teacher calls "Show," students show their answer to their shoulder or face partners (as directed by the teacher) and the teacher writes the answer on the board. Students at this point are praising each other for a job well done or are coaching and redoing the question.

There are more than 200 Kagan cooperative structures to explore and use in classrooms to increase the engagement of your students in learning content areas. Other well-used structures are Quiz, Quiz, Trade; Single RoundTable; and Jigsaw (expert groups). Engaging students goes far beyond what we used to

Time on task is significantly increased.

call "group work" and pulls them into active reinforcement of content through these cooperative activities. Time on task is significantly increased.

For these two math teachers, there was a choice. Do they remain the "stand and deliver" teacher with some students engaged and the goal just to finish the sheet or textbook page? Or do they want the kind of classroom where all students are consistently engaged and accountable to each other? Cooperative learning structures make formative assessment more relevant and assessment *as* learning a priority.

References

Kagan, S. 2014. "Effect Size Reveals the Impact of Kagan Structures and Cooperative Learning." *Kagan Online Magazine* Winter.

Kagan, S, and M Kagan. 2009. *Kagan Cooperative Learning*. San Clemente, Calif: Kagan.

Additional Resources

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Time-Travel Days: Cross-Curricular Adventures in Mathematics

Irene Percival

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A visitor to Carol Pettigrew's Grade 3 class in North Vancouver, Canada, might be forgiven for skepticism when told that a mathematics class is about to start. The children, most wearing sheets pinned around them like tunics, have their heads down on their desks. Music plays while the teacher's voice quickly leads them back through the highlights of two millennia of history. Finally the music stops and Pettigrew announces, "And here we are, in an ancient Greek school." Unlike other imaginary journeys that elementary students take, the main purpose of these "time-travel days" is to learn about the mathematics and mathematicians of previous ages. The children revel in the experience.

A Hive of Activity

Later in the day, that same visitor might again express surprise, but this time at the level and variety of the work in progress. In one corner of the room, three girls in the Hypatia group are drawing ellipses by using a string looped around two pins. Later, they outline a parabola by repeatedly folding a sheet of paper. Their questions about these shapes lead to discussions of wide-ranging topics, from planetary motion to the parabolic reflectors found in satellite dishes. Students in the Archimedes group can be seen balancing cardboard triangles on their pencil points, having first discovered which lines they must draw to locate a triangle's centre of gravity. Later, they investigate their mathematician's "law of the lever," using calculators to explore the data on their activity sheet. They are delighted when the multiplicative nature of the lever principle suddenly becomes apparent, although they are overheard regretting the lack of a teeter-totter on their playground for practical verification.

Nearby, some students in the Pythagoras group strengthen their three-dimensional visualization skills by trying to put four sets of golf balls together to make a pyramid. Two other students concentrate on two-dimensional shapes, looking at the numerical patterns formed by the triangular, square and oblong arrays, which reputedly intrigued Pythagoras and his followers. Other groups work on ideas related to topics attributed to Eratosthenes, Euclid and Thales.

The students' first task was to read stories about the mathematician assigned to them, then write a list of 10 interesting facts based on this material. Some children, however, took the social studies and language arts links a step further. The Archimedes group rehearsed a skit that they performed later in the week with costumes and stage props. Through their reenactment of the famous "Eureka!" story of the king's crown, the performers and their classmates not only learned about Archimedes's discovery but also developed a better understanding of life in ancient Greece.

Background Information

Most of the Greek activities were selected from the Historical Connections in Mathematics series (Reimer and Reimer 1992, 1993, 1995). Pettigrew's resources included many other books on historical and multicultural mathematics, some of which are listed at the end of this article. The idea for time-travel days grew out of her love of history, her wish to make mathematics fun for her students and her recent realization that mathematics can be taught from a historical perspective. She talks enthusiastically about the need to integrate all subjects, including mathematics, and has spent many hours collecting information for the students' trips back in time. The actual time-travel days occur once each month, but the mathematical ideas they generate often spill over into the following days or weeks.

The excursion to ancient Greece was not the students' first time-travel adventure. Pettigrew planned a series of 10 trips that started in the Stone Age, progressed through thousands of years of civilization and ended with a questioning look at math of the future. The structure of the trips varied considerably:

some focused on the work of individual mathematicians and others looked more generally at the mathematics in use at the time. Several activities contained material not usually found in a Grade 3 curriculum but were included for their value in developing skills in communication, problem solving and reasoning (NCTM 2000).

This article shows how Pettigrew's concept came to life. It gives details of the first five expeditions to distant times and places and summarizes the later ones.

Stone-Age Mathematics

Looking at the mathematics of the Stone Age was perhaps the biggest challenge because the only pieces of physical evidence that remain are notched bones, which seem to record ancient attempts at tallying. Before mentioning these, Pettigrew initiated a brainstorming session about early humans' need for mathematics and possible ways that they could record numbers. Then she gave the students bones, taken from plastic Halloween skeletons, on which she had drawn notches. But most of the students ignored the marks and spent their time putting the skeletons together in a spontaneous science lesson. The focus on numbers returned later in the day when the students looked more closely at the bones as well as at pictures of a wolf bone (Wortzman 1996) and the Ishango bone (Zaslavsky 1999). Pettigrew had marked notches in groups of five, in imitation of the wolf bone. The marks on the Ishango bone are in groups of various sizes and have been interpreted either as a calendar or as number patterns, suggesting that their maker had knowledge of doubling and prime numbers. The students were able to spot some of these patterns and enthusiastically discussed the significance of their discoveries.

Mathematics in Mesopotamia

During a visit to the Fertile Crescent, Pettigrew gave her class a new appreciation of the need for numerical records by playing the role of an unscrupulous trader. Dividing her class into several "villages," she traded one group's sheep for another group's goats. Without a written record of the number of animals traded, she was able to cheat the villagers by reporting fewer animals than she had actually received. The Sumerians stopped such fraud by placing into a clay container tokens representing the number of sheep. The vendors then sealed the container and the buyers broke it open (Ifrah 1998). Using Plasticine, a modelling clay, the students discovered how these physical counters were eventually replaced by "written" symbols simply by people making imprints of them in clay.

Quickly moving ahead several thousand years, the children used a "stylus" (a suitably carved chopstick) pressed into a "clay" (Plasticine) tablet to practise writing the Babylonian cuneiform number system that was derived from the earlier number shapes. Pettigrew had used these symbols on the class calendar since the beginning of the month, and most of the students had figured out the system in the 30-day period before the time-travel day (see figure below).

Babylonian symbols represent the date.

Photograph by Irene Percival; all rights reserved

Egyptian Mathematics

On arrival in Egypt, the time-travellers were eager to share their knowledge of the country and its most famous monuments, the pyramids. Pettigrew seized the opportunity to reinforce the previous week's geometry unit by asking about the number of faces, edges and vertices on the structure, strengthening the children's vocabulary. She then gave the students small blocks to build their own pyramids. This project gave valuable insight into the students' level of understanding of three-dimensional shapes. Although the students could describe a square pyramid, none of the groups actually made a square base on which to build their monument, confirming the claim that "the study of geometry in Grades 3–5 requires thinking and doing" (NCTM 2000, 165).

Arithmetic followed later in the day. The students solved a puzzle to determine Egyptian number symbols (see figure on next page), and subsequent comparison of the Egyptian and modern number systems provided a useful review of the present place-value structure.

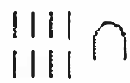
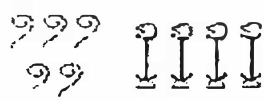
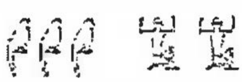
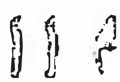
Use the clues at the top of the page to decode the Egyptian numbers. When you know the value of a symbol, draw it in the box at the bottom of the page.

CLUE 1: The Egyptians were good astronomers and knew how many days there are in a year.

CLUE 2: Nârmer captured one hundred twenty thousand prisoners.

CLUE 3: Some Pharaohs lived for many years, but King Tutankhamen died when he was eighteen years old.

CLUE 4: The Great Pyramid of Khufu was built in 2500 B.C. It took one hundred thousand people twenty years to make it, and they used two million, three hundred thousand stone blocks.

- ▲ King Tutankhamen was only  years old when he died.
- ▲ The Egyptian year was  days long.
- ▲ Khufu's pyramid was built about  years ago.
- ▲ It contained  stone blocks.
- ▲ Nârmer took  people captive.

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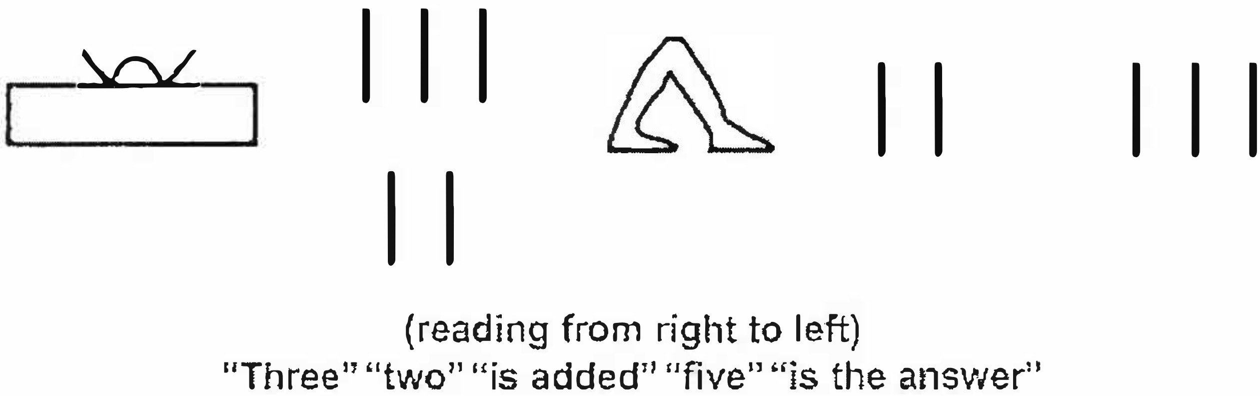
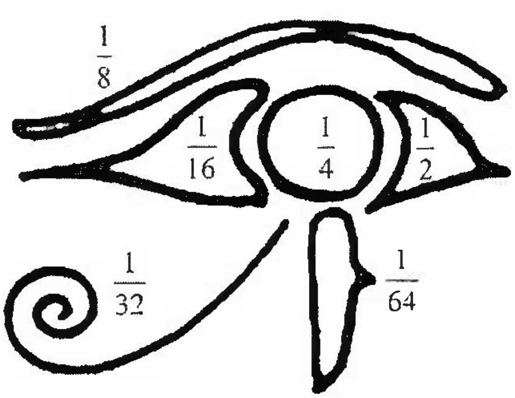


Egyptian addition

The "real-life" origins of the number symbols (Burnett 1999) generated an interesting discussion. The children were delighted when they discovered the meaning of the hieroglyph for *walking*, a pair of legs, which the Egyptians incorporated into their mathematical vocabulary to mean addition or subtraction according to which direction the legs were pointed (see figure above). Students enthusiastically wrote their own mathematical sentences, often using much greater numbers than are commonly encountered at the Grade 3 level. Pettigrew's requirement that they translate their work into modern symbols encouraged the students to discover how to express such numbers in our place-value system, and the Egyptian tally-like representation of numbers helped the students understand the regrouping process that often is necessary in calculations.

To end the day, Pettigrew read Egyptian myths to the class. The myths included the story of the god Horus, whose eye was torn into six pieces in battle but was united by Thoth, the god of wisdom. The students identified the numerical pattern in the values of the fraction symbols derived from the Horus-eye symbol (see illustration below), then developed their understanding of the fractions by answering questions, using Egyptian units of measurement (Burnett and Irons 1996).

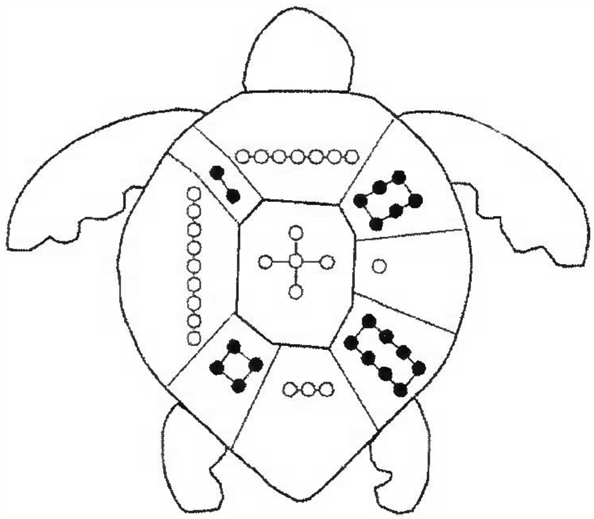


Horus-eye fractions

Activities continued throughout the week. The students made sheets of "papyrus," using strips of construction paper glued together in a crosswise fashion. They used these to write their own Egyptian documents, which contained mathematical sentences and their names spelled out in hieroglyphics. They also explored Senet, a board game seen in the wall paintings of ancient tombs. A precursor of many modern games, Senet involves two players who move their counters around a board. The winner is the first to get all of her or his pieces to the final square. The students constructed their own boards and had many contests. They developed strategies to improve their chances of winning.

Chinese Mathematics

The next destination of the time-travel machine was another cradle of civilization, China. Pettigrew planned this trip to coincide with the Chinese New Year, so the students already were immersed in Chinese culture. They explored two calculating tools: the abacus, which was familiar to some of the students, and the counting board, an earlier device that also employed a base-ten positional system and led to the development of written "stick numerals" (Zaslavsky 2001). The children used toothpicks to imitate the bamboo rods with which the Chinese physically



Lo Shu

represented their numbers on the counting board. Pettigrew's request that her students explain verbally how and where the sticks should be placed not only gave them valuable practice with spatial language but also provided yet another way to reinforce their understanding of place value.

Later in the day, Pettigrew read *Grandfather Tang's Story* (Tompert 1990) to the class, and the students made their own set of tangram shapes by folding and cutting paper. This method, taken from the book *Mathematics: A Way of Thinking* (Baratta-Lorton 1977), is more time-consuming than is simply cutting out shapes from a suitably marked square. However, it led to a rich discussion of geometrical language and gave the students useful practice in carefully following verbal instructions, both of which are important in developing students' ability to communicate mathematically (NCTM 2000). The day ended with Lo Shu, an ancient Chinese depiction of a magic square. Once again, Pettigrew introduced the new topic through storytelling (Irons and Burnett 1995). She described the Chinese legend of the turtle that came out of the river Lo to reveal the puzzle to the emperor (see illustration above). In a class discussion of the illustration, students identified the numerical significance of the patterns of dots and commented on the placement of odd and even numbers. A few students noticed the common sum of the square's rows and columns and suggested that this property explained why the squares were considered "magic." The students then constructed their own magic squares, based on the principles that they had discovered.

The Last Five Time-Travel Days

The students' next time-travel trip took them to Rome to learn about Roman numerals and the abacus on which the Romans did their calculations. Later that day they travelled to India, where they discovered the origin of the Hindu-Arabic numbers that are used today and explored some of the mathematics known to the Hindu priests.

The timeline fastened above the class notice boards had to be scrolled over many centuries to reach the next destination—Europe toward the end of the Dark Ages. Fibonacci, the 13th-century Italian who introduced the Hindu-Arabic numbers to Europe, was the featured mathematician. The children enjoyed exploring his famous series, 1, 1, 2, 3, 5, 8 ..., and were able to demonstrate their understanding of its pattern both verbally and by writing more numbers in the series. Magic tricks based on this series added to the day's fun.

Pettigrew divided the class into six groups for a visit to the Renaissance era. Each group studied a mathematician selected from Historical Collections (Reimer and Reimer 1992, 1993, 1995). The class could choose from a wide variety of activities, the most popular of which was Plot and Swat (Reimer and Reimer 1995). This exercise in coordinate geometry produced the picture of a fly, a reference to the story that Descartes invented his coordinate system while watching a fly crawl around his ceiling. The Galileo group was excited by "that hooky thing"—their name for the square-root sign, which they needed unexpectedly for one of their pattern-recognition activities. The square-root concept was new to them but they were eager to understand it, and they enthusiastically explained the meaning of the symbol to the rest of the class.

Pettigrew set aside one day to teach her students about selected female mathematicians. Again, she chose activities from Historical Connections, but the anecdotes about the women's lives made the greatest impact on the students. These not only showed the difficulties that girls who wanted to study mathematics had encountered but also gave the students examples to challenge the view, still prevalent today, that "girls don't do math."

Pettigrew called her last time-travel day "Math of the Future." Although some of the mathematics that they discussed is already several decades old, the fractal pictures that she showed the children had a futuristic look. A visiting high school teacher demonstrated the graphing calculator, a tool that is likely to be part of the students' future studies.

Reflections

Much has already been written about the benefits of teaching mathematics by using historical or multicultural material (Fauvel 1991; Zaslavsky 1996), but the following points seem particularly relevant.

Time-travel days highlight three important perspectives of mathematics. First, they show mathematics as a human endeavour. Through investigating, or even acting out, incidents in mathematicians' lives, the students come to see the creators of mathematics as real people. Next, learning about mathematics in other countries helps fulfill multicultural goals and enables students who have visited other countries to share their experiences. Third, time-travel days allow mathematics to be part of the present focus on cross-curricular connections. Locating each destination on a time chart and world map are obvious historical and geographic links, but the mathematical topics also led to discussions of the lifestyles of earlier times and motivated many art and language arts activities.

The most visible advantage of this novel approach to mathematics was the high degree of motivation that the students exhibited. The make-believe element of the trips captured the children's imaginations and mentally prepared them to encounter new ideas, several of which stretched their minds to explore higher levels of mathematics than are normally encountered in the Grade 3 curriculum. The enthusiasm with which the students seized such concepts suggested confidence and a lack of concern about "wrong answers."

On their travels, the students encountered mathematics questions that appeared without reference to any specific curriculum topic, allowing true problem-solving experiences to occur. The mathematics was placed in a cultural context. Although teachers could use these activities without the pretense of time travel, many of the activities are so closely related to particular civilizations that not acknowledging this connection would be unfortunate.

Does Pettigrew expect the children to remember all the details of their time travels? Of course not, although end-of-year interviews with the students showed that some of the material had made a big impression on them. The important point of these excursions is that the children began to realize that mathematics has been explored and used since the dawn of civilization by people from all over the world. If students retain this view of mathematics, they might be able to see beyond the boredom of the basic facts of arithmetic, algebra and geometry that are subject to so much drill in our school system. Time-travel days can encourage students to imitate the mathematicians of the past, to ask why and to enjoy the pursuit of an answer, rather than just its attainment.

References

Baratta-Lorton, R. 1977. *Mathematics: A Way of Thinking*. Menlo Park, Calif: Addison-Wesley.

Burnett, J. 1999. *Sights, Sounds, and Symbols*. Narangba, Queensland, Australia: Prime Education.

Burnett, J. and C Irons. 1996. *Egyptian Genius*. San Francisco, Calif: Mimosa.

Fauvel, J. 1991. "Using History in Mathematics Education." *For the Learning of Mathematics* 11, no 2: 3–6.

Ifrah, G. 1998. *From One to Zero: A Universal History of Numbers*. London: Harvill.

Irons, C, and J Burnett. 1995. *Mathematics from Many Cultures*. San Francisco, Calif: Mimosa.

National Council of Teachers of Mathematics (NCTM). 2000. *Principles and Standard for School Mathematics*. Reston, Va.: NCTM.

Reimer, W, and L Reimer. 1992. *Historical Connections in Mathematics*. Vol I. Fresno, Calif: AIMS Educational Foundation.

———. 1993. *Historical Connections in Mathematics*. Vol II. Fresno, Calif: AIMS Educational Foundation.

———Reimer, W, and L Reimer. 1995. *Historical Connections in Mathematics*. Vol III. Fresno, Calif: AIMS Educational Foundation.

Tompert, A. 1990. *Grandfather Tang's Story*. New York: Crown.

Wortzman, R, L Harcourt, B Kelly, P Morrow, R I Charles, D C Burnett, C S Barnett et al. 1996. *Quest 2000*. Grade 3. Don Mills, Ont.: Addison Wesley Longman.

Zaslavsky, C. 1996. *The Multicultural Mathematics Classroom: Bringing in the World*. Portsmouth, NH: Heinemann.

———. 1999. *Africa Counts: Number and Pattern in African Cultures*. 3rd ed. Chicago, Ill: Lawrence Hill Books.

———. 2001. "Developing Number Sense: What Can Other Cultures Tell Us?" *Teaching Children Mathematics* 7, no 6: 312–19

The Genius Plays! A Review of *Genius at Play: The Curious Mind of John Horton Conway*, by Siobhan Roberts

Reviewed by Timothy Sibbald

Siobhan Roberts continues to impress with this book, her second biography of a mathematician. She has clearly put in a lot of legwork and had remarkable access to her subject in her exploration of John Conway. This is an impressive book that details Conway's remarkable career in a fascinating ride through decades of recent math events.

The book begins by detailing how Conway would play games routinely. He seems to have almost used games as a form of procrastination, and yet his mathematical ability shone through at the right moments. The playfulness, in his early years, is punctuated with humorous moments such as his proof that a person's tongue can be folded in four different ways (not to mention the etymological interruptions).

Conway's legacy entails multiple forms of mathematics. Arguably they are interrelated, but they are distinct in layman's terms and it may be the mathematical relatedness that makes the mathematics itself so powerful. Conway's interest in knots appears to have arisen about the same time as his interest in flexagons—paper folded polygons that change their faces when flexed. Are the two related? Then there are the number games, in which we see him asking people for a number and then telling them four squares that add to that value (Lagrange proved that four squares is always enough), or asking for a birthdate and identifying the day of the week.

In some places the mathematical scenery, rather than a single item, becomes the focus. An example is that Cantor's discovery of different infinities is explained along with Conway's generalization to surreal numbers, but the coverage is not as in depth as his multiday presentations would be. This does not detract from the book, because the reader is sure to find a variety of interesting ideas along the way. Not long after surreal numbers are introduced, the game of Dots and Boxes is mentioned—the reader is not trapped in any topic too long. It is this variety that makes the book a cornucopia of ideas that will inspire at multiple levels. A teacher can immediately challenge a class with the sequence 1, 11, 21, 1211, 111221, 312211 and so on, which is explained, and may find the connection between codes and sphere packing to be more a passing interest than a school endeavour.

It is interesting to read about Conway's early adoption of the computer for the Game of Life, an iterative geometric sequence that high school students can find remarkable for the variety of behaviours that can evolve. However, it is surprising to find that Conway has a distaste for the Game of Life because it has diminished the recognition of everything else he has done. An example is FRACTRAN, a row of 14 fractions that, with a very simple algorithm, apparently generates the powers of 2 with exponents that are only prime numbers! This is a curious way of combining fraction skills with the concept of an algorithm that leads to something much more significant. Similarly, Roberts mentions how the Game of Life led to the discovery of another iterative geometric pattern generator, called Rule 110, that is a universal calculator (anything can be calculated by beginning with the appropriate geometric pattern and determining the sequence to the right stopping point).

The overall message of the book is philosophically on the same page as many math teachers. If you tinker with mathematical amusements, the larger concepts will start to emerge through experience. It may well have been the interest in games that led to Conway's interest in codes. That interest, while showing the importance of communication in mathematics, led to the discovery of a sporadic group, which led to Conway's involvement in the Atlas of Finite Groups—a major modern achievement in mathematics. As the book develops we find that there are unresolved questions, such as that no one is really sure of the characteristics of the monster group, with 8×10^{53} elements.

Siobhan Roberts has created a fascinating read and an excellent addition to every mathematically enthusiastic teacher's library. It is also suited to mathematically keen high school students.

Roberts, S. 2015. *Genius at Play: The Curious Mind of John Horton Conway*. New York: Bloomsbury.

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NCTM's Illuminations Website

www.illuminations.nctm.org

Lorelei Boschman

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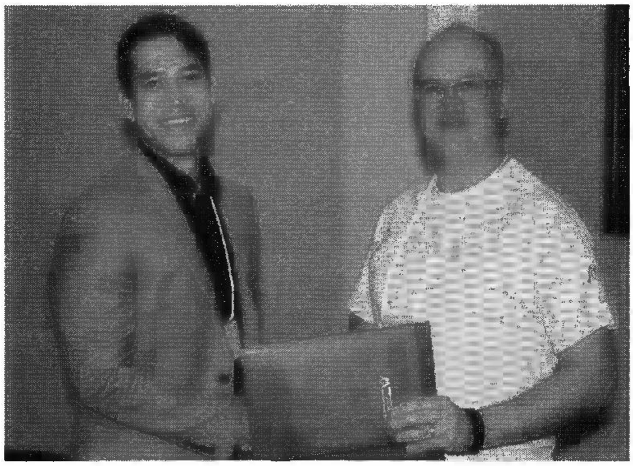
Conference 2015 Highlights

Awards

Dr Arthur Jorgensen Chair Award



This award is presented by the Mathematics Council of the Alberta Teachers' Association (MCATA) to encourage students enrolled in education programs in postsecondary institutions throughout Alberta to pursue and commit to mathematics education.

2015 Recipient

Matthew McDonald,
presented by John Scammell

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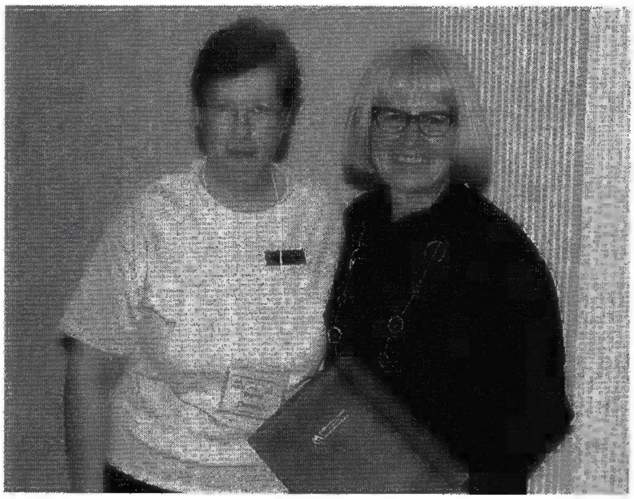
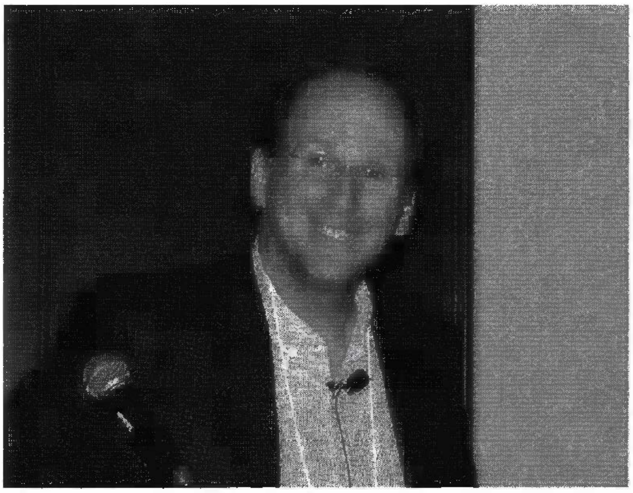


Val Olekshy,
presented by Donna Chanasyk

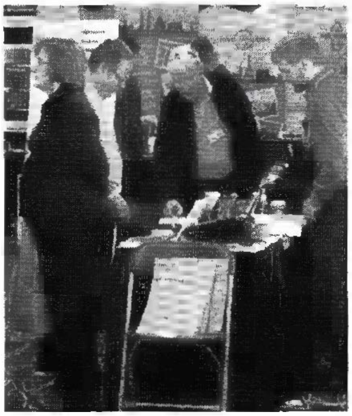
Gladys Sterenberg,
presented by Tancy Whitehouse

Keynote Speakers



Steven Strogatz is an applied mathematician at Cornell University, Ithaca, New York. He has published several best-selling and accessible books on mathematics, including The Joy of x *and* The Calculus of Friendship. *He wrote a series of popular and fun articles on math in the* New York Times; *check them out at http://goo.gl/A880oT.*

Chris Hadfield is a Canadian astronaut, who may be most famous for singing in space. He has just released the album Space Sessions: Songs from a Tin Can. *He has published books including* An Astronaut's Guide to Life on Earth *and* You Are Here.









Diversity • Equity • Human Rights Diversity • Equity • Human Rights

Specialist councils' role in promoting diversity, equity and human rights

Alberta's rapidly changing demographics are creating an exciting cultural diversity that is reflected in the province's urban and rural classrooms. The new landscape of the school provides an ideal context in which to teach students that strength lies in diversity. The challenge that teachers face is to capitalize on the energy of today's intercultural classroom mix to lay the groundwork for all students to succeed. To support teachers in their critical roles as leaders in inclusive education, in 2000 the Alberta Teachers' Association established the Diversity, Equity and Human Rights Committee (DEHRC).

DEHRC aims to assist educators in their legal, professional and ethical responsibilities to protect all students and to maintain safe, caring and inclusive learning environments. Topics of focus for DEHRC include intercultural education, inclusive learning communities, gender equity, UNESCO Associated Schools Project Network, sexual orientation and gender variance.

Here are some activities the DEHR committee undertakes:

- Studying, advising and making recommendations on policies that reflect respect for diversity, equity and human rights
- Offering annual Inclusive Learning Communities Grants (up to $2,000) to support activities that support inclusion
- Producing Just in Time, an electronic newsletter that can be found at www.teachers.ab.ca; Teaching in Alberta; Diversity, Equity and Human Rights.
- Providing and creating print and web-based teacher resources
- Creating a list of presenters on DEHR topics
- Supporting the Association instructor workshops on diversity

We are there for you!

Specialist councils are uniquely situated to learn about diversity issues directly from teachers in the field who see how diversity issues play out in subject areas. Specialist council members are encouraged to share the challenges they may be facing in terms of diversity in their own classrooms and to incorporate these discussions into specialist council activities, publications and conferences.

Diversity, equity and human rights affect the work of all members. What are you doing to make a difference?

Further information about the work of the DEHR committee can be found on the Association's website at **www.teachers.ab.ca** under Teaching in Alberta, Diversity, Equity and Human Rights.

Alternatively, contact **Andrea Berg**, executive staff officer, Professional Development, at **andrea.berg@ata.ab.ca** for more information.

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ISSN 0319-8367
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