delta-k

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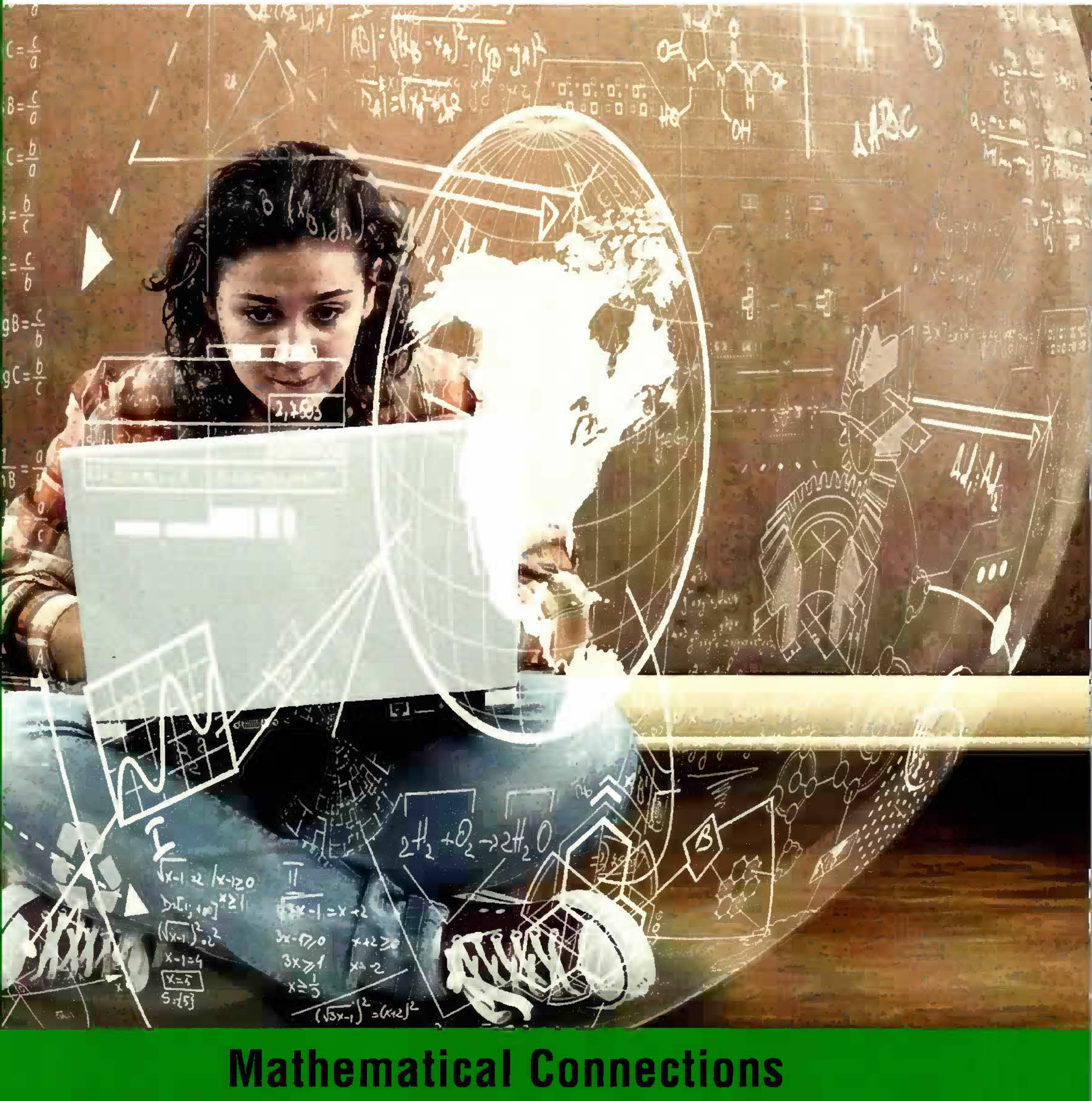

Mathematical Connections

Guidelines for Manuscripts

delta-K is a professional journal for mathematics teachers in Alberta. It is published twice a year to
- promote the professional development of mathematics educators and
- stimulate thinking, explore new ideas and offer various viewpoints.

Submissions are requested that have a classroom as well as a scholarly focus. They may include
- personal explorations of significant classroom experiences;
- descriptions of innovative classroom and school practices;
- reviews or evaluations of instructional and curricular methods, programs or materials;
- discussions of trends, issues or policies;
- a specific focus on technology in the classroom; or
- a focus on the curriculum, professional and assessment standards of the NCTM.

Suggestions for Writers

1. *delta-K* is a refereed journal. Manuscripts submitted to *delta-K* should be original material. Articles currently under consideration by other journals will not be reviewed.
2. If a manuscript is accepted for publication, its author(s) will agree to transfer copyright to the Mathematics Council of the Alberta Teachers' Association for the republication, representation and distribution of the original and derivative material.
3. Peer-reviewed articles are normally 8–10 pages in length.
4. All manuscripts should be typewritten, double-spaced and properly referenced. All pages should be numbered.
5. The author's name and full address should be provided on a separate page. If an article has more than one author, the contact author must be clearly identified. Authors should avoid all other references that may reveal their identities to the reviewers.
6. All manuscripts should be submitted electronically, using Microsoft Word format.
7. Pictures or illustrations should be clearly labelled and placed where you want them to appear in the article. A caption and photo credit should accompany each photograph.
8. References and citations should be formatted consistently using *The Chicago Manual of Style*'s author-date system.
9. If any student sample work is included, please provide a consent form from the student's parent/guardian allowing publication in the journal. The editor will provide this form on request.
10. Letters to the editor, description of teaching practices or reviews of curriculum materials are welcome.
11. Send manuscripts and inquiries to the editor: Lorelei Boschman, c/o Medicine Hat College, Division of Arts and Education, 299 College Drive SE, Medicine Hat, AB T1A 3Y6; e-mail lboschman@mhc.ab.ca.

MCATA Mission Statement

Providing leadership to encourage the continuing enhancement of teaching, learning and understanding mathematics.

delta-k

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From the Editor's Desk

Lorelei Boschman

Yesterday I received a message from a preservice education student who recently completed a mathematics course with me:

"Did you know that a 10 ounce cup of coffee has a splash radius of at least 12 feet? I'm pretty sure that the gravitational force of falling exponentially increases the amount of coffee that was once INSIDE that cup. I've done my part of the research here and am passing it on as a public service to you!"

Of course I smiled in a humorous yet understanding way, visualizing the "research" she participated in. This brought me to thinking about how students perceive the discipline of mathematics in their worlds, as students of mathematics versus teachers who have studied and taught mathematics. Having a student relate, and actually identify purposefully even without a prompt, to the mathematics occurring around her was rewarding for me, the instructor, and I recognized that this may not be a person's main thought as the coffee cup is coming down. But it was. I realized that this symbolizes the effect that we want to have on students of mathematics. We want them to see the value, the usefulness, the evidence and the importance of mathematics. We work daily to this effect. Having students recognize and experience the real relationships of the mathematics within their daily lives and beyond the classroom is notable. We purpose ourselves to creatively teach students and facilitate learning opportunities for them to experience this exact situation and the mathematics that emerges first-hand and actually recognize it! Looking at the world around us and "seeing" all the mathematics that exists and comes into play daily all around us is an aptitude to be encouraged and admired. If and when we build this into students, the math teacher in all of us breathes a contented sigh of accomplishment.

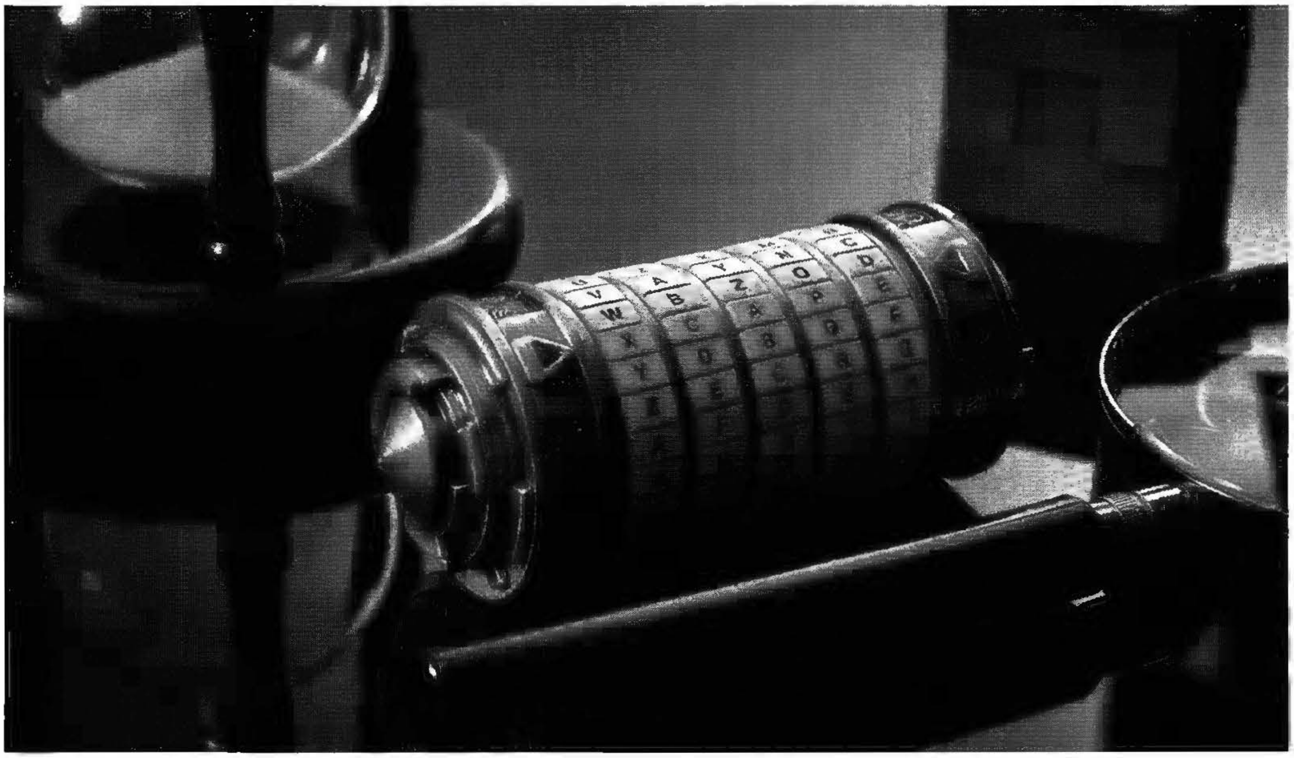
This journal includes many ideas about relating mathematics to our world with purposeful and complementary thoughts, ideas and activities for each level. As you read the journal, consider how to modify the article or activity for students at your level. Mostly, consider how to relate the mathematics you are teaching, exploring and facilitating for students to make those connections to their lives and create that deeper meaning, understanding and usefulness.

I'm quite sure my student's understanding of radius, especially one of 12 feet, was reinforced that day. Perhaps it even brings up more questions and opportunities about the force required to even send coffee that far! I will continue to actively think about this example and consider how to create more of these real mathematical scenarios and absolutely recognize them. My goal is to have my students see the mathematics occurring constantly all around them, and as being innately important and truly complementary to their lives.

Remember to send in any submission you would like to share with others—we value your contributions and see tremendous value in our collective understandings, ideas and practice.

Developing a Passion for Mathematics Through History

Armand Doucet and Jata MacCabe



This article highlights an interdisciplinary collaboration from secondary school that begins in a history classroom. The teacher Armand Doucet invites students to delve into areas that they are passionate about. Jata MacCabe, a student, is passionate about mathematics. Upon hearing of this initiative, an invitation was extended to share their story with the readership. The coauthorship enriches the value. Armand's writing (in italics below) will introduce the context and the background with regards to Passion Projects. Jata will then share her experience with the project and what it meant to discover that she wanted to continue pursuing mathematics as a career path.

My goal as a teacher in the classroom is to develop skills intertwined with curriculum content. Social and Emotional Learning (SEL) and 21st-century skills need to be developed not haphazardly, but purposefully. For this to happen, the culture and design of my classroom and how I approach curriculum outcomes and standards, as well as skills development, is with a combination growth mindset (Carol Dweck) and design thinking process (IDEO-Tim Brown). I try to foster and develop divergent thinking (Sir Ken Robinson) in students who will embrace the problems of the world instead of fearing them because in reality: "The world doesn't care *what you know. What the world cares about is what do you do with it" (Tony Wagner).*

So, I believe that connecting the curriculum to what the students are passionate about is a great way to develop my classroom. With Passion Projects students realize the joys of learning again by following their own path. As you can see in Jata's statement below, when allowed to pursue their own goals in education, students struggle at first. I try to let them explore before giving them stricter guidelines for their creative piece. We conference in order for them to discover what it is they would really like to pursue. As they embrace the core problem of their Passion Projects, resiliency precedes enthusiasm and then enthusiasm leads to pride as students create and subsequently showcase their projects. History comes alive as students gather information and collaborate with the international community. The experience is unique to each student. Tony Wagner (Creating Innovators) states, "the most important thing is allowing students to ask questions and then give them the space to find the answers."

With Passion Projects students realize the joys of learning again by following their own path

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With Jata, she wanted to pursue something in math and women's rights. Her project revolved around proving that women played a key role in World War II with Bletchley Park and this was one of the main reasons that the Allied forces had won the war. At first, she researched a lot on Bletchley Park's role itself, realizing that Mavis Batty played a major part. As her project progressed, she decided to create her own Enigma scavenger hunt. This got her looking at the way the code breakers were using math to break codes and build the Enigma machine. She ended up being able to utilize her precalculus class to help her develop the scavenger hunt and Enigma machine (which was made up of tinfoil and some boxes). However, what she really developed was an understanding of how math, as well as history, are both connected from a perspective of the skills that are needed such as problem solving, critical thinking and creativity. Those higher order skills were pushed to the limit as she continuously tried, innovated and ultimately created her supportive creative piece, namely, the scavenger hunt.

Giving students like Jata a chance to pursue their passion, math in this case, in combination with other subjects like history, within a safe environment to take a chance on a project, analyze, improve and try again, gives them the opportunity to realize if they truly want to chase down those dreams in the future. Jata's project garnered attention from CBC once we posted the results on social media. They attended her presentation, interviewed her and it was shared over 200 times. Also, she received praise from Sue Black, OBE and computer scientist, who was one of the people who helped to save Bletchley Park. Sue was able to share with Jata her connection with Mavis Batty, having known her before she passed away. All these things combined to solidify for Jata that she wanted to continue pursuing math and that it was going to give her numerous avenues for an interesting career.

That, I believe, is my job as a teacher, to help students develop skills while finding who they are so they can succeed in the future. You can visit my template for this type of classroom and other Passion Project examples at www.lifelessonlearning.com.

The idea was to connect something we were passionate about to a revolution in modern history.

My approach to history has always been impersonal. Dates and names have never stuck in my head for longer than they took to go in one ear and out the other. I was kind of into that Roman unit, but my friends tell me watching *Gladiator* doesn't actually count as studying. I was obviously not looking forward to an entire semester of memorization and regurgitation of a subject I didn't particularly care about.

Within the first week of Mr Doucet's class we were introduced to the Passion Project. The idea was to connect something we were passionate about to a revolution in modern history. I was terrified. That very helpful premise narrowed the possible topics down to relatively everything, and the only concrete thing I understood was the deadline. When Mr Doucet suggested researching code breaking during World War II, I finally had some small lifeline to grasp on to. This was a way to explore my passion for mathematics in

a course that I would have otherwise loathed. Besides, what kid isn't intrigued by spies and code breaking?

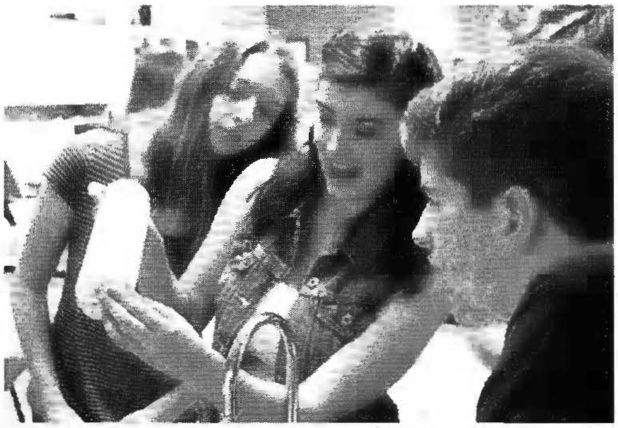
For almost the entire history of the world, battles have been the epitome of concrete and physical. Obey him, protect them, bash and whack the enemy. The major action occurred directly on the battlefield; you simply had to roll with the punches as they came—literally. Espionage had always been field agents infiltrating enemy divisions, overhearing important information and accessing critical documents. However painstakingly won, this information hardly ever majorly impacted the outcome of a battle.

Communication was slow and unreliable; a messenger could be delayed or a letter could be intercepted. Even if the information should have reached someone who might have been able to act upon it, the information was often as unreliable as the methods to send it. In matters of life and death, confusion is not always the preference. Our modern history course taught us of major innovations that were catalysts for revolution.

Very few modern innovations had such a profound effect on military communication, and the world, as radio transmission. During the Second World War, communication was decidedly less tangible. Encrypted messages could pass through brick walls, over enemy camps and across borders. In a game of interceptions, the best encryption won. As tensions and conflicts mounted, it was clear that the Germans had it.

In a game of interceptions, the best encryption won.

Originally, the Enigma machine was a commercial product designed for businesses or firms to encrypt their financial data. The creators were quick to see the machine's potential military use and began approaching federal governments with the product. Ironically, the encryption machines were even



presented to the British government, who chose not to invest. The German government was interested in the product, however. After ramping up the security, the German Enigma resembled the simple commercial product solely on a superficial level. The machine had a standard German keyboard, like a typewriter, and an additional alphabet with illuminated keys. It included a series of rotors that encoded letters and rotated with each additional letter. It also had a switchboard that added an extra layer of security by switching the coded letters for other—seemingly random—letters. By changing the settings of the rotors every night at midnight, the Germans had created a nearly invincible fortress of security. In a total blackout of information, the Allied forces would be subject to almost certain defeat. Britain's Government Code and Cypher School's base for Axis decryptions was at Bletchley Park.

At Bletchley Park, mathematics and problem solving meant lives saved.

Perhaps the most famous name to come out of Bletchley Park is Alan Turing. Before the war, the Polish had developed a method of deciphering Enigma codes with the use of "Bomba" machines. These functioned by checking all possibilities using a series of sheets. The machine was slow, inconsistent and fickle, but it was progress. After being introduced to the Bomba, the concept that a machine could do the quantitative work of a human mind would stay with Turing for the rest of his life. Turing was a theorist, but he couldn't achieve his objective of creating a more efficient version of the Bomba alone. He and Gordon Welchman combined with an Oxford engineering team and created the first Bombe machine. It was not precisely a computer as one still needed to feed the machine a section of code guessed at manually, but Bomba machines could check thousands of possibilities in minutes.

In high school, math seems almost completely unrelated to the world at large. You can barely step into a precalculus classroom without hearing "How will this help me in the real world?" We all want our hard work to mean something more than a number on a test. It was amazingly coincidental that while I was researching how probability was used at Bletchley Park, we had just begun our Combinatorics and Probability unit in precalculus. At Bletchley Park, mathematics and problem solving meant lives saved. Churchill believed that the work conducted at Bletchley shortened the war by two years. Many others believe that the war could not have been won without the park. It is often said that Bletchley was present at every famous battle in the Second World War, stealthily swaying the balance.

The most confusing part of this project was the creative part. For mine, groups of five had to use a tinfoil and pool noodle Enigma machine to decipher the location of their next checkpoint. It was exactly *The Amazing Race* and it certainly wasn't life at Bletchley Park, but teams had to work together to solve problems under pressure, which was my goal.

The world is composed of dichotomies. You're a naive child or a sophisticated adult. You're a dreamer or a realist. You're a mathematician or an artist. Bletchley Park unabashedly disregarded these constraining labels. Academics, translators, debutantes, actors, novelists, athletes and even chess enthusiasts were recruited to aid their country. Major operations included but weren't limited to university students or graduates. Some of the greatest breakthroughs during World War II were interdisciplinary collaborations of many kinds of thinkers.

This project taught me that I already was a mathematician.

I knew before this project that I wanted to be a mathematician. I knew that I loved numbers and I knew that solving a difficult problem made me irrationally happy. This project taught me that I already was a mathematician. Math never was about numbers or formulas on a page. Math has always been about humans solving problems. All of those countless symbols, complex equations and abstract theories have been about the human race learning to understand and manipulate the world around them. So maybe it was weird that I found my passion for mathematics studying people and civilizations and revolutions, but maybe it wasn't that weird at all.

Armand Doucet is a passionate and award winning educator, leader and business professional with a unique combination of entrepreneurial, teaching and motivational speaking experience. He recently received the Prime Minister's Award for Teaching Excellence as well as a Meritorious Service Medal from the Governor General. He is the creator of www.lifelessonlearning.com which leads the way in placing skills development on equal footing to curriculum content in the classroom.

Jata MacCabe is a self-proclaimed math dork who is equally talented in the classroom as on the improv stage or rugby field. As a Grade 12 student this year, she is looking to pursue a career in math while still being passionate about many other subjects.

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Open-Ended Questions

Lorelei Boschman

Open-ended questions encourage students to think about different methods, representations and possible solutions, all the while promoting mathematics understanding and processing. Sharing these possible solutions with peers is also a powerful strategy for teachers.

Some open-ended questions are listed below: Can you use one or adapt one for your math students to see how powerful the conversations/number talks and mathematical thought processes can be through this? Is this something that you could build into your weekly lessons? Think about having students represent their possible solution on a personal whiteboard, vertical nonpermanent surface or through a placemat activity.

Grade 1: You went to the store and bought red and blue candies. There were more red candies than blue candies. How many of each could you have bought? How many candies did you buy altogether? How many more red candies than blue candies did you buy?

Grade 3: You write a number with tens and ones. When you switch the numbers around, your new number increases by more than 20 but less than 30. What could your original and new number be? Can you think of another solution?

Grade 4: Write a four-digit number whose digits total 23. Let your partner check this. What is the greatest/least four-digit number you can make whose digits total 23? 18? Create another one for a partner to try. Can you pick any number for the digits to total or are there only certain numbers that would work?

Grade 5: You buy an item with a $100 bill. You get back four bills and six coins. How much did your item cost?

Grade 7: Add in order of operations to make the following true: 5 __ 3 __ 2 __ 2 = 9. Now create one of your own for a partner to solve.

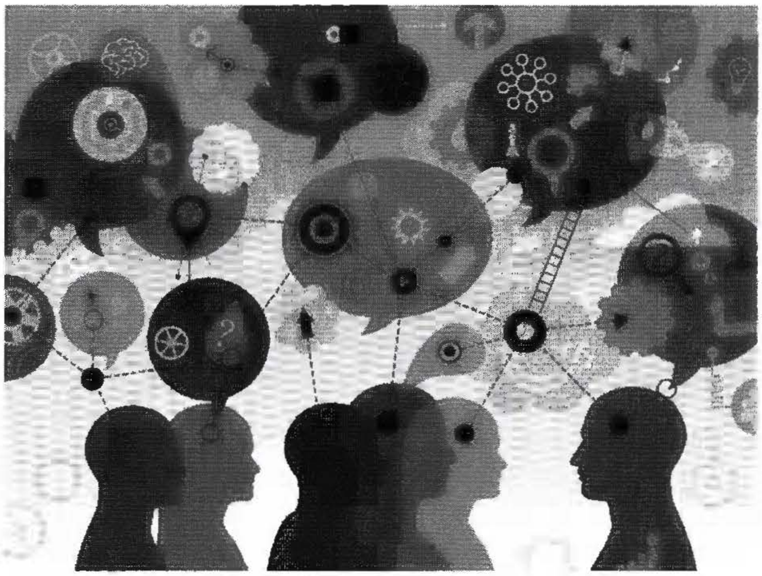
Grade 9: Choose any number that is 10 less than or 10 more than a certain perfect square number. Describe how you could estimate the square root of the number you picked and actually share what your estimate would be.

Research Article

Intersections in Reasoning Within Science and Mathematics

Ashley Pisesky, Janelle McFeetors and Mijung Kim

Elementary school classrooms are rich sites of children's mathematical and scientific thinking. As a preservice teacher (Ashley Pisesky) and researchers who have taught in schools (Janelle McFeetors and Mijung Kim), we are privileged to watch and listen to children's excitement as they make sense of a new mathematical idea or figure out a scientific way of problem solving. Observing colleagues in classrooms, teachers often plan in interdisciplinary ways knowing that children's learning is more meaningful when they connect ideas. With curricula packed with content, integrating content areas also helps to ensure that all outcomes are addressed in a school year. Teachers and students do not necessarily live out artificial distinctions between content areas in their classrooms.



With the advent of a STEM (science, technology, engineering and mathematics) approach, more resources are available for integrating science and mathematics. These resources contain activities students find engaging. However, a critical viewing reveals that much of the early implementation of STEM results in activities that prioritize one subject area over another where either mathematics serves the scientific ideas with technical skills or a mathematics idea is dressed up in a scientific context. This results in a coordinate approach (Babb et al 2016) being supported, rather than integration. Additionally, interdisciplinary teaching of science and mathematics is not assumed in curricular documents written for separate subject areas.

On one hand, teachers are balancing the expectations and realities of children's learning. While on the other hand, resources and curricula provide nominal support for integration of science and mathematics. We see an area with great potential for growth, given thoughtful design of opportunities for children to experience synchronicity in thinking across multiple subject areas to support integration. As there are no boundaries among disciplines in everyday problems, children as problem solvers do not experience separation or differences in mathematical and scientific reasoning; that is, children's reasoning processes intersect and integrate across disciplines, seeking answers and solutions to problems.

We hoped research-based literature would help us find intersections between mathematics and science learning. Our main intention was to move beyond tasks where mathematics and science coexist and to examine in finer detail how children think within the subject areas. As we reflected and discussed possible intersections, reasoning arose as an interesting site to explore. We framed our inquiry around the question: To what extent is the process of reasoning a possible intersection between mathematical thinking and scientific thinking in elementary school classrooms?

Because of the vast quantity of studies depicting children's reasoning both in mathematical and in scientific contexts in elementary school, we chose to first pursue this inquiry by understanding current research literature. The literature review would inform our understanding of how reasoning is referred to in mathematics and science in order to identify possible intersections.

Reasoning as Characterized in Curricula

To understand any intersections that may exist between science and mathematics, we needed to know how researchers were discussing reasoning in both subjects independently. The Alberta program of studies is a good place to look for working definitions regarding reasoning.

According to the mathematics program of studies, "mathematical reasoning helps students think logically and make sense of mathematics" (Alberta Education 2016, 6). While the benefits of students using reasoning are explicit, what defines reasoning is ambiguous. Reasoning, rather, is characterized by the actions students carry out in the process of reasoning and problem solving. For example, "analyze observations, make and test generalizations from patterns . . . . use a logical process to analyze a problem, reach a conclusion and justify or defend that conclusion" (2016, 6). Broad in nature, these actions could be woven throughout all of the content strands as children describe and support their mathematical thinking.

A commonality between both characterizations and emphases is that of problem solving.

Similarly, the science program of studies has no direct definition of reasoning, yet comparable language describes the qualities of reasoning. For example, the science "program provides a rich source of topics for developing questions, problems, and issues, that provide starting points for inquiry and problem solving" (Alberta Education 1996, A.2). As developing critical thinking skills is a main goal of science education, the science program of studies clearly emphasizes critical thinking with "evidence." The importance of evidence is shown in General Learner Expectations as follows: "critical-mindedness in examining evidence and determining what the evidence means" and "a willingness to use evidence as the basis for their conclusions and actions" (p B.24). The program of studies clearly emphasizes critical thinking and evidence-based reasoning as part of scientific thinking.

A commonality between both characterizations and emphases is that of problem solving. In the problem-solving process, children observe, collect data and information, analyze, and generalize with and for patterns. Interestingly, even though the science program of studies provides a similar characterization as to the definition of reasoning in the mathematics program of studies, the term *reasoning* is never formally defined. This might speak to some of the issues that arise when disciplines use different subsets of languages that have similar definitions.

Reflecting on the characterizations of reasoning from the respective programs of studies only gave us a general starting place. To continue in our inquiry on reasoning as a possible intersection between scientific thinking and mathematical thinking in children, we needed to locate more finely nuanced descriptions of reasoning. Framed by the curricular understandings of reasoning, we undertook the following inquiry.

Inquiry Process

Much has been written about reasoning in both mathematics education and science education. To begin, we scanned a few seminal readings in both mathematical thinking and reasoning (for example, English 1997; Mason, Burton and Stacey 2010; Polya 1954) and scientific reasoning and argumentation (for example, Erduran and Jimenez-Aleixandre 2007; Kuhn 2010; McNeill 2011; Osborne, Erduran and Simon 2004) to contextualize current research.

We then searched for current journal articles in databases, such as JSTOR, EBSCOHost, ProQuest, ERIC and the University of Alberta library catalogue. The search terms, in combination with either mathematics or science, included *elementary, reasoning, argumentation* and *proof*. The list of articles was substantial, and eventually searching with various keywords did not produce any new articles beyond what was already collected.

To collect a manageable group of readings in each discipline, we delineated the bounds for searching through the following selection criteria. Our selection focused on journal articles and excluded conference proceedings and books, as articles are usually the venue through which researchers share their most current findings. We looked for peer-reviewed reports of empirical studies published in academic and professional journals. To use the most recent research available, we used a date range of 2,000 to the present. In the end, we used about 40 papers for this literature review.

We did the initial analysis by reading all the papers to see how reasoning was defined and discussed within each discipline to ascertain the range of ideas. We found that researchers explained their understanding of reasoning through various examples that provided insight into characterizations initially outlined by them. We kept detailed notes on what type of reasoning the researchers explored, how they defined it, how they observed children developing reasoning and noteworthy findings. Throughout the reading and summary writing, prominent words began to emerge and were used to categorize articles. For each category, an overall analysis was written.

Major Themes of Mathematical Reasoning

After reading about 20 articles focused on mathematical reasoning, we identified 10 general themes regarding how researchers discuss reasoning in mathematics. These general themes can be sorted into two broader categories: processes of reasoning and forms of reasoning, depicted in Table 1.

Processes of Reasoning	Forms of Reasoning
Conjecturing	Deductive
Justifying	Inductive
Specializing	Plausible
Problem solving	By analogy and metaphor
Creating proofs	By contradiction

Table 1. *Ten themes within two categories for mathematical reasoning.*

Processes of reasoning encompass the ways in which children engage in acts of reasoning, also described as the verbs of mathematical reasoning (McFeetors and Palfy 2017). Conjecturing and justifying are integral processes often explored in literature. Forms of reasoning refers to logical chains of statements and their structural aspects that are conventions within mathematics leading to proofs. Interestingly, Polya's early work on deductive (demonstrative) and plausible reasoning has maintained high importance in recent literature. Rather than exploring all of the themes below, we describe two themes from each category that represent the best possibilities for intersection between mathematical reasoning and scientific reasoning in elementary school classrooms.

Processes of Reasoning

Conjecturing can be defined as offering "a statement which appears reasonable, but whose truth has not been established" (Mason, Burton and Stacey 2010, 58). Often children will express a conjecture based on a pattern that is emerging in their mathematical thinking, some initial sense they are making of a mathematical problem akin to a guess or hunch. Sharing a conjecture with others allows for investigation that could lead to justification or modification, where mathematical reasoning "often begins with explorations, conjectures" (NCTM 2009, 4). As a specific example for classrooms, Houssart and Sams (2008) had upper elementary school children play Lines, a game similar to Connect Four. One student pointed out a good starting place and conjectured about the value of the move, "because it's right in the middle and we could go up across, diagonal, loads of different ways" (p 62). Even though many students were not convinced initially, by the end of the sessions they had tested the conjecture sufficiently to show that they had a better chance of winning with a central start. Interestingly, Lane and Harkness (2012) noted that when students skip the process developing conjectures through exploring the problem context, they are unable to justify solutions convincingly. These examples demonstrate that it is important for children to form initial conjectures, evaluate the conjectures and continue to modify or offer new conjectures to lead toward convincing solutions to mathematical problems.

Justification is another key process in children's use of mathematical reasoning. In fact, many researchers refer to reasoning interchangeably with justification. They state, "mathematical reasoning . . . involves justifying" (Thom 2011, 234) or define reasoning as "the ability to justify choices and conclusion" (Johnsson et al 2014, 20). Staples, Bartlo and Thanheiser (2012, 448) see justification as "an argument that . . . uses . . . mathematical forms of reasoning," while Mason, Burton and Stacey (2010) see it as convincing yourself and others of why a conjecture or solution works all the time. As a specific example for in Grade 6 classrooms, Mueller and Maher (2009) used tasks with Cuisenaire rods, which focused on fractional relationships among the differing lengths. The researchers elicited justifications from students by asking, "How can you convince the whole class?" (p 112). In one instance, students defended their answers of why a rod of length 9 did not have any corresponding half lengths by lower and upper bounds: "The yellow is a little bit more than a half, and the purple is shorter than a half" (p 113). By contraction, "Here is not a rod that is half of the blue rod because there are nine little white rods; you can't really divide that into a half, so you can't really divide by two because you get a decimal or remainder" (p 113). This example demonstrates that elementary school children are capable of justifying their thinking and need their teachers' support through questioning to regularly express their reasoning in many ways. Additionally, the way justifications are constructed and expressed warrants more discussion in the following section.

Forms of Reasoning

Deductive reasoning is one of the defining forms of mathematical reasoning, typically described as being able to draw a conclusion from pre-established facts (Reid 2002a). The prominence deductive reasoning plays in mathematics as a discipline is not surprising as it is the primary form of constructing proofs (Flegas and Charalampos 2013; Reid and Zack 2009). Moving beyond a broad categorization, Reid (2002b) describes different types of deductive reasoning, such as "simple one-step deductive reasoning . . . multistep deductive reasoning . . . [and] hypothetical deductive reasoning" (pp 235–36). While the first two types refer to the complexity of chains of reasoning, the last type signals making inferences from the hypotheses generated during problem solving (Stylianides and Stylianides 2008). Furthermore, Komatsu (2016) emphasizes the importance of deductive thinking in students by explaining, "deductive guessing can be regarded as an authentic mathematical action because . . . it [can] overcome counter-examples" (p 159).

...elementary school children are capable of justifying their thinking and need their teachers' support through questioning to regularly express their reasoning in many ways.

Reasoning by counter-examples is not an exhaustive approach to proving, so the shift in students' use of deductive guessing in the reported research showed a shift in students' invocation of reasoning within problem solving. In other words, children show more sophistication in their reasoning as they move beyond using counter-examples to justify a conjecture toward creating chains of reasoning using established facts. The observable improvement in reasoning helps to further the idea that deductive reasoning is an essential skill that students should be developing. As a specific example for classrooms, Wanko (2009) introduced a variety of Japanese puzzles into his classroom to help foster deductive reasoning. He explains the value of using these puzzles in that "when students learn to provide deductive arguments for their puzzle-solving strategies, they are laying the foundation for good mathematical practices" (p 271). This statement emphasizes the essential nature of deductive reasoning in the mathematics classroom. Puzzles, like Sudoku, require students to use given information with completed cells and rules for placements to fill in the missing cell values.

...children show more sophistication in their reasoning as they move beyond using counter-examples to justify a conjecture toward creating chains of reasoning using established facts.

Plausible reasoning, as complementary to deductive reasoning, is important to solving mathematical problems and is a component of reasoning in daily life. Plausible reasoning (Polya 1954) is based on explorations that do not follow a prescribed pathway, is bound up with conjecturing through use of inferences, acknowledges personal knowing, coincides with mathematical thinking, and does not demand the same rigour and aim of absolute certainty as in deductive reasoning. Leading to developing mathematical ideas, plausible reasoning incorporates generalizing through pattern-noticing within inductive reasoning while relying on connections made to similar structures within analogic reasoning. Put in another way, Polya (1954) states that "it is reasonable to try the simplest case first" and how "even if we return eventually to a closer examination of more complex possibilities, the previous examination of the simplest case may serve as a useful preparation" (p 194). The following example further demonstrates this, wherein Sumpter and Hedefalk (2015) analyzed preschool children's reasoning through play. When a young child suggested measuring the height of a rock, the children collectively offer reasoning based on inferences. For example, "Yes, but the house is bigger than the rock" (p 5). Or where a conclusion is offered based on measuring as evidence, "It is bigger than me anyway [walks and stands next to the rock and looks up, using her own body as a measure]" (p 5). The informal reasoning implied by plausible reasoning is a wonderful starting place in the early years of elementary school, where children can be asked to provide defenses that are connected to their experiences and reasonable to the problem-solving context.

Major Themes of Science Reasoning

Forms and Skills of Scientific Reasoning

Several prominent themes emerged from the literature on science reasoning, and we have selected the most comprehensive descriptions and definitions. One major theme is deductive reasoning, which is also described as a means of reasoning in mathematics. Deduction, as a key skill for scientific reasoning (Van der Graaf, Segers and Verhoeven 2015), is often discussed with a hypothesis-based approach in science. For instance, researchers emphasized hypothetico-deductive reasoning whereby deduction is combined in an overall process of reasoning alongside hypothesizing (Chen and She 2015; Lawson 2008).

When students made a hypothesis, they were also challenged to give their reasoning and, where appropriate, to provide evidence to support their statements, that is, deductive reasoning.

The process of hypothetico-deductive reasoning in classrooms occurs when students make a hypothesis based on their experiences and knowledge to an unknown situation, deduce what would happen if their hypothesis was correct, design a test based on the deduced ideas and finally test it to verify or falsify it. If it is false, they will make another hypothesis. Lei et al (2009) explicitly states that "scientific reasoning ability . . . focuses on . . . reasoning skills such as the abilities to . . . formulate and test hypotheses" (p 586). The skills of scientific reasoning, such as hypothesizing and fair testing, are essential components of understanding scientific reasoning as an entirety, because they aid in describing the big picture of scientific problem solving and knowledge development. As a classroom example, Tytler and Peterson (2003) asked students to hypothesize which whirlybird would fall and spin faster. The whirlybirds had three different wingspans: short, medium and long. When students made a hypothesis, they were also challenged to give their reasoning and, where appropriate, to provide evidence to support their statements, that is, deductive reasoning. Deductive reasoning is also described as a reasoning skill that scientists often engage in (Wasserman and Rossi 2015).

Inductive reasoning is used to describe and discuss scientific reasoning and is often mentioned with reference to observed patterns. Lawson (2005) viewed it as a primary component of scientific reasoning. Wasserman and Rossi (2015) explain the significance of induction in scientific reasoning by describing how "one of the primary modes of reasoning in science is induction" (p 23). Wasserman and Rosi (2015) also found that "science teachers . . . were more prone to us[ing] inductive methods of reasoning" (p 32). Duschl (2003) further supports this by stating that "scientific inquiry . . . [is] an inductive process." A classroom example is an electric conductor and indicator activity. Students test various materials, such as a wood stick, metal spoon, nail, plastic pen, paper, rubber band and so on, in an electric circuit to determine that metal materials are conductors (induction). This approach is common in hands-on science inquiry. This science concept through inductive reasoning often continues to develop with deductive reasoning when teachers provide everyday materials, such as a key, a coin or a metal glass frame, and ask if the items would pass an electric current or if wearing rubber gloves would be safe during electricity repair. These further questions will help develop students' deductive reasoning (for example, the key is metal, metal is a conductor, conductors pass electricity, therefore, key passes electricity).

The collaboration of claims, evidence and justification in argumentation empowers students' scientific reasoning.

Another key theme to explain science reasoning is argumentation, which is a means through which scientific reasoning is developed. For example, it is seen as an essential aspect of "prompting scientific reasoning" (Driver, Newton and Osborne 2,000; Duschl and Osborne 2002; Roberts and Gott 2010). Argumentation is used to develop and evaluate claims based on data and evidence. When students encounter conflicting claims, they need to search for evidence to justify which claim is more convincing to reach an agreement or conclusion. For instance, when students propose two conflicting claims: (1) platypus is a mammal, and (2) platypus is an amphibian, they need to find sufficient evidence to justify their conclusion. The collaboration of claims, evidence and justification in argumentation empowers students' scientific reasoning (Osborne, Erduran and Simon 2004).

The Essence of Scientific Reasoning: Evidence

In the process of scientific reasoning, linking theory and evidence, that is, understanding the covariation between theory and evidence is critical

(Kuhn and Pearsall 2,000). For instance, in hypothesis testing, students use scientific data or information as evidence to support or refute their hypothesis. In an inductive approach of scientific experiments, a conclusion must be drawn from data collected, that is, evidence-based data analysis. In the processes of argumentation, a claim must be justified with evidence to be persuasive and convincing. Thus, "argument[ation] in the science classroom . . . can help students develop science skills . . . [such as] using evidence to defend a point of view" (Thier 2010, 70). In any type of scientific reasoning and problem-solving process, students are challenged to connect their claims, explanations and conclusions to evidence to make their ideas scientific, justifiable and, thus, persuasive. So important is evidence in scientific reasoning that Tytler and Peterson (2004, 98) state, "A key aspect of scientific reasoning is the ability to suggest and make judgments about evidence." McNeill and Krajcik (2008) also explained the important role of evidence in science: "When scientists explain phenomena and construct new claims, they provide evidence and reasons to justify them or to convince other scientists of the validity of the claims" (p 121). This description of the importance of evidence and its role in science facilitates the concept that evidence-based thinking in science is critical.

Scientific reasoning can be broadly defined as *intentional* coordination of theory and evidence (Mayer et al 2014, italics added). As science reasoning requires one's intention, practice and skills to coordinate theory (claim) and evidence (data) in scientific explanation, for students to think and process material from a truly scientific perspective, we must provide the tools for this to become a reality. Helping students to learn evidence-based means of thinking will help to facilitate this into a reality. Hardy et al (2010) discuss the concept of evidence-based reasoning (EBR) and how it potentially "contribute[s] to the development of individual students' abilities in scientific reasoning" (p 198). They categorized evidence-based reasoning into three levels: (1) data-based reasoning—students' ideas (claims and statements) are supported by a single property or observation, (2) evidence-based reasoning—students' ideas are supported by a contextualized relationship between two or more data or evidence, and (3) rule-based reasoning—students' ideas are supported by a generalized relationship or principle (Hardy et al 2010). Evidence- and rule-based reasoning are higher and more sophisticated levels of reasoning than data-based reasoning in terms of evidence-claim evaluation and knowledge generalization and application. Another notion discussed in the literature is that of scientific literacy, viewed in relation to evidence. For example, Brown et al (2010, 124) state how "students who are scientifically literate should be able to make judgments based on the evidence supporting or refuting [an] assertion." This only further assists in demonstrating the critical nature of evidence-based thinking as it is viewed through this definition of scientific literacy as an essential component of it. The concept of scientific literacy is further backed by McNeill and Krajcik (2008), who claim that "students need to be able to critically read . . . by evaluating the evidence and reasoning presented . . . [this] allows students to make informed decisions" (p 121). That critical and evidence-based thinking are integral components to thinking scientifically is clearly a common theme throughout the literature.

...for students to think and process material from a truly scientific perspective, we must provide the tools for this to become a reality.

Discussion and Reflection

In elementary mathematics and science classrooms, reasoning is an important foundation for students to form a significant and thoughtful understanding of the processes that underlie these subjects and to apply and develop disciplinary content knowledge. For instance, claims and hypotheses are made, and data and evidence are evaluated as plausible or implausible based on children's current knowledge (Sadler and Zeidler 2005). When children's current knowledge does not support observed phenomena, such as discrepant events or cognitively conflicting situations, they need more plausible and fruitful knowledge to explain the phenomena in the justification process where teachers can expect conceptual change and development. Because of this significance, it is essential to understand how reasoning is understood within each discipline, as with that knowledge we can begin to develop stronger links between the two subjects that can facilitate increasing student understanding both in the individual subjects and between both subjects.

Reasoning as it was discussed in the mathematics literature primarily focused on the keywords that one typically may conjure up when thinking about reasoning from a more standard perspective—terms, definitions and examples of deductive, inductive and plausible reasoning were common themes in the realm of mathematics reasoning. Some of these key words and definitions were also demonstrated within the literature on scientific reasoning, in particular, deductive and inductive reasoning. In the discussion of deductive reasoning in science, hypothesis is a key idea whereby students' hypothesis testing often includes deductive reasoning. As a distinction within the commonality of deductive reasoning is that in mathematics constructing a proof is seen as the purpose of deductive reasoning. From the literature, we found conjecture in mathematics and hypothesis in science seem to share some degree of commonality where students make a claim based on their prior experiences, observation and knowledge to explain what is going to happen in an unknown situation.

...we can begin to develop stronger links between the two subjects that can facilitate increasing student understanding.

Interestingly, the prevalent theme of the topic of evidence and the essential role that a variety of authors viewed it to have in scientific reasoning, and how the understanding of reasoning with an emphasis on evidence was not prevalent in the literature on mathematics reasoning. However, although evidence was not necessarily a common theme that arose in the mathematics literature, other keywords were often referenced, which have similar meaning to evidence, such as justification through specific examples and specializing to convince with a smaller problem. We believe that even though the literature refers implicitly to the concept of evidence in the mathematics literature, the idea of evidence may be a commonality these two disciplines share about reasoning, and one that deserves further exploration to benefit future teachers and students.

Overall, commonalities of mathematical and scientific reasoning lie in the area of observing, analyzing and justifying in a problem-solving process. To understand and solve the problem, children observe, collect data (evidence) and analyze the observed data to come up with answers. In mathematics classrooms, teachers commonly use *conjecturing* and *justification* to explain this problem-solving process, and in science classrooms, teachers use the terms *making claims, seeking evidence* and *justification.* In this problem-solving process, inductive, deductive, hypothetico-deductive and plausible reasoning are complexly intertwined, yet whichever reasoning students call on, their solutions must be justified with evidence. Even though students' mathematics and science reasoning share many commonalities, in literature review, they are explained with different terms and language; thus, it seemed they were separate cognitive skills in children's thinking.

Reflection

In this section, we share our reflections on children's reasoning in elementary classrooms based on our own perspectives and experiences as a preservice teacher (Pisesky) and teacher educators (McFeetors and Kim).

Ashley Pisesky

These findings have been very helpful to me as a preservice teacher, and they would aid other elementary school preservice and current teachers. For example, the time-intensive lesson planning was a challenge while completing my practicums. Since elementary school generalist teachers are responsible for instructing about five subjects daily, lesson planning becomes overwhelming; few explicit cross-curricular connections between the subjects are taught in postsecondary preparation. Having more explicit connections specific to the school subjects demonstrated that this kind of preparation may have made lesson planning easier. Some of the mathematics and science lessons may have been linked together, using one lesson and one time block to instruct both sets of content.

The focus should be on the processing that students are engaging in.

Alongside this, students would benefit from having more of the subjects linked across the curriculum. I was a strong believer of this throughout my practicums, and I often looked for ways to link students' learning. However, many of the links that I found were more superficial in nature, such as how doing writing in science class links both language arts and science. Alternatively, linking content in subjects, such as a learning outcome in mathematics and in science, may also be viewed by some as more of an artificial connection. Although it is good to point out the two similarities and to reinforce one subject through another, a fundamental missing link between subjects at a deeper level in order to better understand and facilitate student processing is currently a deficit that should be included in preservice teacher training. A prime example of how this could be better integrated into preservice preparation is the research gathered through this literature review. With STEM being an increased focus in schools, both in the classroom and in extracurricular activities, it is essential that teachers know and understand the deeper meaning as to why and how these subjects are related to one another in order to better implement learning in the classroom. From my experiences, a better understanding of how students engage in the process of reasoning in both subjects will help to foster greater understanding in both. I therefore believe that linking the subjects of mathematics and science with students in the elementary classroom is something that not only could be but should be reasonably practised by preservice and practising teachers.

One revelation from this process was when I discussed the intersections between science and mathematics reasoning with my supervisors. Janelle and Mijung mentioned that in science reasoning we discuss the hypothesis-verification process to develop reasoning, but mathematics reasoning is developed through the use of conjectures. They proceeded to explain that conjectures and hypotheses essentially point to the same phenomenon; however, they are each used in their respective field. I think that this is something that should change in the future, as we look toward creating more cohesive and comprehensive learning opportunities for students. We should use both words interchangeably in both fields so that students do not get left behind in the language of the topic. The focus should be on the processing that students are engaging in. If we allow this to be the focus of teaching and learning, we will see increased student understanding in both domains. We will reduce the disparity that exists between students who excel in each domain but struggle in the other. All of these are important effects that students would benefit from.

Janelle McFeetors and Mijung Kim

Reasoning in general involves logic thinking. When children encounter a puzzling question, they try to find solutions by retrieving and reorganizing their thoughts, experiences and knowledge. We educators want to support students in constructing reasonable solutions developed through logical thinking processes. Through various pedagogical strategies, educators strive to enhance children's thinking and reasoning processes, which help them construct solutions, which also develops knowledge application. For instance, in mathematical problem solving, children learn to conjecture, specialize, justify and create proof, and in scientific problem solving, they learn to evaluate and justify claims with evidence to draw conclusions. In this process, children's knowledge is reflected, examined and developed to solve the current problem. However, often the particular terminologies for these cognitive actions are used in a way that teach children to see reasoning as if they were different and isolated within content areas. We seldom question what children do differently during *conjecturing* in mathematics class and *hypothesizing* in science classrooms. Children try to make sense of the current situation at hand (for example, a puzzle, question, discrepant event and so on) using their knowledge, experiences and creativity to come up with a possible explanation, which is *conjecture* in mathematics and *hypothesis* in science. We acknowledge these terminologies are unique in each disciplinary tradition, thus need to be acknowledged and respected. Yet when separately taught to preservice teachers and further to children in classrooms, they could become confusing and seemingly isolated cognitive processes.

Reasoning in general involves logic thinking.

In this study, we teacher educators looked at mathematics and science reasoning not from a subject disciplinary lens but from the perspectives of a child and a teacher who might not distinguish reasoning processes in two different subject areas. We believe there is a need for understanding how reasoning in mathematics and science could be integrated and taught, such as in STEM-oriented classrooms. In a STEM approach, students are engaged in problem solving, which requires integration of knowledge and skills among different disciplines and the boundaries of disciplines often disappear. Once the problems are identified and goals are shared in the problem-solving community, disciplinary traditions and knowledge and reasoning skills are all complexly intertwined and integrated in collective levels. Students create, justify, evaluate and negotiate their ideas to reach the best solutions to problems. Which mathematical reasoning and scientific thinking do students use in a STEM problem-solving process? One might find this question difficult and not necessary as children's reasoning and problem-solving process are intertwined and integrated without the boundaries of subjects, which motivated our interest in this study.

To illustrate, we offer a specific example of a STEM approach, where students are challenged to solve a problem, such as building a boat with material and time constraints. The boat needs to meet with certain criteria, such as (1) holding a certain weight, and (2) reaching a certain point as fast as possible when a fan is blowing. In this problem-solving situation, students must understand the relationship of density, buoyancy, geometrical shapes, friction of materials, measurement of distance and loading strategy. To prove their design, they would test their boat with a certain load and a fan blowing on water. When the load gets heavier, they would conjecture the maximum load before it sinks. In this problem process, children's reasoning is complexly intertwined with various types of reasoning. Thus, it is neither possible nor meaningful to indicate mathematical and science reasoning separately. An implication for classroom practice is that mathematics and science content be addressed simultaneously through intriguing problems for students, where reasoning is elicited in their actions and discourse. Rather than labelling these actions with discipline-specific terminology, teachers can celebrate the understandings students develop as they offer tentative explanations, explore the context and ultimately justify their ideas. This is where we feel the gap exists between theory of cognition and everyday practice.

During our reading and conversations, we questioned how we could develop more integrated ways of teaching. We reflected on our own classrooms in our teacher education program in subject-specific curriculum courses and our own teaching at the university. We recognized that it is also very isolated as we perpetuate distinctions using different terms for similar reasoning processes. This led us to examine the terminologies of reasoning that we use in each discipline and how we introduce them to preservice teachers. As we realize that students in schools and citizens in everyday life integrate knowledge and skills without disciplinary boundaries similar to a STEM approach, it was worthwhile questioning how reasoning is discussed in research, curriculum and in our own classes as an initiative of developing an integrated approach for mathematics and science teaching.

As a result of this inquiry, we have more questions and challenges as we start to reflect on our own classrooms at the university. The current teacher education program has perpetuated the separation between science and mathematics through its subject-based program design. Also, as the specific terms of reasoning, such as *conjecture* and *hypothesis*, are the means of communicating among educators and researchers within the subject disciplines, they will be continuously used in the communities of mathematics and science education. As we realize the need for an integrated approach in today's classrooms, how we introduce these terms without creating confusion and resistance becomes a challenge. Creative and collective efforts will be required in further conversations.

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Leibniz's Heuristic Derivation of the Product Rule and Quotient Rule

Indy Lagu

The standard derivations of product rule and quotient rule are algebraic and involve adding zero in a clever way. These proofs are mathematically correct, but not pedagogically illuminating. We present Leibniz's heuristic derivation of product rule. A heuristic derivation of quotient rule, based on Leibniz's idea, is also given. While not mathematically rigourous, we believe an approach based on these ideas to be pedagogically superior.

Introduction

Let $x = 100$ and $y = 100{,}000$. Then $xy = 10$ million. Now let $x_1 = 97$ (3 per cent less than x) and $y_1 = 101{,}000$ (1 per cent more than y), and think about the following question:

Is xy greater than x_1y_1?

It turns out that $x_1y_1 = 9{,}797{,}000$, so xy is greater. In fact, xy is greater than x_1y_1 by 2.03 per cent.

That relative difference, 2.03 per cent, is a bit suspicious, and looks a lot like the sum of the relative difference in x (–3 per cent) and the relative difference in y (1 per cent).

As it turns out, that coincidence has an explanation, and that explanation gives us a heuristic proof of the product rule and of the quotient rule. The following is essentially Leibniz's derivation of product rule.

Product Rule

Given two numbers, x and y, change x by Δx and y by Δy. Now, the difference between the product of the new x and y and the old x and y is

$$\Delta(xy) = (x + \Delta x)\,(y + \Delta y) - xy,$$

and a routine calculation shows

$$\Delta(xy) = y\Delta x + x\Delta y + (\Delta x)(\Delta y),$$

and hence

$$\frac{\Delta(xy)}{xy} = \frac{\Delta x}{x} + \frac{\Delta y}{y} + \left(\frac{\Delta x}{x}\cdot\frac{\Delta y}{y}\right). \qquad (1)$$

Now if the relative changes in x and y are small, the third term in equation (1) is negligible and we have

$$\frac{\Delta(xy)}{xy} \approx \frac{\Delta x}{x} + \frac{\Delta y}{y}\,. \qquad (2)$$

In other words, *the relative change in a product is the sum of the relative changes in the factors.*

Using this heuristic, the product rule is almost immediate. For functions, the analogue of equation (2) is

$$\frac{(fg)'}{(fg)} = \frac{f'}{f} + \frac{g'}{g},$$

and multiplication by fg yields

$$(fg)' = f'g + fg',$$

which is the product rule.

Quotient Rule

Since the relative change in a product is the sum of the relative changes in its factors, it stands to reason that the relative change in a quotient is the difference of the relative change in the numerator and denominator. In fact, this is true. We leave it to the reader to verify that

$$\frac{\Delta(x/y)}{x/y} = \frac{y}{y+\Delta y}\cdot\frac{\Delta x}{x} - \frac{\Delta y}{y+\Delta y}.$$

Therefore, if the relative changes in x and y are small, one obtains

$$\frac{\Delta(x/y)}{x/y} \approx \frac{\Delta x}{x} - \frac{\Delta y}{y}. \qquad (3)$$

The continuous analogue of equation (3) is

$$\frac{(f/g)'}{(f/g)} = \frac{f'}{f} - \frac{g'}{g},$$

which is equivalent to the quotient rule.

To be sure, our derivations (or should we say Leibniz's derivations) are not mathematically rigourous. We would argue the standard proofs presented in calculus classes are not entirely rigourous either, since they rely on an intuitive understanding of how one calculates limits, rather than definitions involving ϵ and δ. We believe Leibniz's idea has greater pedagogical value.

Backing Up and Moving Forward in Fractional Understanding

Angela T Barlow, Alyson E Lischka,
James C Willingham and Kristin S Hartland

A well-crafted opening problem can provide preassessment of students' fraction knowledge and assist teachers in determining next steps for instruction.

After watching a demonstration lesson that exposed students' misunderstandings regarding division of fractions, a teacher shared this sentiment:

> When I read a standard, I think about what that standard says I have to teach and I find a way to teach it. I don't think about how far I need to back up. (Pamela, a Grade 5 math teacher)

In a discussion of the lesson among colleagues, two key ideas surfaced. First, standards such as those presented in the Common Core State Standards for Mathematics (CCSSM) (CCSSI 2010) "identify the end goal of a unit of instruction that encompasses more than a skill that may be taught in one or two lessons" (Barlow and Harmon 2012, 500). Second, carefully crafted word problems provide a means for identifying students' misconceptions (Barlow 2010) and guide the teacher in knowing how far to back up along the path of the learning trajectory. This process of backing up begins with using responses to a word problem to identify categories of students' understandings in relation to the expectations of the standard and using this information to make instructional decisions. In some instances, students will provide evidence of meeting or exceeding lesson expectations; instructional decisions, therefore, will need to advance their thinking. Instructional decisions for other students, who are working toward lesson expectations, should help them connect prior knowledge to new concepts. Students who are lacking fundamental understandings require instruction aimed at filling gaps in prior knowledge. The purpose of this article is to demonstrate this backing-up process—by examining categories of student work taken from a carefully crafted problem—and suggesting instructional decisions.

The Backing-Up Process

Use students' responses to a carefully crafted word problem to identify categories of understandings and to make instructional decisions.

1. When students exceed expectations, instructional decisions should advance their thinking toward an identified end goal. Build students' understanding, perhaps by using a different problem context or by using numbers that are more complicated.
2. When students meet lesson expectations, they are ready to begin exploring the new concept. Guide group discussions to attend to key aspects of the context to allow students to move deeply into the concept.
3. When students are working toward meeting lesson expectations, instructional decisions should help them connect their prior knowledge to new concepts. Supply supportive tasks that will prepare students.
4. When students lack fundamental understandings, aim instruction at filling in gaps in students' prior knowledge before expecting them to work toward the lesson expectations.

The Measuring Scoops Problem

To begin thinking about the backing-up process, we present a problem used in a professional development project that elicited students' fraction understandings. The first author created the Measuring Scoops problem using a problem-creation framework (Barlow 2010) with a goal of engaging students in interpreting the remainder of a division problem that involved repeated subtraction of a fractional quantity. The problem, which follows, is significant in terms of CCSSM content standard 6.NS.1 (CCSSI 2010).

> Chef Frederick is mixing ingredients to bake a dessert. His recipe calls for 2 1/2 cups of sugar. The only measuring scoop that Chef Frederick has measures 1/3 cup. How many measuring scoops of sugar will Chef Frederick need?

In thinking about this problem, several key features emerged that we considered important in terms of its ability to meet our instructional goal:

- Students are likely to be familiar with measuring scoops and will relate to the context of the problem.
- Measuring scoops represent different fractional amounts and can support students in counting with a fractional amount as the "unit."
- By using 2 1/2 and 1/3, students can represent the problem in a variety of ways, including drawings and manipulatives.
- The remainder of 1/2 can be identified visually, supporting students in making sense of the remainder. More specifically, they can see that the remainder is half of what they are counting.

To solve this problem, we anticipated students representing 2 1/2 cups with pictures or pattern blocks. Recognizing that they need to know how many scoops of size 1/3 cup are in 2 1/2 cups, students would divide their cups into thirds (representing the scoops) and then count the scoops or thirds. We expected a rich discussion regarding the remaining partial scoop's value. Is the remainder 1/2 or 1/6? We anticipated encouraging students to think about what they were counting to support making sense of this.

In our professional development project, we use demonstration lessons as a means for enhancing participants' knowledge of content and instructional strategies. During a demonstration lesson, one project team member teaches a lesson while the project participants observe. For this demonstration lesson, the first author implemented the Measuring Scoops problem in a participant's fifth-grade class of 20 students. About 50 project participants observed the lesson. Although the problem aligns with a standard from the sixth-grade curriculum, we felt it was appropriate for the fifth-grade class, given that the lesson occurred near the end of the school year. In addition, we were interested in the ideas brought by students who had not yet been taught the standard, which would likely not have been the case had we been in a sixth-grade classroom.

Although the Measuring Scoops problem was designed to support students in interpreting the remainder of a division problem involving fractions, the student work it generated supplied vital insights into students' understandings. This analysis of student work led Pamela to express the sentiment shared at the beginning of this article. Next, we will share this student work and demonstrate how the problem supported project staff and participants in thinking about the backing-up process.

Examining Student Work

Considering the purpose of the backing-up process, we deliberately made the choice to engage students in solving the Measuring Scoops problem even though they had no prior instruction on interpreting the remainder in fraction division. This allowed us to preassess students' understandings on the topic and gauge their readiness to learn, which is the intent of the backing-up process. Although previous student experiences included working with models as well as the algorithm for dividing fractions, we did not expect to have students who would meet the expectation of interpreting the remainder in fraction division. Doing so at this time would result in students exceeding our expectations for this lesson. Ideally, we expected students to make sense of the context of the problem, generate appropriate representations of the quantities involved, and select a reasonable approach to solve the problem. In reviewing students' responses to the problem, we found it useful to group the students' work into four categories:

1. Exceeding lesson expectations
2. Meeting lesson expectations
3. Working toward lesson expectations
4. Lacking fundamental understanding

We begin with an example of students who exceeded the lesson expectations and then move through the remaining categories.

Exceeding Lesson Expectations

Although students had not received instruction on the topic, we unexpectedly had a few students who were able to correctly interpret the remainder in the Measuring Scoops problem (see Figure 1). These students correctly modelled 2 1/2, separated it into 1/3 pieces (the scoops) and correctly counted 7 1/2 scoops. By correctly interpreting the remainder in this way, students exceeded our expectations for the lesson. We hypothesized, however, that the problem context supported these students with interpreting the remainder. Therefore, a teacher might offer these students the opportunity to interpret the remainder in a division problem in a different context, perhaps with less simple numbers. In this way, students would be able to engage in reasoning and recognizing patterns and thus build understanding.

Meeting Lesson Expectations

In general, students who meet the lesson expectations are ready to begin thinking about the new content contained in the standard (that is, the interpretation of a remainder). For the Measuring Scoops problem, students who meet the lesson expectations should demonstrate their ability to model fractions and use the fraction models to solve a division problem. Students began by drawing models for 2 1/2 and 1/3 (see Figure 2). Next, they drew 2 1/2 again but this time divided the wholes into thirds and labelled each third. Although they did not label the remainder with 1/6 in the model, we see on the right side of the poster that they used 1/6 in their check as well as in their solution statement. In thinking about this remainder piece, however, students did not attend to the problem context or the unit being counted (that is, thirds or scoops). As a result, they did not present evidence of meeting the expectations of the standard. They are ready, though, to begin thinking about interpreting the remainder. A teacher might use this example to facilitate a whole-class discussion regarding the meaning of the remainder for the Measuring Scoops problem. Such questions as the following might be useful in guiding this discussion.

- How can we deal with the fact that Chef Frederick has only a 1/3 scoop if he needs 1/6 of a cup of sugar?
- How can you report your solution in terms of one unit?
- What are you counting?

Working Toward Lesson Expectations

In some instances, students who are working toward lesson expectations will provide evidence of possessing foundational understandings but an inability to connect these to the problem context. Such students are not ready to think about the new mathematics contained in the standard but rather need support to be ready to learn it. For the Measuring Scoops problem, students must be able to model division of a whole number by a unit fraction as well as division of another fraction by the unit fraction. Figure 3 presents an example of

Figure 1. *Students had not been instructed on interpreting remainders. Nevertheless, some met lesson expectations, exceeding what the authors anticipated.*

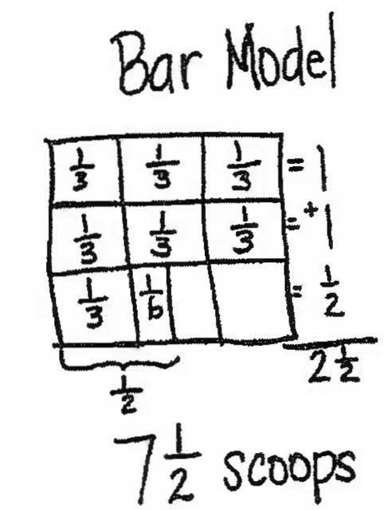


Figure 2. *Although this work shows students' alternative interpretation of remainders and their readiness to interpret remainders, failure to attend to the problem context and the unit being counted show a lack of evidence of meeting expectations of the standard.*

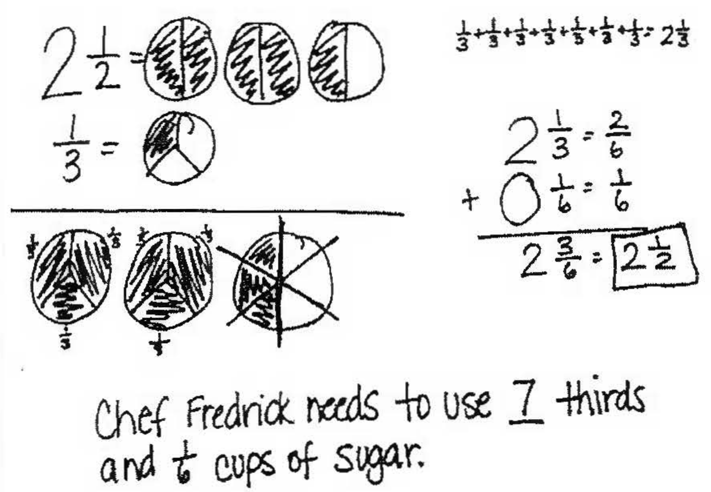


Figure 3. *For the Measuring Scoops problem, students must be able to model division of a whole number by a unit fraction as well as division of another fraction by the unit fraction.*

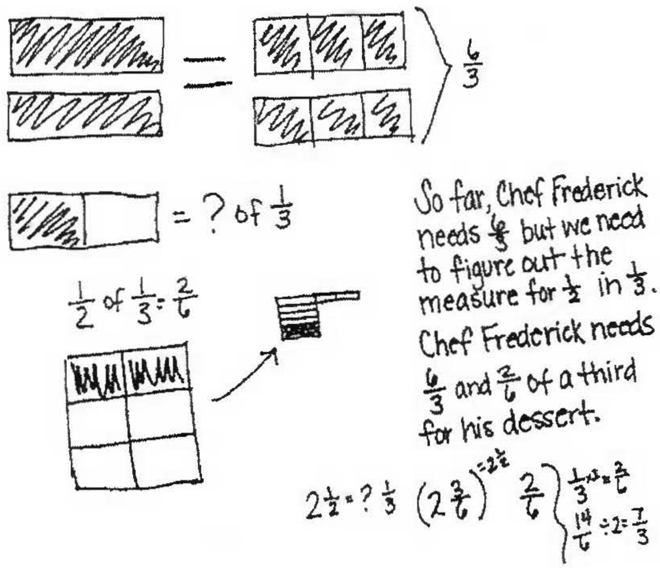
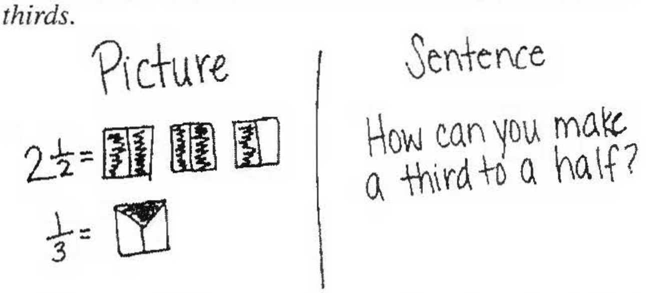


Figure 4. *This student work fails to accurately model thirds.*



this; students demonstrated 2 divided by 1/3 but were unable to extend this to modelling 1/2 divided by 1/3. The statement on the right side of the poster reads, "So far, Chef Frederick needs 6/3, but we need to figure out the measure for 1/2 in 1/3 (thirds)." This statement indicates that students were attempting to think about how many one-thirds are in one-half. However, as they attempted to find their solutions, students appeared to have gotten lost in their computations and models.

A teacher might ask questions concerning what the students were attempting to count during their fraction-by-fraction division or questions leading to a different model by which students might make sense of the problem. Follow-up tasks involving fraction-by-fraction division on appropriately marked grids may help these students progress in their thinking (Battista 2012) and eventually become ready to attend to interpretation of the remainder in fraction division.

Lacking Fundamental Understanding

Most classrooms will inevitably have students who lack fundamental understandings, which prevents them from being able to meaningfully engage in the intended topic. The ability to accurately model fractions is fundamental to modelling and solving division problems. On the left side of Figure 4, students have correctly modelled 2 1/2 and incorrectly modelled 1/3. Interestingly, the sentence on the right asks, "How can you make a third to a half?" indicating that they recognize the goal of the problem (that is, determining how many thirds are in 2 1/2). Their inability to model thirds, however, seems to be a stumbling block for beginning the solution process.

For these students, returning to basic understanding of fractions is essential. The introduction or reintroduction of manipulatives, such as pattern blocks, and returning to visual modelling of fractions can allow students entry into this problem (Battista 2012). However, expecting students to make sense of fraction operations, such as those represented in the Measuring Scoops problem, is unreasonable without first addressing these fundamental gaps.

Anticipating Roadblocks in the Backing-Up Process

The goal for using the Measuring Scoops problem was to preassess students' readiness for interpreting the remainder in fraction division by eliciting and understanding students' thinking. When examining student work in this way, though, a "roadblock" may sometimes be encountered if the work does not clearly align with one of the previously described categories. In these instances, additional questioning of the students is needed to better understand their readiness for interpreting the remainder. To help the reader anticipate potential roadblocks, we describe two examples in which students' work provided inconclusive evidence about students' understandings or misunderstandings related to the division of fractions, in general. In both cases, students produced work that held the potential for modelling division of fractions, but to draw conclusions regarding their understanding of fraction division would require too many assumptions on our part.

Anticipated Roadblock One

In some instances, students get lost in their work and lose sight of the problem goal. We see this in Figure 5. Here, students began by representing the problem with a bar model twice. They correctly drew and labelled thirds as well as sixths. In the process, though, they seem to have forgotten that they were counting thirds (for scoops). Instead, they began "putting the thirds back together" and announced that their answer was 2 1/2. In reviewing this work, it was problematic for us to determine what these students understood about fraction division and the remainder, making it difficult to categorize the work.

Anticipated Roadblock Two

A second roadblock involves students generating algorithmic statements using the numbers in the problem without considering the problem context. In Figure 6, students appear to have performed numerous calculations with the numbers that have been extracted from the problem. They began by subtracting 2 1/2 minus 1/3 in multiple ways. Then students began repeatedly adding thirds, arriving at 2 1/3, for which they then drew a bar model.

Figure 5. *Sometimes students lose sight of the problem goal. The authors had difficulty determining from students' work below what they understood about fraction division and remainders.*

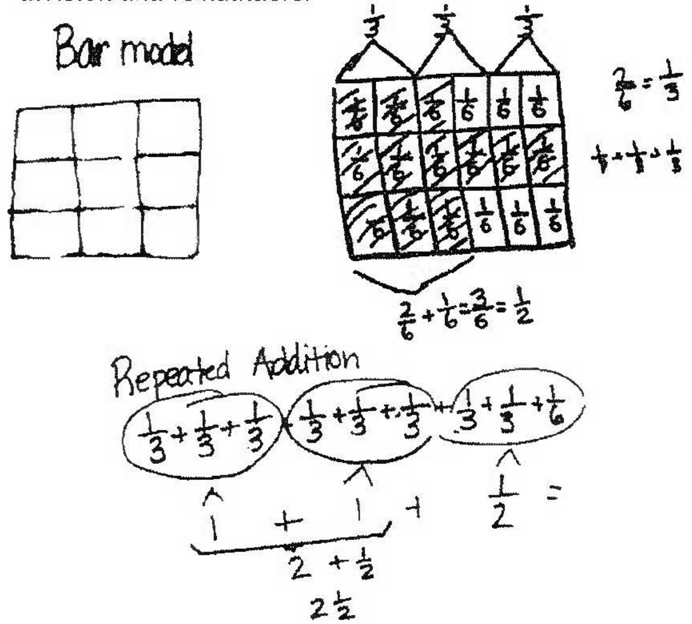
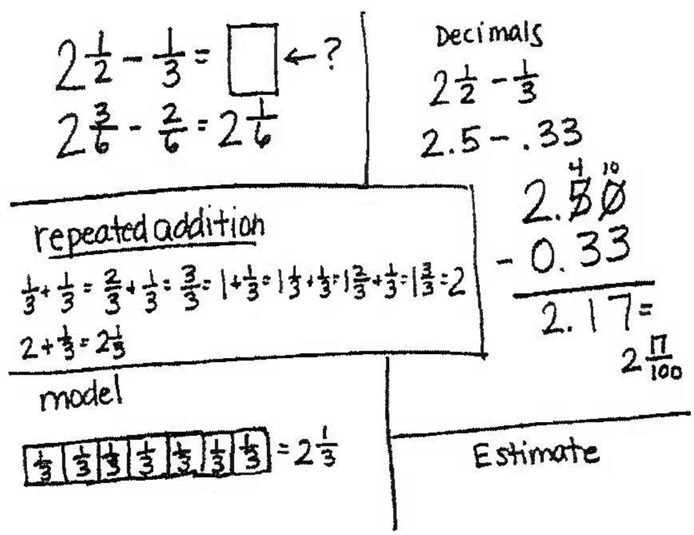


Figure 6. *This work focuses on calculations with extracted numbers but shows no evidence that students considered the problem context.*



Although their calculations appear to be correct, they have not provided evidence of ability to model division with fractions and to begin thinking about the meaning of the remainder. We could even hypothesize that these students did not recognize the problem as one involving division. However, the work alone does not clearly indicate an appropriate categorization.

Backing Up as Formative Assessment

This use of student work as formative assessment and as a driving force for instruction supports both the Teaching and Learning Principle and the Assessment Principle described in *Principles to Actions: Ensuring Mathematical Success for All* (NCTM 2014). Assessing student work in this fashion allowed project participants to see the reality of students' understandings of fraction-related concepts and to think about instructional strategies that would support student learning within each category of student work (that is, exceeding lesson expectations, meeting lesson expectations, working toward lesson expectations and lacking fundamental understandings). We began by posing a problem beyond students' current knowledge that allowed for multiple solution methods, provided opportunities to connect to prior knowledge and promoted productive struggle. By doing so, we

> embrace[d] a view of students' struggles as opportunities for delving more deeply into understanding the mathematical structure of problems and relationships among mathematical ideas, instead of simply seeking correct solutions. (NCTM 2014, 48)

As project staff and participants considered students' work within each category, we acknowledged that the final goal of understanding for all students could be accomplished only through incremental movement. Determining the instruction and intervention needed to facilitate this movement is one of our primary roles as mathematics teachers. This assessment-driven process for making instructional decisions is crucial in advancing our students' understanding of fractions. By starting with a carefully crafted problem, we were able to identify student understandings and misconceptions and make instructional choices by which we could guide students to our goal.

Common Core Connections

3.NF.1

5.NF.7

6.NS.1

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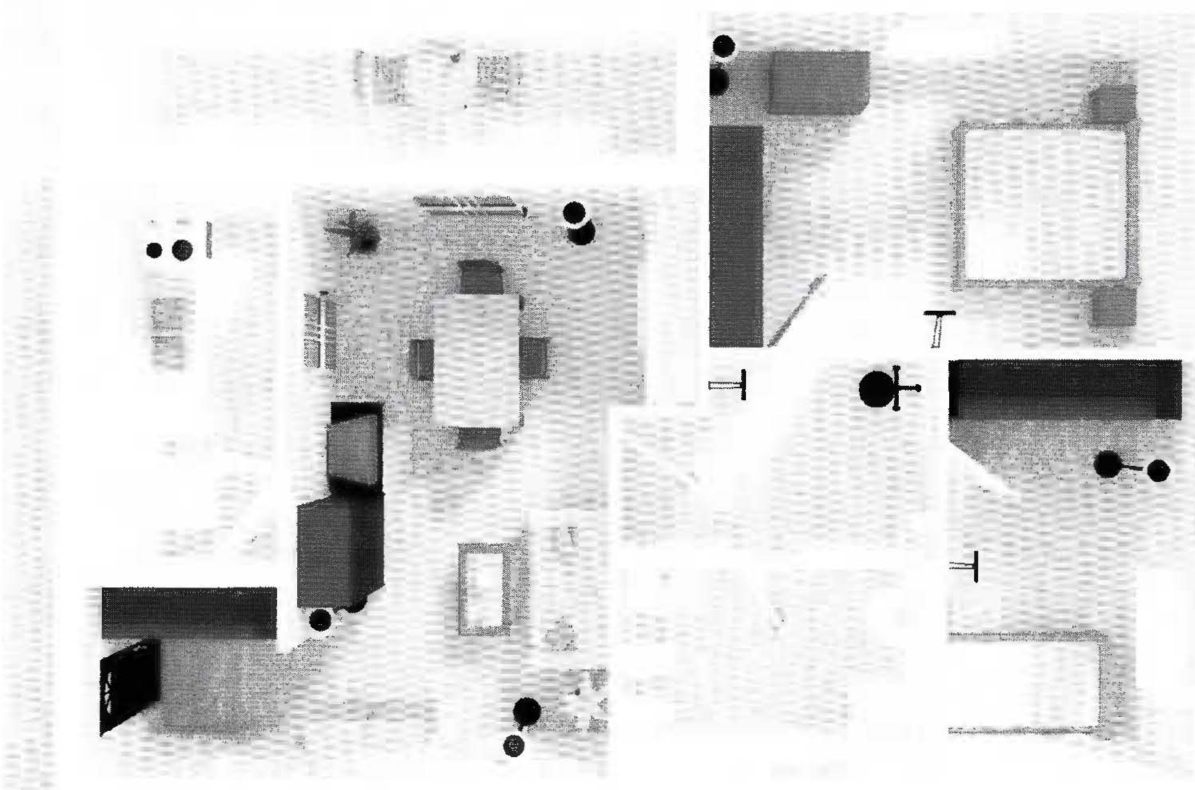
An Architecture Design Project: Building Understanding

Sarah B Bush, Judith Albanese, Karen S Karp and Matthew Karp

Seventh-grade students investigate area, surface area, volume, proportional thinking, number sense and technology.

Middle school students need relevant, meaningful contexts to apply emerging mathematical ideas. In this project, through the context of an architecture investigation, seventh-grade students engaged in mathematics involving area, surface area, volume, ratios and proportional thinking, number sense, and technology integration. Students, working in mixed-ability groups, were given an occupant scenario, which they used to build a home designed to meet the needs of their unique residents. After initial drawings of plans followed by critiques from a practising architect, they finalized designs and carried out mathematical tasks related to their plans. As a culminating event, student groups presented their home plans to local stakeholders, including peers, an architect who designed the school building, the district's mathematics curriculum specialist, and teachers from the school, who provided valuable feedback. Throughout the project, students completed a math log to record their mathematical thinking. Our project was tested in two seventh-grade classes taught by one of the authors.

This project aligns primarily to one cluster in the seventh-grade geometry domain of the Common Core State Standards for Mathematics (CCSSM), which is to "solve real-life and mathematical



problems involving angle measure, area, surface area, and volume" (CCSSI 2010, 7.G.6, 50). The appendix (online) describes specific alignment to both sixth- and seventhgrade content standards as well as connections to solving real-world and mathematical problems in ways that connect to two seventh-grade domains: ratios and proportional relationships, and the number system. Additionally, this project addresses two of the Common Core's eight Standards for Mathematical Practice (SMPs).

SMP 4, Model with mathematics, states that "Mathematically proficient students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace" (p 7). Students also accessed SMP 5, Use appropriate tools strategically, employing technology to build their architectural designs. Furthermore, NCTM's *Principles to Actions: Ensuring Mathematical Success for All* describes eight high-leverage Mathematics Teaching Practices that guide teachers to effectively implement instruction. This activity provides an example of Practice 1 and Practice 7:

- "Implement tasks that promote reasoning and problem solving." *As students engage in this project, which allows for a variety of solution strategies, they must reason mathematically.*
- "Support productive struggle in learning mathematics" (NCTM 2014, 10). *Students authentically wrestle with using design technology to create home plans that are responsive to the needs of occupants as well as conforming to established building constraints.*

Because this transdisciplinary project was an authentic convergence of design, art, aesthetics, engineering, community planning and mathematics, the teacher had to move between the realm of the mathematics and other subjects to truly address objectives from each field of study. Transdisciplinary teaching supports students in

> exploring content areas by foregrounding a problem or issue using multiple inquiry processes, which naturally connect the disciplines through the problem to be solved. (Herro and Quigley 2016, 2)

You will notice that as a truly integrated project, it is neither a mathematics project that touches on some small aspect of engineering nor an art project that touches on a trivial aspect of mathematics—it is a blend. Therefore, in some sections of the work described below, the focus will, for example, shift to design. We have found it important in our work that middle school students witness how learning can truly cross over into multiple disciplines, as this is what they will experience in the real world.

Figure 1. *Students had to work within these project constraints.*

Directions: The following is a list of constraints that your group needs to be aware of when designing your home. Your design must have the following:

- One story—no basement
- Between 2,000 square feet, ±20 per cent
- At least 15 feet from the road
- At least 15 feet from all property lines
- Front door
- Back door
- Kitchen
- At least 1 bathroom
- Use standard conventional dimensions for doorways, hallways, ceilings and so forth.
- A closet for each bedroom
- Do not consider electricity, heat and plumbing

Introduction and Brainstorming

On the first day of this exploration, students watched a video in which the architect on our author team provided home-design constraints (see Figure 1). Some elements of community planning and design were easily understood, but some required more detail, such as the description of public versus private space. To get students thinking, the architect asked, "Where would be the ideal location to position the bathroom in the home?" and "What distance from the front door of your house do you want to make your bedroom?" When considering the connections between interior and exterior spaces, he asked students to reason about "Which actively used rooms might have windows to look out into the neighbourhood and "What would you want people to see if they looked into your house from the street."

Next, students were placed in groups and given a unique occupant scenario card (see Figure 2) and quiet time to individually brainstorm and create initial conceptual plans for their homes (see Figure 3 for an example). At this point, students were not yet focused on the precise dimensions of each room. Day 1 concluded with students working in groups to discuss the responsiveness of their individual sketches to the hypothetical residents' needs and to work on questions in their math logs (see Figure 4).

Figure 2. *Each student group received a unique occupant scenario card.*

Scenario A: Your challenge is to design the ideal space for a family of four. This family includes a mom and dad in their 40s, a daughter age 7, and a son age 9. They also have a pet pig. Mom likes to do yoga, dad likes watching sports, the daughter wants to be a scientist and the son loves to play basketball. Their pet pig needs a place to stay cozy outside, but the family would also like a designated space in the house where the pig could stay.

Scenario B: Your challenge is to design the ideal space for a newlywed couple. They have two cats and a cockatiel. She needs an office space in the house, and he wants a man cave. He also has a motorcycle.

Scenario C: Your challenge is to design the ideal space for three elderly sisters. One sister has a walker, and one sister loves to cook. They all think they are the "ruler of the house" and deserve the biggest space.

Scenario D: Your challenge is to design the ideal space for three college students—two males and a female. The female wants her privacy. All three of them are avid road bikers and have a combined five bikes and accessories.

Scenario E: Your challenge is to design the ideal space for a family of three, soon to be four. The couple already has a four-year-old boy and just found out they are expecting another boy. They are very musically inclined, and they want the four-year-old to learn how to play the piano.

Scenario F: Your challenge is to design the ideal space for a couple who have grown children. Although they have been empty nesters for the last five years, they recently found out that their daughter and granddaughter, age 13, are moving back in. The teenager is not excited about moving in with her grandparents.

Scenario G: Your challenge is to design the ideal space for two brothers. One plays the drums, and the other is an exercise enthusiast and has lots of equipment, including weights. Both work out of the home with technology jobs that require workspace and the ability to e-conference.

Scenario H: Your challenge is to design the ideal space for a young couple with newborn twin girls. Both parents work long hours, so the husband's mom is moving in with them to help out. The couple wants to make sure the husband's mom has a small kitchenette in or near her bedroom.

Scenario I: Your challenge is to design the ideal space for a single man.

In question 1, students tapped into empathy in considering the occupants' needs. Two student responses highlight how the scenarios played a critical role in their design decisions for the proposed residents:

> Our occupants had 5 bikes, so we knew that we'd need a garage. Also, since the people are college students, we inferred they would require study space. The girl got her own room for privacy, and we had to incorporate a large living room for parties. Since there are 3 people, they would probably need a laundry room for all their clothes.

> The occupant of the house needs includes the man that lives there who is handicapped and enjoys gardening. This scenario influences our design choices because we can't use stairs and there has to be a big backyard.

Question 2 required students to explore the mathematics as they thought through such project constraints (see Figure 1) as room dimensions and square footage. Students were challenged by how to handle the "extra inches" in the calculations of square feet. When a measure was 18 ft 10 in by 23 ft 11 in, they realized after discussion that finding the area was made easier by converting those measures to decimals. Students used the Internet to find standard conventions for such dimensions as length and width of doors, width of a hallway and area of a laundry room.

Creating SketchUp Designs

For two days, student groups fine-tuned their original paper-and-pencil designs using SketchUp (2016) software to simulate the authentic work of an architect. Students reviewed several tutorials about SketchUp to help them understand how to effectively use such features as the dimension tool (find tutorials under the SketchUp website's Learn tab). Groups' SketchUp designs were based on the best ideas from each individual sketch. At the end of these two days, student group designs were sent electronically to the architect, who then provided written feedback to strengthen their final home designs (see Figure 5). SketchUp is free software that can be downloaded onto either Windows or Mac. Teachers may also wish to explore other free software, including GeoGebra or Tinkercad design software. Although creating the group designs by using software that helps students visualize the house three dimensionally has many advantages, this project can be completed effectively with group designs constructed using paper and pencil.

Although creating SketchUp designs were a key part of this project, the mathematics could get lost without explicit attention to focusing on students' thinking. Therefore, students were also responsible for completing questions 3–6 in their math logs. Question 3 sparked the most interesting discussions. Some groups had difficulty recalling the meaning of surface area, which provided an opportunity to review this key concept. For some groups, we brought out a three-dimensional (3-D) solid of a rectangular prism and asked students to imagine it as a bedroom: "What parts of this figure would we need to paint if we were painting the walls and ceiling?" and "How could you determine how much paint you need?" Although some students immediately wanted to use the traditional algorithm for finding surface area, instead, we asked students to consider "What makes sense?" as highlighted in the following typical discussion:

Teacher: What do you really need to paint in this bedroom?

Student 1: Four walls and the ceiling.

Teacher: How is this different from finding the overall surface area of a 3-D solid?

Student 2: We don't need to paint the floor; we only need to paint five sides total.

Teacher: Is there anything else we need to account for?

Student 3: The door and windows.

Teacher: Good thinking; how could your group account for the fact that you aren't painting the door or windows?

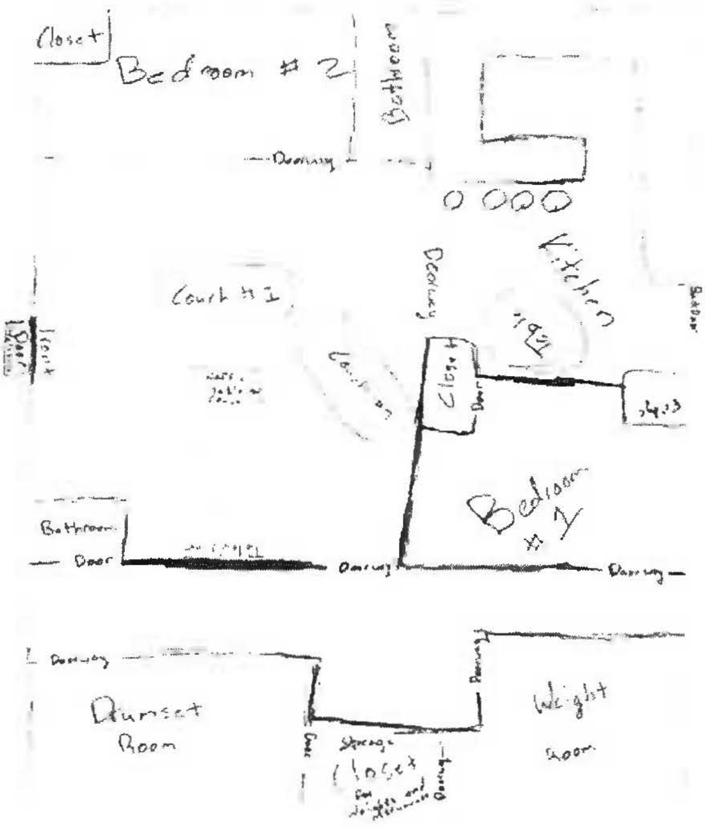


Figure 3. *This sample student plan involved scenario G.*

Student 1: We could find the area of this wall (the one across from the wall with the door) and then subtract the area of the door.

Some students also started to confuse surface area and square footage. While calculating the surface area of the bedrooms, students referred to the original list of constraints and started to panic, thinking that because their surface area was more than 2,000 square feet that they had exceeded the constraint of 2,000 square feet ± 20 per cent for the house. Again, using the 3-D solids as well as the classroom space as examples, we held discussions about the difference between surface area (such as in question 3) and square footage of a room or entire home.

As students continued to test and retest whether their overall home design was between 2,000 square feet ± 20 per cent, we found it interesting how students made sense of ± 20 per cent. The conversation below displays evidence of students' mathematical sense making:

Student 1: What is "± 20 per cent"?

Figure 4. *Students worked on their Math Logs throughout the project.*

1. Describe your consideration of occupant needs. How did your scenario card influence your design decisions?
2. What will be the dimensions of your actual house? The total square footage?
3. Suppose you wanted to paint the walls and ceiling of all the bedrooms. What is the surface area of these spaces? Explain your thinking.
4. You may consider getting air conditioning and base the size of the air conditioner on the amount of space it must cool. What is the volume of each room in your house?
5. Formulate a rationale on how and why your home fits the needs of your occupants. What particular features did you include as a response to your scenario card?
6. As you work on your prototype in SketchUp, how did you use the SketchUp tools? Describe your thinking using mathematical words, drawings, and symbols.
7. Who will be responsible for each part of the presentation? What questions should you be prepared to answer (e.g., consider your audience: the architect, the principal, and so on)?
8. During your presentation, how will you explain to the stakeholders the important mathematics related to your design?
9. How did the feedback from the architect change your thinking about your design? Be specific.
10. Architects often consider the surface area to volume ratio of a house using the surface area of the home exterior and the volume of the entire house. What is this ratio for your group's house? Show all work for finding both the exterior surface area and volume, as well as the ratio.
11. What ideas did you gain from being critiqued by the stakeholders and fellow classmates?
12. Describe the challenges you faced adhering to the constraints of the project.
13. What essential mathematics must architects know to do their job?
14. If you were hiring an architect to design your house, what mathematics questions would you ask to determine if he or she was qualified for the job?

Teacher: It means it is acceptable to have 20 per cent more or 20 per cent less than the 2,000 square feet.

Student 1: How would I know how much that is?

Teacher: Good question. How would you figure that out?

Student 2: Could we try 2,000 multiplied by 20/100?

Student 3: Oh, that would be 400 because 4 + 4 + 4 + 4 + 4 = 20, so 20 per cent of 2,000 would be 400.

Teacher: Interesting; so what range of square feet could you have?

Student 3: 2,400.

Teacher: I agree that is the max.

Student 1: Oh, so you could have anywhere from 1,600 to 2,400 square feet.

Teacher: Let's go back to the idea of 4 + 4 + 4 + 4 + 4 = 20, so 20 per cent of 2,000 would be 400." Can you say more about your thinking here?

Student 3: Out of 100 per cent, which is five 20 per cents, so I knew out of 20, there are five 4s. The 4s and the 20s would be the same thing as the 400 and 2,000. So 400 + 400 + 400 + 400 + 400 = 2,000.

Teacher: What do you call a relationship that is not the same but that has the same scale (trying to link their thinking to proportional relationships).

Student 2: This is like simplifying a fraction.

Student 3: When you take the fraction

$$\frac{400}{2{,}000} = \frac{4}{20}$$

As students grappled with these various mathematical concepts, they were able to work through question 3 (see Figure 6). As they moved to the next question, students had less difficulty finding the volume. As students worked on question 5, they fine-tuned their previous ideas from question 1. On question 6, students described their selection and use of tools (see Figure 7), which connected mathematics, art and engineering design concepts.

Preparing for Project Culmination

Students spent two days completing these tasks: addressing architect feedback to finalize their home designs, responding to questions in their math logs and creating their presentations. Architect feedback was in the form of general considerations for design, in every case resulting in students improving their home design through multiple iterations. For example, one group received feedback about their kitchen being only about six feet wide. After we prompted this group to get a yardstick and measure six feet, they quickly realized that this measurement "won't work because you need enough space for counters, an oven and to walk through past someone." Architect feedback also focused on design and proportional thinking (see Figure 8), causing students to reconsider and improve some of their previous responses in their math logs.

Once designs were finalized, groups worked to complete questions 7–14 in their math logs. Some questions were designed to help organize students' presentations, and other questions called for students to summarize changes made from the architect's feedback. Question 10 allowed us to formatively reassess students' understanding of surface area and volume; we found that as the week progressed, students had gained a better conceptual understanding of these ideas.

Finally, students created PowerPoint presentations guided by a template (see online) that included needed presentation components.

Figure 5. *The architect gave students feedback to strengthen their final home designs.*

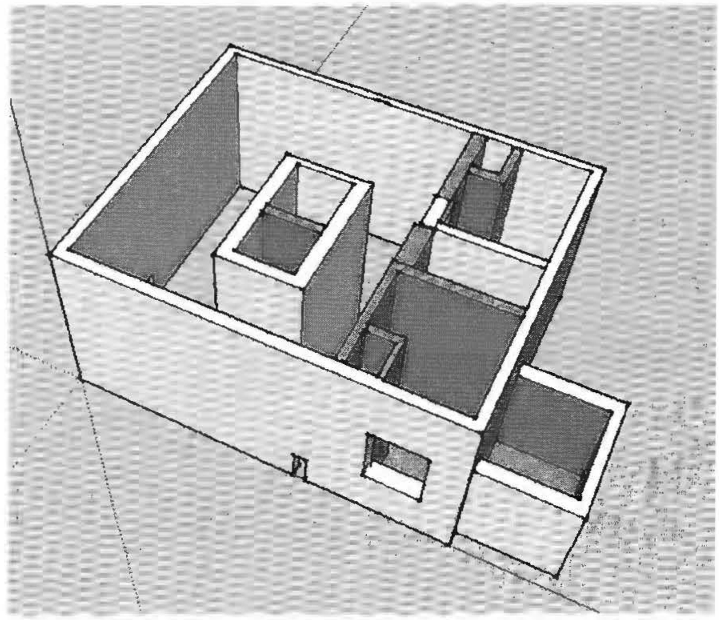
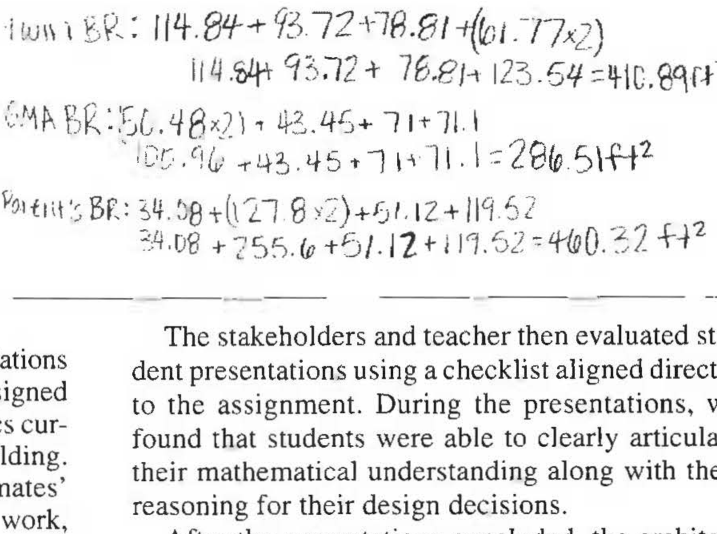


Figure 6. *The computation below is a sample student response to question 3.*



Presentation Day with Final Reflections

On presentation day, group members set up stations that were visited by peers, an architect who designed the school's building, the district's mathematics curriculum specialist and teachers from the building. Groups were also given time to view classmates' presentations. As they critiqued their peers' work, they completed feedback sheets that included "a plus and a wish." Student feedback showcased a new understanding of architecture and included these comments: "The kitchen is too long and skinny," "I would have the man cave separate from the living room," "They should have explained more how they found the dimensions, etc," and "More open space is needed from kitchen to living room."

The stakeholders and teacher then evaluated student presentations using a checklist aligned directly to the assignment. During the presentations, we found that students were able to clearly articulate their mathematical understanding along with their reasoning for their design decisions.

After the presentations concluded, the architect gave feedback to help improve the overall designs. He suggested that students could measure the dimensions of their own home so that they could better understand the typical ratio between the area of a kitchen and a living room. Another suggestion was to allow students to look at blueprints of a house to deepen their understanding of scale and proportions

Figure 7. *Question 6 answers connected mathematics, art and engineering.*

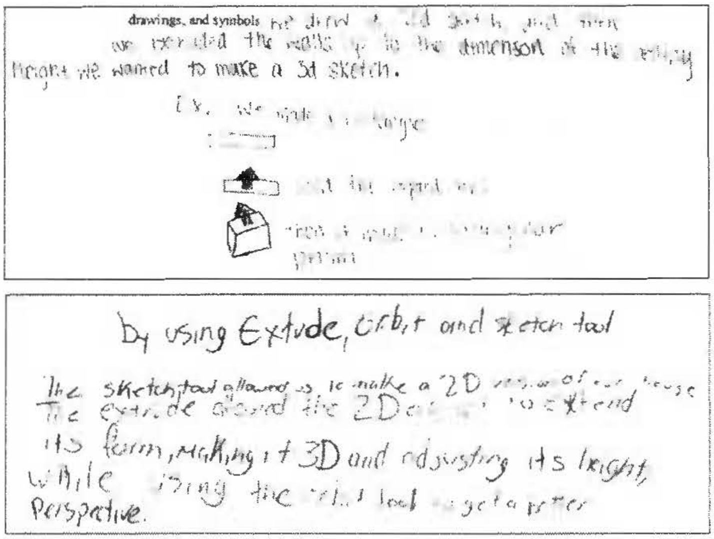
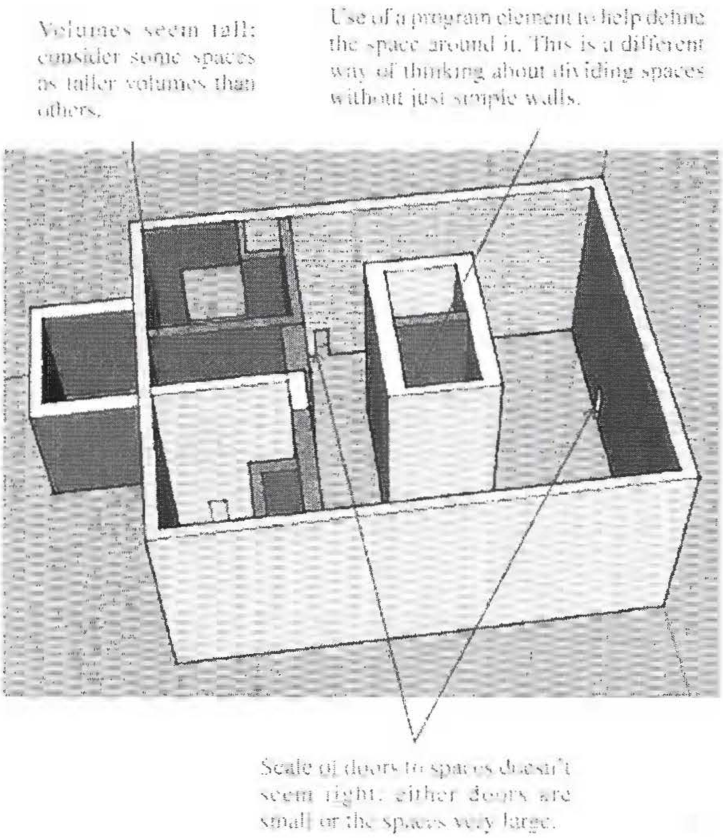


Figure 8. Architect feedback helped students improve previous math log responses.



before using SketchUp. Following the presentations, students reflected on what they had learned from this project (supported by Math Log questions 11–14). These culminating questions addressed several take-aways, including the advantages of continuously using feedback to make iterative improvements to one's design. To tie the work to the mathematics, students were asked to articulate what mathematics architects must know, questions 13 and 14 (see Table 1).

A Meaningful Context for Integrating Technology

This Architecture Design Project provided a meaningful context for working with area, surface area, volume, ratios and proportional thinking, number sense and the integration of technology. Students were motivated and engaged, and they greatly valued the video information and feedback from a practising architect. The practising architect on our author team emphasized to students the importance of being able to use SketchUp and other technology tools as an important skill for their future careers in the 21st century. In addition to the focus on mathematics, this transdisciplinary project incorporated key elements of engineering design, art and technology, and it offered an avenue for the classroom teacher to showcase the work of her students to multiple stakeholders. We are hopeful that reading about this project inspires other middle-grades teachers to explore architecture and integrate mathematics with other content areas in support of authentic mathematics applications in concert with individuals working in these professions.

Table 1. *Student responses to questions 13 and 14 focused on the mathematics that architects must know to do their jobs.*

13. What mathematics do architects need to know to do their job well?	14. What mathematics questions would I ask an architect to determine whether he or she should be hired for the job?
Ratio, area, surface area, volume, width, length, height, thickness (of walls), dimensions and so on.	• What are the dimensions of the house? • What is the volume of each room? • How would you scale the house in a model?
The essential mathematics architects must know how to do their job is how to find the square footage, volume and making the rooms the right size for its occupants.	• How do you find dimensions? • How do you find the surface area? • How tall do doors need to be?
• Square foot • Cubic foot • Conversion between measurements • Ratios	How big do you think the master bathroom should be in relation to the master bedroom?
They need to know standard dimensions and the height walls should be in relation to people.	What square footage and volume would meet the standards of the area (plot of land) the house will be built on?

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Instruction and Learning Through Formative Assessments

Teachers Can Use Rich Mathematical Tasks to Measure Students' Conceptual Understanding

Michael J Bossé, Kathleen Lynch-Davis, Kwaku Adu-Gyamfi and Kayla Chandler

Assessment and instruction are interwoven in mathematically rich formative assessment tasks, so employing these tasks in the classrooms is an exciting and time-efficient opportunity. To provide a window into how these tasks work in the classroom, this article analyzes summaries of student work on such a task and considers several students' solution strategies to exhibit the usefulness of these tasks in assessment, learning and teaching in the classroom. This article also provides some guidance on implementing these tasks in the classroom.

The literature is replete with descriptions, uses and effects of rich mathematical tasks. These tasks draw on students' prior understanding; create conceptual connections among mathematical ideas; provide students with the opportunity to engage in activities that require them to attend to precision, use tools appropriately, model with math and critique the reasoning of others; provide interwoven assessment and learning experiences; direct students' attention to precise mathematical concepts rather than skills; engage students to creatively investigate and communicate concepts; and provide teachers with opportunities to assess student understanding, misunderstandings and gaps in knowledge (Arbaugh and Brown 2005; Boesen, Lithner and Palm 2010; CCSSI 2010; Henningsen and Stein 1997; Herbst 2003; Smith and Stein 1998).

It is commonly recognized that formative assessments provide opportunities for teachers to assess student understanding through "evidence of students' reasoning and misconceptions to use in adjusting instruction" (NCTM 2013, para 1). However, through well-designed formative assessment tasks, students can also learn the mathematics inherent in the task. Thus, formative assessments through mathematically rich tasks can have multifold effects of assessing student understanding and misunderstandings and discovering gaps in student understanding; providing information through which teachers can adjust instruction; offering student feedback to support their own learning; and being an engaging task through which the mathematics at hand can be encountered and learned (Black and Wiliam 2009; Clark 2011; Hobson 1997; Long, Clark and Corchran 2,000; NCTM 2013; Pryor and Crossouard 2008).

In concert, rich mathematical formative assessments possess a number of recognizable characteristics. They address conceptual understanding of precise mathematical concepts recognizable by both the teacher and the student; assess student understanding of particular mathematical concepts and also serve as springboards through which the associated concepts can be investigated and learned; can be generated to address any grade-appropriate mathematical concept; can be differentiated quite easily to address students of differing ability levels; often address Krutetskii's (1976) three processes of reversibility, flexibility and generalizability; and are solvable through multiple heuristics.

A Sample Task and Classroom Context

The task shown in Figure 1 was designed to pinpoint student conceptual understandings and misunderstandings regarding constructing and comparing function models (CCSS.Math.Content.HSF.LE.A.2) (CCSSI 2010). While seemingly straightforward and unambiguous, this rich task encompasses numerous notions associated with the concept of polynomials, including the definition of polynomial functions and their continuous nature; the role of the leading coefficient and the degree of a polynomial on the graph's extreme behaviours; the definitions of factors, linear factors and a factored polynomial; the graphical effects of roots of odd and even multiplicity; and the association of zeros, roots, factors and x-intercepts between the polynomial function and its graph.

Beginning with concepts from introductory algebra, this task intersects precalculus through the generalized solution

$$y = K(x - a)^{odd}(x - b)^{even}(x - c)^{odd}(x - d)^{even}$$

where $K \in R^+$ and $a, b, c, d \in R$.

This specific task addressed three of the basic processes identified in Krutetskii's (1976) model of mathematical abilities (that is, reversibility, flexibility and generalizability). It required that students reverse their thinking about polynomials and factoring in a direction counter to what they typically experienced during instruction; flexibly solve a problem in more than one way and understand more than one solution; and generalize from specific cases to make deductions from given or known facts.

Below is a truncated graph of a polynomial (All the behaviour near the x-axis is shown.) There is no scale for the y-axis. Write the equation of a polynomial function that would produce this truncated graph.

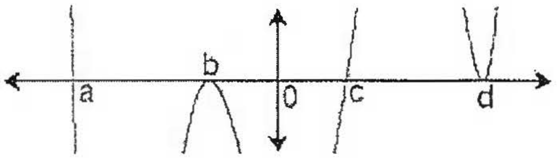


Figure 1. *This is an example of a mathematically rich formative assessment task.*

This mathematically rich formative assessment task was selected for a number of reasons. All students were from the same high school class under the same teacher and had previously investigated polynomial functions and graphical and algebraic representations in their high school precalculus class. They had all experienced identical content, instructional practices and extended problem-solving challenges. The task served as a means through which student knowledge, gaps and misunderstandings could be observed. The classroom teacher assessed the task as both challenging to most students and doable by all.

All students were given up to two hours to complete the three tasks; most took less time (10 to 90 minutes), as they were either able to solve the problem quickly or struggled to persevere through the problem-solving process. Students were primarily left alone to demonstrate what they knew and to learn through the activity while the researchers observed student work and assessed student understanding. The findings and summaries are addressed below in two parts: assessment and learning.

Assessment

The following are syntheses of narrative accounts of students' activity as they worked independently on the task. These summaries abbreviate much fuller transcripts; omitted materials were deemed as not furthering the findings. (See Bossé, Adu-Gyamfi and Chandler [2014] for a more detailed description of the associated study.)

Student 1 holds a course grade of C. The teacher believes that he will be able to do the task, albeit with a struggle. Trying to create a correct graph, Student 1 unsuccessfully uses trial and error, entering values and polynomial functions into the calculator. He does not know what "truncated graph" means and struggles, unnecessarily, to predict the behaviour of the graph above and below the x-axis. He claims that polynomials are in the form $x^2 + 2x + 3$ and does not understand "polynomial in factored form." Through trial and error, he unsuccessfully plugs numbers in for a, b, c and d into

$$y = ax^3 + bx^2 + cx + d.$$

He claims that a graph is the answer to a problem, not the beginning point. After he is shown (x – a), (x – b), (x – c) and (x – d) as factors, he is unsure how these are connected to the graph. When he is told that

$$y = (x - a)(x - b)^2(x - c)(x - d)^2$$

represents a possible solution, he tries to rewrite it in the form

$$px^n + qx^{n-1} + ... + rx + s$$

and shows no understanding that the leading coefficient has to be positive. Throughout, he is continually frustrated.

Although the teacher expected him to struggle some, she expected him to do better. She was surprised that he struggled with the vocabulary, linear factors and polynomials and that she had not seen this before.

The remainder of the work of Student 1 (beyond the summary provided) demonstrates that he perceives the polynomial function and graph as mostly disjointed and unconnected. He does not recognize zeros on the graph and does not understand the corresponding $(x - \_)$ binomial in the factored form of the polynomial or consider the far-left and far-right behaviour of the graph in respect to either the degree of the polynomial or the sign of the leading coefficient. He recognizes "polynomial" only in the form $y = px^3 + qx^2 + rx + s$. Altogether, this student has significant gaps in his knowledge that were revealed to the teacher through the implementation of this task. The teacher recognizes that much effort will be needed to bring him to satisfactory understanding and that most concepts will need to be readdressed in novel ways.

Student 2 holds a course grade of C. The teacher believes that this student will be quite successful with the task. Student 2 writes down expressions

$$(x + 2)(x + 1^2)(x - 1)(x - 2^2),$$
$$(x + 2)(x + 1)^2(x - 1)(x - 2)^2,$$
$$\text{and } -3x^3 + -2x^2 + 2x + 3,$$

superficially analyzes them, and then attempts to graph the function. She struggles as to whether a and b should be represented by $-a$ and $-b$. She repeatedly attempts to graph polynomials entered in general form and some in factored form. She recognizes that the roots are squared at a and c, but does not know how to represent that condition in factored form. She tries values for a, b, c and d in polynomials of the form $ax^3 + bx^2 + cx + d$. She remembers that a, b, c and d must be inside parentheses, but does not remember how to do this. She claims her confusion is because they are variable and not numbers. She struggles to determine if the linear factors should be $(x - a^2)$ or $(x - a)^2$ and decides on the example

$$(x + 2)(x + 1^2)(x - 1)(x - 2^2).$$

Her continued investigation (with numerous brief pauses) is full of inquisitiveness and problem solving, without any semblance of frustration.

The teacher is relatively pleased with the student's work but is surprised by her lack of understanding linear factors, positive and negative roots, and the position of the exponent.

Through this and additional work (beyond the transcripts provided), Student 2 recognizes a number of aspects of the graph itself, including the far-left and far-right behaviour of graphs of polynomial functions; the association of zeros, roots, and x-intercepts between the graph and the equation; and the nature and effects of roots of odd and even multiplicity. However, the specific nature of linear factors together with their multiplicities remains an obvious gap in her knowledge; she is unsure if the factors should be $(x - a) \cdot (x - b)$ or $(x + a) \cdot (x + b)$, and she is confused regarding whether the exponentiation should be inside or outside the parentheses. Notably, she attempts to map a, b, c and d from the graph to the equation without understanding the interconnection of zeros and intercepts on a graph and zeros and real roots of a function. While this student has significant gaps in her knowledge, they are less so than for Student 1, and the teacher comes to better understand precise concepts with which the student struggles. Now the teacher recognizes the particular concepts that need to be addressed to complete the student's understanding.

Student 3 has an A+ average in the course. The teacher expects that he will fully master all the concepts in these tasks. Almost immediately, Student 3 recognizes that the polynomial is of even degree (at least 6) with a positive leading coefficient. He claims that a, b, c and d represent the zeros of the function and writes

$$y = (x - a)(x - b)^2(x - c)(x - d)^2,$$

then rewrites the expression as

$$y = e\,(x - a)^{odd}(x - b)^{even}(x - c)^{odd}(x - d)^{even},$$

where $e > 0$.

Student 3 has a strong understanding of mathematical concepts embedded in this task. He fluently understands both representations and can communicate such without effort. The context of the problem immediately directs him to the structures that are most important in both representations. Through observing this student perform the task, the teacher recognizes that she has not sufficiently challenged the student in respect to his ability and current understanding. She decides to provide him additional mathematically rich tasks targeted to additional concepts.

Assessment Summary

As seen in some summaries, the teacher was surprised at the understanding, misunderstandings and knowledge gaps that she was able to observe through student work and communication on the task. Even though these students had passed her previous traditional assessments on this topic, she was surprised by the degree to which they struggled in general and on which concepts in particular. Specifically, she was pleased by the targeted way the task revealed individualized precise concept understanding among the students and prescribed similarly precise and differentiated instruction to help each and all be successful.

Learning

The following excerpt describes Student 2's progress.

Approximately 45 minutes later, Student 2 realizes that the polynomial has to be raised to an even power to produce the correct left and right behaviour, but she does not know how to use parentheses to accomplish this. She decides to graph

$$y = (x + -3)(x + -1)(x - 1)(x - 3)$$

and other such cases. Through protracted trials, she recognizes that

$$y = (x + 3)(x + 1)(x - 1)(x - 3)$$

implies

$$y = (x - a)(x - b)(x - c)(x - d).$$

After more investigation, she recognizes that the graph reveals some single and some double roots; struggles to know which are which; recognizes the need to distinguish these through $(x - b^2)$ or $(x - b)^2$; writes

$$(x + 2)(x + 1)^2(x - 1)(x - 2)^2;$$

and after more thought and experimentation, rewrites this into

$$(x - a)(x - b)^2(x - c)(x - d)^2$$

and finally to

$$(x - a)^{odd}(x - b)^{even}(x - c)^{odd}(x - d)^{even}.$$

The teacher is pleased that the student learned through only one task within one class period, since after days spent previously covering the associated mathematical topics in class the student had not gained sufficient understanding.

This student received no assistance from the teacher or the interviewer, but was given sufficient time to work through the investigation. Fortunately, since she had previously experienced time-intensive problem-solving tasks, she was able to persevere through this task. The progression from Student 2's previous transcripts to this transcript (over the total span of about 90 minutes) demonstrates a growth from misunderstandings and knowledge gaps to understanding most of the associated concepts. Moreover, the concepts learned are now strongly interconnected both within each representation and between the two representations, rather than being treated disjointedly. Altogether, the teacher was pleased at the rapidity, efficiency and thoroughness of the student's learning and credited this success to the nature of the mathematically rich task and the protracted time allowed for its investigation.

Learning Summary

While Student 1's extensive misunderstandings and knowledge gaps significantly slowed his learning of the concepts, extended transcripts reveal that he learned many of the mathematical concepts, but at a slower pace than Student 2. The teacher was pleased with the learning of Student 1, but she stated that she believed that if a simpler version of the task had been provided before this one, the students would probably have done better on both tasks. The teacher decided to create concept-similar tasks that would scaffold to this type of example for this student (for example, begin with quadratic functions). Additionally, the teacher decided that she would allow this student to work with another student on some future tasks to simultaneously scaffold his learning and diminish his frustration. Since Student 3 was already familiar with most of the mathematical content associated with the task, the transcripts show little gain in understanding. The teacher decided that she could create parallel tasks (using transcendental functions) to challenge this student and lead him to more advanced concepts.

Implications for Instruction

As demonstrated above, the mathematically rich formative assessment task served its dual purposes of assessing student understanding, misunderstandings and knowledge gaps while providing them with an effective learning experience. As students responded to the task, their understanding and connections of mathematical concepts deepened. Through these tasks, teachers can assess much more than whether or not students can answer questions or perform mathematical calculations; student conceptual understanding of numerous embedded notions can be assessed, and teachers can use that information to plan further instruction.

Students with greater gaps in understanding tend to learn much from rich mathematical tasks, albeit at a slower pace than others. Initially, they balk at these unusual tasks in which they are not given explicit direction on how to complete the task or what the correct response may be. However, as these tasks become more common, students will warm to them. For these students, it may be best to initially scaffold their experiences by using versions of tasks that are differentiated for their specific needs before employing more complex tasks. These students may need to complete a greater number of these tasks than may their classmates. Since these students are often more prone to be frustrated in problem solving and have difficulty persevering in such, care must be taken to not break their spirits. Thus, it is valuable to limit the duration of the tasks initially and increase the duration of tasks as is tolerable. Allowing students to work with others, rather than independently, may also help them avoid being overly frustrated.

Students who are comfortable with more advanced mathematics should be given tasks that also meet their needs. Most mathematically rich tasks are easily modified to be deeper and more challenging. These students often enjoy such tasks. Students can be given these tasks prior to instruction on particular topics; they can learn through these tasks, sometimes even independently of an instructor. Also, students can be asked to create and solve their own rich mathematical tasks, leading to tremendous learning experiences.

What to Expect in the Classroom

Mathematically rich formative assessment tasks may seem more difficult initially than traditional classroom instructional questions, particularly if they are seen as unusual or unfamiliar. These tasks address or assess

Selecting the Mathematical Tasks

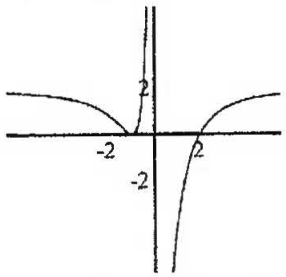
For any mathematical topic at any level, rich mathematics tasks are available. We provide additional examples applicable to high school. For each example, a variation differentiates the problem to be either more or less mathematically complex.

1. The following functions are equivalent, but in different algebraic forms. What information regarding the function is revealed or hidden in each of the forms?

$$f(x) = 2x^2 + 3x + 1$$
$$g(x) = x(2x + 3) + 1$$
$$h(x) = (2x + 1)(x + 1)$$

   To make the task simpler: Provide options such as showing the function is a quadratic; showing its factors; revealing its roots; revealing its y-intercept; showing that it is concave up.

2. Explain why the accompanying function and graph are inconsistent.



$$f(x) = \frac{(x-2)^2(x+1)}{x^2(x+1)}$$

   To make the task simpler: Use polynomial functions.

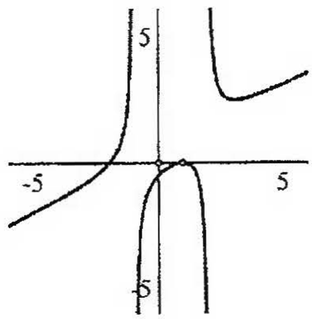
3. For $f(x) = 2x + 3$ and $g(x) = (x - 3)/2$, we find that $f(g(x)) = g(f(x))$. Is it usually the case that $f(g(x)) = g(f(x))$? Explain why or why not.

   To make the task more complex: Explain necessary conditions for $f(g(x)) = g(f(x))$ to be true.

4. For ___x + ___ = ___x +___, fill in the blanks such that the equation has one solution; no solution; an infinite set of solutions.

   To make the task more complex: Create an equation including a quadratic and a linear function.

5. Without converting the graph below to an equation, explain everything you can about the graph and its respective function.



   To make the task simpler: Use a polynomial function.

precise mathematical concepts and cause students to think more deeply about the mathematics at hand and the interconnectedness among mathematical concepts and representations. Although students may balk at these tasks at first, many students quickly come to enjoy the challenge and heartily participate in classroom discussions.

Classroom time must be planned for students to struggle with and learn through a mathematically rich task. Combining formative assessment and instruction focused on conceptual understanding can break the cycle of skills-based instruction, assessment, follow-up instruction and further assessment. Teachers must place some trust in students as learners and communicate high expectations to them. As students work through these tasks, their conceptual understanding can grow at an exceptional pace. When students show significant misunderstanding or knowledge gaps, teachers can intervene with instruction directed at particular concepts and scaffold understanding while not forfeiting time globally addressing concepts that students may have mastered.

There is a delicate balance between allowing students to persevere through the problem-solving task and providing them assistance before they become too frustrated and shut down. Classroom teachers must know their students well, adjust the task or the time allotted for the task appropriately for individual students and the class, and know when to intervene. We recommend that they allow learning to happen organically and not provide hints too quickly; jumping in to assist skews interpretations regarding what students know or learn.

Most of these tasks are excellent fodder for collaborative assessment and instruction. This practice elicits rich communication and dialogue among students, giving teachers greater access to student thinking and giving students access to greater learning. Teachers also enjoy students' robust mathematical dialogue.

The most obvious question for any teacher may now be, "But I have 30 students in my classroom! How can I possibly do this?" First, no one educational practice—even using mathematically rich tasks—is a panacea for all student learning. These tasks should supplement, and not completely replace, other instructional techniques. (See sidebar, Selecting the Mathematical Tasks.) Second, novel instructional techniques take time and practice to master. Third, when initially using these tasks, it may be beneficial to try them as either instructional or assessment tasks rather than integrating both.

We hope that this brief introduction to rich mathematical formative assessments will evoke interest in these tasks and encourage teachers to try them in their classrooms. The authors have used these tasks with great results. We hope others see their worth and enjoy them also.

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Alberta High School Mathematics Competition 2016/17

Part 1

1. If it is 10:00 AM on a Tuesday, which day would it be 2016 hours later?

(a) Tuesday (b) Wednesday (c) Thursday (d) Friday (e) Saturday

Solution:
Since $2016 = 32 \times 9 \times 7 = 12 \times 7 \times 24$, the answer is 10:00 AM on a Tuesday (12 weeks from the original day). The answer is (a).

2. If $x > 0$, $x \neq 1$, and $(\log_2 x)^2 = \log_4 x$, then:

(a) $0 < x < 1$ (b) $1 < x < 2$ (c) $2 \leq x < 4$ (d) $4 \leq x < \infty$ (e) the situation is impossible

Solution:
The equation can be written as $(\log_2 x)^2 = \frac{1}{2}\log_2 x$, and since $x \neq 1$, this equation is equivalent to $\log_2 x = \frac{1}{2}$ with the solution $x = \sqrt{2} \in (1,2)$. The answer is (b).

3. A ring of 10 grams is 60% gold and 40% silver. A jeweller wants to melt it down, add 2 grams of silver and add enough gold to make it 70% gold. How many grams of gold should be added?

(a) 4 (b) 5 (c) 8 (d) 9 (e) more than 9

Solution:
Let x be the grams of gold which should be added. Then

$$\frac{7}{10} = \frac{10 \times \frac{6}{10} + x}{10 + 2 + x}$$

Solving the equation one obtains $x = 8$.

Alternative solution: The original ring contains 4 g of silver and 6 g of gold. The new ring will contain $4 + 2 = 6$ g of silver, which must account for 30% of the total. Thus the new ring must weigh $6 \times \frac{10}{3} = 20$ g, of which therefore 20-10-2=8 g must be added gold. The answer is (c).

4. A quadratic polynomial $f(x) = ax^2 + bx + c$, where a, b and c are integers, satisfies $f(2) = 4$ and $f(3) = 9$. The number of such polynomials is:

(a) 0 (b) 1 (c) 2 (d) 3 (e) more than 3

Solution:
Using the given condition we get $4a + 2b + c = 4$, $9a + 3b + c = 9$ from which, solving for b, c in terms of a, one obtains $b = 5(1 - a)$, $c = 6(a - 1)$, that is, infinitely many integer solutions. The answer is (e).

5. For any integer n, the expression $n^2 + 3n + 2$ cannot assume the value

(a) 0 (b) 2 (c) 110 (d) 375 (e) 420

Solution:
Since $n^2 + 3n + 2 = (n + 1)(n + 2)$ it must be even. Thus, 375 is not attainable. The other four numbers in the list can be attained using $n = -1$, $n = 0$, $n = 9$ and $n = 19$, respectively. The answer is (d).

6. Two straight lines with nonzero x and y-intercepts have the following property: the x-intercept of the first line equals the y-intercept of the second line, and the x-intercept of the second line equals the y-intercept of the first line. If the slope of the first line is m, then the slope of the second line is

(a) m (b) $-m$ (c) $\frac{1}{m}$ (d) $-\frac{1}{m}$ (e) none of these

Solution:
It is given that if $(a, 0)$ and $(0, b)$ lie on the first line then $(0, a)$ and $(b, 0)$ lie on the second line. The slopes of the lines are then $m = -b/a$ and $-a/b = 1/m$, respectively. The answer is (c).

7. The angles of a triangle when measured in degrees are all prime numbers. The smallest possible size of the largest angle is:

(a) 61° (b) 67° (c) 79° (d) 89° (e) the situation is impossible

Solution:
Let $A \leq B \leq C$ be the measures in degrees of the angles of $\triangle ABC$. Since $A + B + C = 180°$ one of the angles should be even and hence $A = 2°$. On the other hand, $178° = B + C \leq 2C$, hence $C \geq 89°$. Since 89 is prime, we can take $B = C = 89°$, $A = 2°$. The answer is (d).

8. How many three-digit numbers can be written after 523 to yield a six-digit number which is divisible by each of 7, 8 and 9?

(a) 0 (b) 1 (c) 2 (d) 3 (e) 4

Solution:
Since 7, 8 and 9 are pairwise relatively prime, the six-digit number must be a multiple of $7 \times 8 \times 9 = 504$. When 523999 is divided by 504, the remainder is 343. To get a multiple of 504, we subtract 343 from 999 to obtain 656. This is one of the answers, and we have $523656 = 7 \times 8 \times 9 \times 1039$. We can get another answer by subtracting 504 from 656 to obtain 152, and we have $523152 = 7 \times 8 \times 9 \times 1038$. This subtraction cannot be repeated without reducing the difference below 523000. Hence 656 and 152 are the only possible answers.

Alternative Solution: The six digit number should have the form $504n$ where n is a positive integer. The conditions of the problem lead to the inequality $523100 \leq 504n \leq 523999$ or equivalently $1038 \leq n \leq 1039$, and hence $n = 1038$ or $n = 1039$. With these two values of n, one obtains two three-digit numbers having the requested properties. The answer is (c).

9. When $x^4+x^5+x^{10}+x^{17}+x^{100}$ is divided by x^2-1 the remainder is

(a) $x+2$ (b) $x+3$ (c) $2x+3$ (d) $3x+1$ (e) $3x+2$

Solution:
The remainder must be a first degree polynomial $Ax+B$ and if $Q(x)$ is the quotient then

$$x^4+x^5+x^{10}+x^{17}+x^{100}=Q(x)(x^2-1)+Ax+B$$

for any real x. Taking $x=\pm 1$ in the above equation we obtain $A+B=5$ and $-A+B=1$ hence $A=2, B=3$ and thus the remainder is $2x+3$. The answer is (c).

10. Let D be an arbitrary point on the side BC of the equilateral triangle ABC. Points E and F are on AB and AC respectively so that $DE\perp AB$ and $DF\perp AC$ and E_1, F_1 are points on BC such that $EE_1\perp BC$ and $FF_1\perp BC$. If E_1F_1 has length $\frac{1}{2}$ then the length of BC is

(a) $\frac{2}{3}$ (b) $\frac{4}{5}$ (c) 1 (d) $\frac{3}{2}$ (e) $\sqrt{3}$

Solution:
Let $BC=a$. We have $DE_1=DE\cos 30^\circ=\frac{\sqrt{3}}{2}DE$ and $DF_1=DF\cos 30^\circ=\frac{\sqrt{3}}{2}DF$, hence $E_1F_1=\frac{\sqrt{3}}{2}(DE+DF)$. On the other hand $DE\cdot AB+DF\cdot AC=2\text{Area}(ABC)$ that is, $a(DE+DF)=\frac{a^2\sqrt{3}}{2}$ and hence $DE+DF=\frac{a\sqrt{3}}{2}$. Therefore, $E_1F_1=\frac{3a}{4}$. Since $E_1F_1=\frac{1}{2}$ we must have $a=\frac{2}{3}$. The answer is (a).

11. A box contains two red balls, two green balls and two yellow balls. If you randomly remove three balls from the box without replacement, what is the probability that you have removed one of each colour?

(a) $\frac{1}{8}$ (b) $\frac{2}{5}$ (c) $\frac{1}{2}$ (d) $\frac{4}{5}$ (e) none of these

Solution:
There are a total of $\binom{6}{3}=20$ possibilities and only 8 are favourable. The requested probability is $\frac{8}{20}=\frac{2}{5}$. The answer is (b).

12. Let $f:\mathbb{R}\to\mathbb{R}$ be a function such $xf(x)+(1-x)f(-x)=x^2+x+1$ for any real number x. The greatest real number M for which $f(x)\geq M$ for all real numbers x, is

(a) $\frac{3}{4}$ (b) $\frac{5}{6}$ (c) $\frac{7}{8}$ (d) $\frac{9}{10}$ (e) $\frac{11}{12}$

Solution:
If x is replaced by $-x$ in the given equation then

$$-xf(-x)+(1+x)f(x)=x^2-x+1.$$

Using the given equation and the one that is obtained above, one obtains by subtraction that $f(-x)=f(x)+2x$, so $xf(x)+(1-x)(f(x)+2x)=x^2+x+1$ and thus

$$f(x)=x^2+x+1-2x(1-x)=3x^2-x+1=3\left(x-\frac{1}{6}\right)^2+\frac{11}{12}$$

and hence $f(x)\geq\frac{11}{12}$. If $x=\frac{1}{6}$ then we get $f\left(\frac{1}{6}\right)=\frac{11}{12}$. The answer is (e).

13. In a school's math club, the number of different 3-person committees that could be formed containing two girls and one boy is 2016 more than the number of different 3- person committees containing two boys and one girl. The number of girls in the club is:

(a) 1 (b) 63 (c) 64 (d) 2016 (e) not uniquely determined

Solution:
Let m, n denote the number of girls and respectively boys in the club. The condition of the problem is

$$n\binom{m}{2} - m\binom{n}{2} = 2016 \iff mn(m-n) = 2 \times 2016 = 63 \times 64 = 2^6 \times 3^2 \times 7$$

Let $k = (m, n)$ be the greatest common divisor of m, n. Since k is a divisor of $m, n, m-n$, then k^3 will be a divisor of $2^6 \times 3^2 \times 7 = mn(m-n)$ hence $k \in \{1, 2, 4\}$. If $k = 1$ the only convenient solutions are $m = 64, n = 1$ and $m = 64, n = 63$, otherwise $m > 64$, which is not possible. If $k = 2$ then $m = 2m_1, n = 2n_1$ with $(m_1, n_1) = 1$, and $m_1 n_1 (m_1 - n_1) = 8 \times 9 \times 7$. No convenient integer values for m_1, n_1 can be found in this case. Similarly, if $k = 4$ then $m = 4m_2, n = 4n_2$ with $(m_2, n_2) = 1$, and the equation can be simplified to $m_2 n_2 (m_2 - n_2) = 9 \times 7$, which does not have integer solutions.

Alternative Solution: The solutions of the equation $mn(m-n) = 2^6 \times 3^2 \times 7$ can be found using another approach. First one can remark by AGM inequality that

$$4032 = mn(m-n) \leq m\left(\frac{n+m-n}{2}\right)^2 = \frac{m^3}{4}$$

hence $m^3 \geq 16128$ and thus $m \geq 26$. On the other hand if $m > 64$ then

$$64 \cdot 63 = mn(m-n) > 64n(64-n))$$

hence

$$63 > n(64-n) \iff (n-1)(n-63) > 0$$

and thus $n > 63$, which is not possible since $mn(m-n) = 63 \cdot 64$.

We conclude that $m \in \{28, 32, 36, 42, 48, 56, 63, 64\}$. The only convenient value is $m = 64$ which gives the equation $n(64 - n) = 63$ and hence $n = 1, n = 63$.

The answer is (c).

14. The product of all real numbers x that are solutions of the equation $\sqrt[3]{x^2 + x + \sqrt[3]{x^2 + x + 3}} = 3$ is

(a) −26 (b) −24 (c) 4 (d) 20 (e) 26

Solution:
The function $f(t) = \sqrt[3]{x^2 + x + t}, t \in \mathbb{R}$ is increasing hence $f(f(t)) = t \iff f(t) = t$ that leads to the conclusion that the real solutions of the given equation and $\sqrt[3]{x^2 + x + 3} = 3$ are the same. The solutions of the last equation are just the solutions of the quadratic equation $x^2 + x - 24 = 0$, for which the product of the solutions is -24.

Alternative Solution: Let $y = \sqrt[3]{x^2 + x + 3}$ then the given equation can be written as

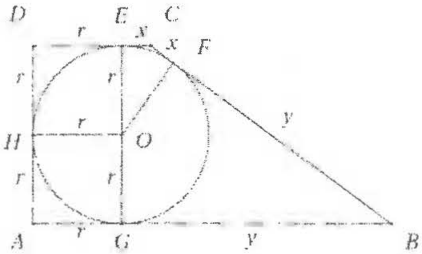
$$y^3 - 3 + y = 27 \iff (y-3)(y^2 + 3y + 10) = 0 \iff y = 3$$

and then $x^2 + x - 24 = 0$. The answer is (b).

15. The area of the trapezoid $ABCD$ with $AB \parallel CD$, $AD \perp AB$ and $AB = 3CD$ is equal to 4 . A circle inside the trapezoid is tangent to all of its sides. The radius of the circle is

(a) $\frac{\sqrt{3}}{5}$ (b) $\frac{\sqrt{3}}{4}$ (c) $\frac{\sqrt{3}}{3}$ (d) $\frac{\sqrt{3}}{2}$ (e) none of these

Solution:



Using the notations from the above diagram and the conditions from the problem one obtains:

$$3(r+x) = r + y \iff y = 3x + 2r$$

and

$$(x+y)^2 = 4r^2 + \left((r+y) - (r+x)\right)^2 \iff xy = r^2.$$

Hence

$$x(3x + 2r) = r^2 \iff x = \frac{r}{3}$$

and consequently $y = 3r$. On the other hand the area of the trapezoid $ABCD$ is 4, thus

$$(r + x + r + y)r = 4.$$

Substituting for $x = \frac{r}{3}$ and $y = 3r$ we get $r = \frac{\sqrt{3}}{2}$. The answer is (d).

16. A quadrilateral is called convex if its diagonals intersect inside the quadrilateral. A convex quadrilateral has side lengths 3, 3, 4, 4 not necessarily in this order, and its area is a positive integer. The number of non-congruent convex quadrilaterals having these properties is:

(a) 12 (b) 24 (c) 28 (d) 35 (e) none of these

Solution:

(i) Assume that the sides of the quadrilateral are of lengths 3,4,3,4, in this order. The quadrilateral is a parallelogram (so it is convex) . Let $\alpha \in (0, \frac{\pi}{2}]$ be the measure in radians of an angle of the parallelogram and S its area. Then we have

$$S = 12 \sin\alpha \iff \sin\alpha = \frac{S}{12}.$$

Since $0 < \sin\alpha \leq 1$ there are twelve convenient integral values of S, namely $1, 2, \cdots 12$, and thus twelve distinct values of α. Therefore we get twelve non-congruent parallelograms with area a positive integer.

(ii) Assume that the sides of the quadrilateral are of lengths 4,4,3,3, in this order. Let α be the measure in radians of the angle between two sides of lengths 4 and 3. The quadrilateral is convex if $\alpha \in (\beta, \pi)$ with $\sin\beta = \frac{\sqrt{7}}{4}$. Note the the lower limit β for the angle α occurs when the sides of length 3 are both on the same line and hence, the quadrilateral degenerates to an isosceles triangle of sides 4,4,6.
As above, one obtains $\sin\alpha = \frac{S}{12}$. If $\alpha \in [\frac{\pi}{2}, \pi)$ there are 12 integer values for S for which we get twelve distinct values for α and hence one obtains twelve non-congruent quadrilaterals. If $\alpha \in (\beta, \frac{\pi}{2})$ then $\sin\beta < \sin\alpha < 1 \iff \frac{\sqrt{7}}{4} < \sin\alpha < 1 \iff \sqrt{63} < S < 12$. There are four integer values for S in $(\sqrt{63}, 12)$ and thus four distinct values for $\alpha \in (\beta, \frac{\pi}{2})$ such that $\sin\alpha = \frac{S}{12}$ for which one obtains four non-congruent convex quadrilaterals. The number of requested convex quadrilaterals is 12+12+4=28.

Alternative Approach: The largest parallelogram of sides in the order 3,4,3,4 is clearly the rectangle (as it has largest altitude) of area 12, and so other parallelograms in this family can have areas 1 to 11. Similarly the quadrilaterals with sides in the order 4,4,3,3 are all composed of two congruent triangles with two of the sides being 4 and 3, with area at most 6, so the largest such quadrilateral will also have area 12. As the angle between the sides 4 and 3 becomes greater than 90°, we get 11 more convex quadrilaterals of area 1 to 11. When this angle is less than 90°, we stay convex as long as the two sides of length 3 do not align, which happens when the quadrilateral becomes an isosceles triangle of sides 4,4,6 of area $\sqrt{63} < 8$. Thus we get four more convex quadrilaterals of areas 8,9,10 and 11 as well, for a total of 28. The answer is (c).

Part 2

Problem 1

Suppose for some real numbers x, y and z the following equation holds:

$$2x^2 + y^2 + z^2 = 2x(y+z).$$

Prove we must have $x = y = z$.

Solution:

Rewriting the equation gives $(x-y)^2 + (x-z)^2 = 0$ implying $x = y$ and $x = z$.

Alternative Solution: The given equation can be rewritten as $2x^2 - 2(y+z)x + (y^2+z^2) = 0$, and hence

$$x = \frac{2(y+z) \pm \sqrt{4(y+z)^2 - 8(y^2+z^2)}}{4} = \frac{y+z \pm \sqrt{-(y-z)^2}}{2}.$$

Since x must be real, $(y-z)^2 \le 0$ which means $y = z$, and then $x = \frac{2y+0}{2} = y$.

Problem 2

Two robots R2 and D2 are at a point O on an island. R2 can travel at a maximum 2 km/hr and D2 at a maximum of 1 km/hr. There are two treasures located on the island, and whichever robot gets to each treasure first gets to keep it (if both robots reach a treasure at the same time, neither one can keep it). One treasure is located at a point P which is 1 km west of O. Suppose that the second treasure is located at a point X which is somewhere on the straight line through P and O (but not at O). Find all such points X so that R2 can get both treasures, no matter what D2 does.

Solution:

Using Cartesian coordinates, we put $O = (0,0)$, $P = (-1,0)$ and $X = (x,0)$ for some real number $x \neq 0$. The treasure located at point P will be denoted P, and similarly for the treasure located at X. First note that if $x < 0$, then R2 can travel west in a straight line and get both treasures, one after the other, before D2. Now suppose that $x > 0$.

(a) If *R2 travels west at a maximum speed to pick up P and then returns east to pick up X*, it needs $\frac{1+1+x}{2} = \frac{2+x}{2}$ hours. D2 has no chance to get P, hence it should travel east for at least x hours and try to pick up X. If $\frac{2+x}{2} < x$ or equivalently $x > 2$, R2 will get both treasures.

(b) If *R2 travels east at a maximum speed to pick up X and then returns west to get P*, it needs a $\frac{x+x+1}{2} = \frac{2x+1}{2}$ hours. D2 should travel west to pick up P, for which it needs at least one hour. If $\frac{2x+1}{2} < 1$ or equivalently $x < \frac{1}{2}$, R2 will get both treasures.

(c) If $x \in (\frac{1}{2}, 2)$ there is no winning strategy for R2. This is equivalent to showing that always D2 can prevent R2 getting both treasures. Here is D2's strategy: *until R2 gets one treasure, D2 moves so that its position is always on the other side of the point O from R2's position, but at half the distance from O that R2 is.* Once R2 gets one of the treasures, R2 is at least twice as far from the other treasure than D2; then D2 heads straight to the other treasure and gets there before R2.

If $x = 2$ or $x = \frac{1}{2}$, *D2* can prevent *R2* to get both treasures by using the same strategy as above. In this case both robots reach one treasure at the same time, so neither one can keep it.

From (a),(b) and (c) we conclude that R2 has strategies to get both treasures no matter what D2 does if and only if $x \in (-\infty, 0) \cup (0, \frac{1}{2}) \cup (2, \infty)$.

Problem 3

One or more pieces of clothing are hanging on a clothesline. Each piece of clothing is held up by either 1, 2 or 3 clothespins. Let a_1 denote the number of clothespins holding up the first piece of clothing, a_2 the number of clothespins holding up the second piece of clothing, and so forth. You want to remove all the clothing from the line, obeying the following rules:

(i) you must remove the clothing in the order that they are hanging on the line;

(ii) you must remove either 2, 3 or 4 clothespins at a time, no more, no less;

(iii) all the pins holding up a piece of clothing must be removed at the same time.

Find all sequences $a_1, a_2, \ldots, a_n$ of any length for which all the clothing can be removed from the line.

Solution:

We claim that the clothing can be removed for all sequences *except* for 1, 131, 13131, and so on; that is, the exceptional sequences are of the form

$$a_1, a_2, \ldots, a_n = 1, 3, 1, 3, \ldots, 1, 3, 1,$$

where the 1's and 3's alternate, starting and ending with 1. Call such a sequence a *bad* sequence.

If $a_1, a_2, \ldots, a_n = 1, 3, 1, 3, \ldots, 1, 3, 1$, then in your first step you are forced to remove the first two pieces, using 1 and 3 pins respectively, because you cannot remove just one pin and cannot remove 5 pins at a time. This continues right to the end, till there is only one pin left, which you cannot remove. Thus all the bad sequences result in clothing left on the line.

Now we prove that any non-bad sequence $a_1, a_2, \ldots, a_n$ can be removed. Of course the 1-digit sequence 1 (which is bad) cannot be removed, while the non-bad sequences 2 and 3 can be. We proceed by induction. Choose a non-bad sequence $a_1, a_2, \ldots, a_n$ of 1's, 2's and 3's, and suppose that all shorter non-bad sequences can be removed.

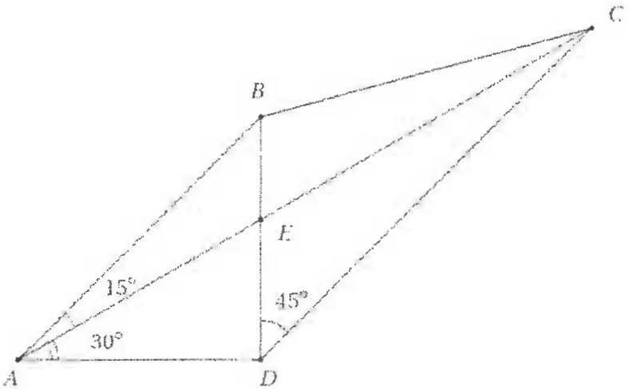
If $a_1 = 2$ or 3, and the sequence $a_2, a_3, \ldots, a_n$ is not bad, then we remove a_1 by itself, and the remaining sequence can be removed by induction. If $a_1 = 2$ or 3, and the remaining sequence $a_2, a_3, \ldots, a_n$ is bad, then we remove a_1 and $a_2 = 1$ (which add up to 3 or 4), and the remaining sequence $a_3, \ldots, a_n$ is not bad so can be removed by induction.

If $a_1 = 1$, and the sequence $a_3, a_4, \ldots, a_n$ is not bad, then we remove a_1 and a_2 (which add up to 2, 3 or 4), and the remaining sequence can be removed by induction. If $a_1 = 1$ and the sequence $a_3, a_4, \ldots, a_n$ is bad, and $a_2 = 1$ or 2, then we remove a_1, a_2 and $a_3 = 1$ (which add up to 3 or 4), and the remaining sequence can be removed by induction. Finally, if $a_1 = 1$ and the sequence $a_3, a_4, \ldots, a_n$ is bad, and $a_2 = 3$, then the sequence $a_1, a_2, \ldots, a_n$ is in fact bad, which is a contradiction.

Problem 4

$ABCD$ is a convex quadrilateral such that $\angle BAC = 15°$, $\angle CAD = 30°$, $\angle ADB = 90°$ and $\angle BDC = 45°$. Find $\angle ACB$.

Solution:



Let $AD = a$, then $DB = a, DE = \frac{a}{\sqrt{3}}, BE = a\left(1 - \frac{1}{\sqrt{3}}\right), AB = a\sqrt{2}$. $\triangle AEB$ is similar to $\triangle CED$ thus $\frac{AB}{DC} = \frac{EB}{ED}$, and hence $DC = a\frac{\sqrt{2}+\sqrt{6}}{2}$. In $\triangle BDC$, by using cosine law we get

$$BC^2 = DB^2 + DC^2 - 2DB \cdot DC\cos 45° = a^2 + \frac{a^2(8+4\sqrt{3})}{4} - 2a^2\frac{\sqrt{2}+\sqrt{6}}{2} \cdot \frac{\sqrt{2}}{2} = 2a^2$$

thus $BC = a\sqrt{2}$, which leads to $\triangle ABC$ is isosceles, hence $\angle BCA = \angle BAC = 15°$.

Problem 5

Find the minimum value of $|x + 4y + 7z|$ where x, y, z are **non-equal** integers satisfying the equation

$$(x - y)(y - z)(z - x) = x + 4y + 7z.$$

Solution:

If an integer m is a multiple of 3 let us write $m = \mathcal{M}3$. If x, y, z have different remainders when they are divided by 3, then $x = \mathcal{M}3 + r_1$, $y = \mathcal{M}3 + r_2$, $z = \mathcal{M}3 + r_3$ where $\{r_1, r_2, r_3\} = \{0, 1, 2\}$. One obtains that $x + 4y + 7z = \mathcal{M}3 + r_1 + r_2 + r_3 = \mathcal{M}3$ while $(x - y)(y - z)(z - x) = (\mathcal{M}3 + r_1 - r_2)(\mathcal{M}3 + r_2 - r_3)(\mathcal{M}3 + r_3 - r_1) = \mathcal{M}3 + (r_1 - r_2)(r_2 - r_3)(r_3 - r_1) \neq \mathcal{M}3$ which is a contradiction. Therefore at least two remainders are equal and hence

$$3|(x - y)(y - z)(z - x) \iff 3|(x + 4y + 7z) \iff 3|(x + y + z) \iff 3|(r_1 + r_2 + r_3).$$

Since two of the remainders are equal, $3|(r_1 + r_2 + r_3)$ if and only if all three remainders are equal. Therefore

$$27|(x - y)(y - z)(z - x) \Longrightarrow 27|x + 4y + 7z.$$

Take $x = y + 3a$, $y = z + 3b$ and $x + 4y + 7z = 27k$ where a, b, k are integers. Since x, y, z are distinct, one obtains that none of the integers a, b or $a + b$ is equal to 0. By using these notations, the given equation $(x - y)(y - z)(z - x) = x + 4y + 7z$ can be written as $-ab(a + b) = k$. It is clear that $|k| \geq 2$. The value $|k| = 2$ could be obtained if we take $a = 1, b = 1$, for which we get $x = 0$, $y = -3$ and $z = -6$. We conclude that the minimum value of $|x + 4y + 7z|$ is $27 \cdot 2 = 54$.

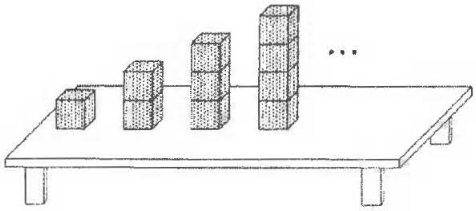
Calgary Junior High School Mathematics Competition 2016/17

Part 1

A1 If you place one die on a table. you can see five faces of it (the front, back, left, right and top). If you stack two dice on a table, then the number of visible faces is nine. In a stack of three dice. the number of visible faces is thirteen, and so on. How many dice do you need to stack on a table (in a single stack) so that the number of visible faces is 101?

A1

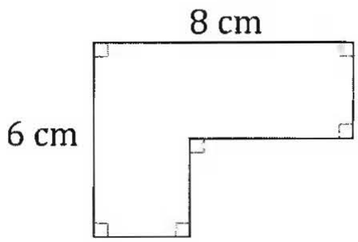
25



A2 What is the perimeter (in cm) of the following figure?

A2

28



A3 The integer 5 has the property that it is prime and one more than it (i.e.. 6) is twice a prime ($6 = 2 \times 3$). The next integer with this property is 13. since 13 is prime and one more than it (i.e., 14) is twice a prime ($14 = 2 \times 7$). What is the next integer after 13 with this property?

A3

37

A4 Srosh can jog at 10 km per hour in sunny weather and at 6 km per hour in rainy weather. She jogs 20 km in 3 hours. How much time (in hours) during her run was it raining?

A4

$2\frac{1}{2}$

A5 A number is multiplied by $\frac{3}{2}$ and $\frac{3}{2}$ is then added. The result is divided by $\frac{3}{2}$ and finally the original number is subtracted. What is the answer?

A5

1

A6 In the game of pickleball. the winner scores 9 points while the loser gets between 0 and 8 points (inclusive). Ruby plays 6 games and gets a total of 50 points. What is the smallest possible number of games she won?

A6

2

A7 Mary is 24 years old. She is twice as old as Ann was when Mary was as old as Ann is now. How old is Ann?

A7

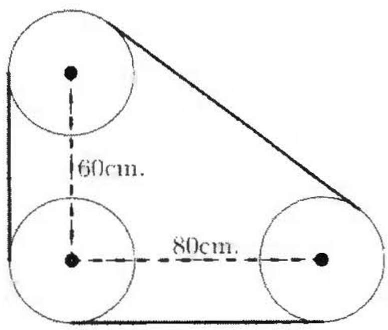
18

A8 A belt runs tightly round three pulleys. each of diameter 40 cm The centre of the top pulley is 60 cm vertically above the centre of the second pulley. which is 80 cm horizontally from the centre of the rightmost one.

What is the total length in cm of the belt?

A8

$40(6+\pi)$
or 365.6



Solution. The straight portions of the belt have lengths 60 cm.. 80 cm.. and (by Pythagoras's theorem) 100cm. The curved portions comprise the circumference of one of the pulleys. length 40π cm. Total $240 + 40\pi = 40(6 + \pi) = 365.663706\ldots$ cm.

A9 Rahat has a jar with ten red balls. ten blue balls. and ten yellow balls. He picks one ball at random and puts it in his pocket. Then he picks another ball at random from the remaining 29 balls in the jar. What is the probability that the two balls Rahat selected have different colour?

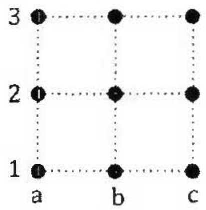
A9

20/29

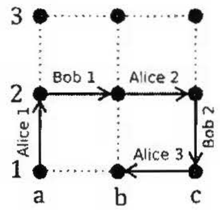
Part 2

B1 In the game Worm, Alice and Bob alternately connect pairs of adjacent dots on the shown grid with either a vertical line or a horizontal line. Subsequent segments must start where the previous one ended and end at a dot not used before, forming a *worm*. The player who cannot continue to build the worm (without it intersecting itself) loses.

For example, if Alice's first move is a1 a2, Bob may then continue with either a2 – a3 or a2 – b2. Suppose Bob plays a2 – b2, and Alice then plays b2 – c2, followed by Bob playing c2 – c1. Then Alice will win with the move c1 – b1 since Bob has no remaining moves to continue building the worm.



The Grid



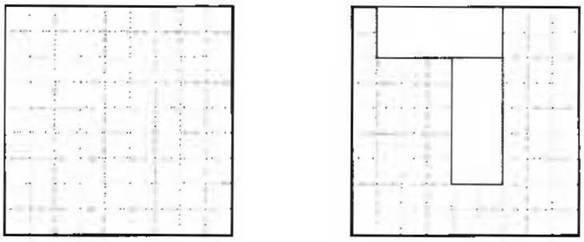
Sample Game: Alice wins

If Alice plays first, can she always win if she plays well enough? If so, how?

Solution. Alice can guarantee a win if she plays well enough. For example, Alice could first play a2 – a1, and Bob is then forced to play a1 – b1. Alice then plays b1 – c1 forcing Bob to play c1 – c2. Alice then plays c2 – c3 forcing Bob to play c3 – b3. Alice then wins with either b3 – b2 or b3 – a3.

Other solutions are possible but may require case work.

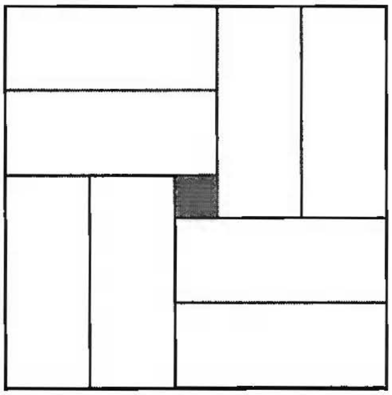
B2 We say that a 2 by 5 rectangle fits *nicely* into a 9 by 9 square if the rectangle occupies exactly ten of the little squares in the 9 by 9 square.



The diagram on the right shows the 9 by 9 square with two non-overlapping rectangles nicely placed in it.

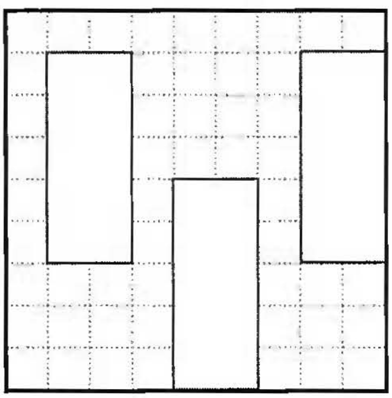
(a) How many 2 by 5 rectangles can you fit nicely into a 9 by 9 square without overlapping? The more rectangles you succeed in fitting into the square, the better your score will be.

Solution. The maximum number of rectangles that can nicely fit into a 9 by 9 square is eight. One such configuration is shown below.



(b) Show how to fit some 2 by 5 rectangles nicely into a 9 by 9 square so that no further 2 by 5 rectangles can be fit nicely into the 9 by 9 square. The *fewer* rectangles you use, the better your score will be.

Solution. The minimum number is three (one can justify why two is not possible by considering the cases of two horizontal rectangles, two vertical rectangles or one horizontal and one vertical rectangle, then analyzing the empty space left over). One solution demonstrating that three is possible is shown below.



B3 (a) Write 2017 as a sum of two squares of positive integers.

Solution. One solution is $2017 = 81 + 1936 = 9^2 + 44^2$ (in fact, it can be shown that this solution is unique).

In order to reduce trial and error, consider the following observations:

- Since 2017 is odd, one square must be odd, the other even.
- Odd squares end in 1, 5 or 9; even squares in 0, 4 or 6. Therefore the two squares must end in 1 and 6.
- An exploration of numbers then gives the answer. Alternatively, one can notice that odd squares are 1 more than a multiple of 8 and since 2017 is one more than a multiple of 16, the squares must be of the form $(4x)^2$ and $(8y \pm 1)^2$.
- Thus, $(4x)^2 + (8y \pm 1)^2 = 2017$ implying $x^2 + 4y^2 \pm y = 126$. Then x and y are of the same parity, both odd, or both even with y singly even.

Alternatively, one could compute a table of squares, subtract each from 2017 and check if the result is a square number.

n	n^2	$2017 - n^2$	check
1	1	2016	not a square
2	4	2013	not a square
3	9	2008	not a square
4	16	2001	not a square
5	25	1992	not a square
6	36	1981	not a square
7	49	1968	not a square
8	64	1953	not a square
9	81	1936	is a square

(b) Write 2017 as a difference of two squares of positive integers.

Solution. One solution is $2017 = 1009^2 - 1008^2$ (in fact, it can be shown that this solution is unique). One method to deduce this is as follows.

$$\begin{aligned} 2017 &= 2017 \times 1 \\ &= (1009 + 1008) \times (1009 - 1008) \\ &= 1009^2 - 1008^2 \end{aligned}$$

B4 Greg and Joey decide to race each other on an 800 metre track. Since Joey is faster than Greg, the two decided to give Greg a head start. In the first race, Greg was given a 20 metre head start, however, Joey still won and finished 2 seconds earlier than Greg. In the second race, Greg was given a 38 metre head start, and this time Greg won and finished 1 second ahead of Joey. Assuming both Greg and Joey ran at uniform speeds in both races, determine the speeds (in metres per second) of both runners.

Solution. The answer is that Greg runs at 6 metres per second and Joey runs at 6.25 metres per second.

Solution 1. Suppose Joey ran 800 metres in t seconds. Then Greg ran 780 metres in $t+2$ seconds and 762 metres in $t-1$ seconds. Since Greg ran at uniform speed in both races (by assumption), we have

$$\frac{780}{t+2} = \frac{762}{t-1}.$$

Cross-multiplying gives $780(t-1) = 762(t+2)$, thus, $t = 128$. This implies that Joey runs at $800/128 = 6.25$ metres per second, and Greg runs at $780/130 = 6$ metres per second.

Solution 2. Suppose Greg runs at x metres per second. Then Greg finished the first race in $780/x$ seconds and the second race in $762/x$ seconds. Joey finished the first race in $\frac{780}{x} - 2$ seconds and the second race in $\frac{762}{x} + 1$ seconds. By assumption, Joey ran at uniform speed in both races, and since he ran 800 metres in each race he must have finished both races in the same amount of time. Thus,

$$\frac{780}{x} - 2 = \frac{762}{x} + 1.$$

This implies, $780 = 762 + 3x$, hence, $x = 6$.

Thus, Greg finished the first race in 130 seconds and the second race in 127 seconds. This implies that it takes Joey 128 seconds to run 800 metres, that is, Joey runs at 6.25 metres per second.

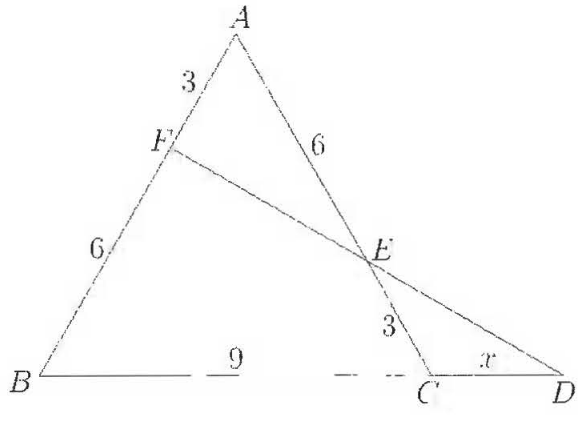
B5 Every day Tom puts on his socks, shoes, shirt, and pants. Of course he has to put his left sock on before his left shoe, and his right sock before his right shoe. He also must put on his pants before he puts on either shoe. Otherwise he can put these six articles on in any order. In how many orders can he do this?

Solution. Suppose that Tom puts his socks and shoes on in the order (sock, shoe, sock, shoe). There are only two ways to do this, namely Tom starts off with either his left sock or his right sock, and then he has no choice for the other three items. Then he must put his pants on either before he puts on the first sock or immediately after, so he has two choices for when he puts on his pants. This gives $2 \times 2 = 4$ ways to put on everything but his shirt in this case.

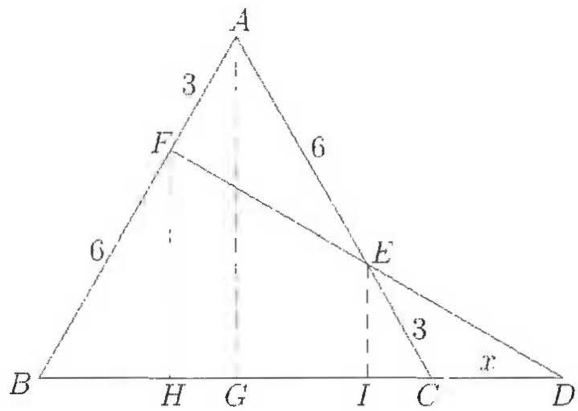
Suppose instead that Tom puts his socks and shoes on in the order (sock, sock, shoe, shoe). He again has two choices for which sock he puts on first, and this time he also has two choices for which shoe he puts on first, so he has $2 \times 2 = 4$ ways to put on his socks and shoes in this case. He can put on his pants either before the first sock, or between the two socks, or immediately after the second sock, so he has 3 choices for when to put on his pants. Thus he has $4 \times 3 = 12$ ways to put on everything but his shirt in this case.

Thus Tom has $4 + 12 = 16$ ways to put on everything but his shirt. He can put on his shirt at any time, so he has 6 choices for that (before the first sock, after the last shoe, or anywhere in between). So the total number of ways he can put on all six items is $16 \times 6 = 96$.

B6 A straight line is drawn across the equilateral triangle ABC of side-length 9, cutting the sides AB and AC at points F and E, as shown. What is the length of CD?



Solution. Let G, H and I be the feet of the perpendiculars from A, F and E, respectively onto BC.



Then $BG = \frac{1}{2}BC = \frac{9}{2}$. Triangles BHF and BGA are similar triangles, thus,

$$\frac{BH}{BG} = \frac{BF}{BA} \quad \rightarrow \quad \frac{BH}{9/2} = \frac{6}{9}$$

implying $BH = 3$. Finally, $ID = HI = \frac{9}{2}$, thus, $BD = 12$ implying $CD = 3$.

Edmonton Junior High School Mathematics Competition 2016/17

Part 1

1. A 4-digit number uses each of the digits 3, 4, 5, and 6 exactly once. If the digits are placed randomly, what is the probability that the 4-digit number is a multiple of 6?

A. $\frac{1}{6}$ B. $\frac{1}{3}$ C. $\frac{2}{3}$ D. $\frac{1}{2}$ E. $\frac{5}{6}$

Answer: D

Solution:

There are 4 x 3 x 2 x 1 = 24 ways to write a 4-digital number. The 4-digital number is already divisible by 3 regardless of the positions of the digits. To be divisible by 6, the number must end with either a 4 or a 6. There are 3 x 2 x 1 = 6 ways to write the first three digits. This gives the probability of $\frac{2 \times 3 \times 2 \times 1}{4 \times 3 \times 2 \times 1} = \frac{1}{2}$

2. Two analog clocks run at the correct rate of speed. Both clocks show the correct time when it is 9:45 PM However, as the hands on one clock run forward, the hands on the other clock run backward. When will both clocks next show the same time?

A. 4:15 AM B. 3:45 AM C. 3:45 PM D. 4:15 PM E. 9:45 AM

Answer: B

Solution:

Since the two clocks run at the same speed, the two clocks would display the same time exactly 6 hours later. This gives 3:45 PM.

3. Cellphone company Apple has no monthly fee but charges:

- Local calls at $0.10/min, plus
- Long Distance calls at $0.50/min, plus
- Text Messages at $0.20/text beyond 75 texts, plus
- Data at $10/GB past 3 GB.

Cellphone company Banana charges $125/month for unlimited usage.

Jaime's typical use per month is:

- Local calls: 500 minutes, plus
- Long Distance calls: 10 minutes, plus
- Text Messages: 250
- Data: 5 GB

Based on Jaime's usage, which statement is true?

A. Jaime saves less than \$200/year using company Apple.
B. Jaime saves more than \$200/year using company Apple.
C. Jaime saves less than \$200/year using company Banana.
D. Jaime saves more than \$200/year using company Banana.
E. Both companies would charge Jaime the same amount.

Answer: A

Solution:

Using Jaime's data usage, we have $500(0.1) + 10(0.5) + 0.2(250 - 75) + 10(5 - 3) = 50 + 5 + 35 + 20 = \$110$. Each year, Jaime saves $12(125 - 110) = \$180$ using company Apple.

4. It will take me 2% of 8 hours to finish folding my laundry. It will take me 55% of 20 minutes to unload the dishwasher. Which task will take me longer to complete, and by how many more seconds?

A. Folding laundry by 84 seconds.
B. Folding laundry by 54 seconds.
C. Unloading the dishwasher by 84 seconds.
D. Unloading the dishwasher by 54 seconds.
E. Both tasks take the same amount of time.

Answer: C

Solution:

Folding laundry requires $0.02(8)(60)(60) = 576$ seconds. Unloading the dishwasher requires $0.55(20)(60) = 660$ seconds. Unloading takes longer by $660 - 576 = 84$ seconds.

5. I started a game with an even number of points, and played 3 rounds. In the first round, I lost half of my points. In the second round, I won back twice the number of points that I had started the game with. I ended the third round with half the number of points that I had started that round with. I ended the game with 15 points. Which describes how the number of points I ended the game with compares to the number of points I started the game with?

A. I ended the game with half the points that I started the game with.
B. I ended the game with double the points that I started the game with.
C. I ended the game with 3 more points than what I started the game with.
D. I ended the game with 3 less points than what I started the game with.
E. I ended the game with the same number of points that I started the game with.

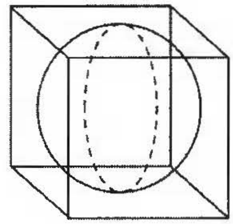
Answer: C

Solution:

Start of round	Win or lose	End of round
2n		
2n	n	n
n	4n	5n
5n	2.5n	2.5n = 15

Start of the game = 12 points. Therefore, Jaime earns 3 more points than the start of the game.

6. Given that the formula for the Volume of a Sphere is: $\mathbf{V} = \left(\frac{4}{3}\right)\pi r^3$



A cube has the same height as the diameter of a sphere. The Surface Area of the cube is 216 cm^2. Rounded to the nearest whole cm^3, how much larger is the volume of the cube compared to the volume of the sphere?

A. 96 B. 103 C. 108 D. 127 E. 216

Answer: B

Solution:

The length of one side of the cube is $\sqrt{\frac{216}{6}} = 6\ cm.$

The difference in volume is $(6 \times 6 \times 6) - \frac{4}{3}(\pi)(3^3) \cong 103cm^3$

7. A package contains 4 chocolate, 3 vanilla and 3 lemon cupcakes. How many chocolate cupcakes, represented by x, must be added to the package so that it will contain 60% chocolate cupcakes?

Which of the following equations could be used to solve this problem?

A. $\frac{x-10}{x-4} = \frac{60}{100}$ B. $\frac{x+10}{x+4} = \frac{60}{100}$ C. $\frac{x}{x+10} = \frac{0.6\,x}{1}$

D. $\frac{x+4}{x+10} = \frac{60}{100}$ E. $\frac{x}{0.6} = x + 10$

Answer: D

Solution:

The total number of chocolate cupcakes would increase by 4 while the total number of cupcakes also increases by 4. The proportional statement $\frac{number\ of\ chocolate\ cupcakes}{total\ number\ of\ cupcakes} = \frac{x+4}{x+10} = \frac{60}{100}$ gives the correct expression.

Part 2

8. Each person in a room shook hands once with each other person in the room. If the total number of handshakes was less than 1000, then what is the most number of people that could have been in the room?

Solution:

Total number of handshakes is best dealt with using the series (n − 1) + (n − 2) + (n − 3) + … + 3 + 2 + 1 where n is the number of people in the group. For example, if n = 5 people, there would be 4 + 3 + 2 + 1 handshakes in total. Pairing the front and back each time yield a sum of n. There are exactly $\frac{n-1}{2}$ pairs giving a sum of $n\left(\frac{n-1}{2}\right)$. Solving the inequality $n\left(\frac{n-1}{2}\right) < 1000$, we have n = 45.

9. The sum of two rational numbers is 1. Amy add the larger number to the square of the smaller number. Beth add the smaller number to the square of the larger number. What is the difference of the two values?

Solution:

Let the larger of the number be n and the smaller number be $(1-n)$.

We have $(n + (1-n)^2) - (n^2 + (1-n)) = (n + 1 - 2n + n^2) - (n^2 + (1-n)) = 0$.

10. Although Jen has no savings, she wants to earn enough money in 4 months to buy a puppy. On the first month, Jen earns half of the total cost. On the second month, Jen earns one-third of the amount she still needs. On the third month, she earns \$80. After 3 months, she has earned 75% of the total cost of the puppy. How much money must Jen earn in the fourth month to have enough to buy the puppy?

Solution:

Let n be the cost of a puppy

We have the equation $\frac{3n}{4} = \frac{n}{2} + \frac{n}{6} + 80$. Solving for n yields $n = 960$. Jen needs to earn $\frac{1}{4} \times 960 = \$240$

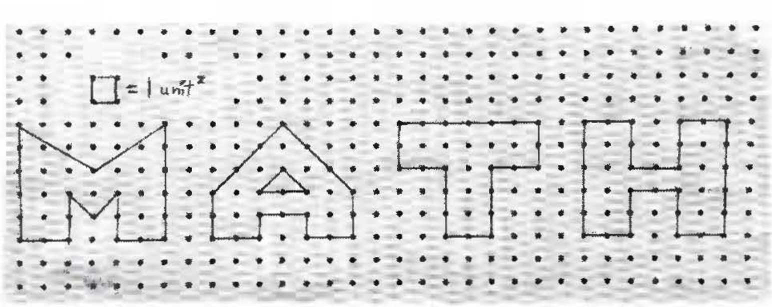
11. Xiang's age is 10 less than the sum of Yvonne's age and Zoe's age. The ratio of Xiang's age to Yvonne's age is 3:2. Zoe is 2 years older than Yvonne. What is the sum of the ages of the three people 4 years from now?

Solution:

Let Y be Yvonne's current age. It follows that $x = \frac{3y}{2}$ and z = y + 2.

Solving the equation $\frac{3y}{2} = y + (y + 2) - 10$, we have $y = 16$, $x = 24$ and $x = 18$. In 4 years, we have 20 + 28 + 22 = 70.

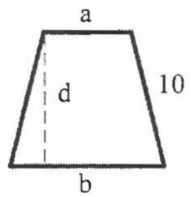
12. What is the sum of the interior areas, to the nearest unit^2, of the letters used to spell the word "MATH"?



Solution:
M = 21, A = T = 18, H = 22.
The total area = 79 unit^2

13. What is the area, in square centimeters, of an isosceles trapezoid, given the following clues?

- Its perimeter is 64 cm
- Each of the 2 congruent sides is 10 cm
- The difference in the lengths of the parallel sides is 12 cm



Solving the two equations $a + b + 20 = 64$ and $b - a = 12$, we have $a = 16$ and $b = 28$.

$d = \sqrt{10^2 - 6^2} = 8$. The area is $\frac{(16+28)(8)}{2} = 176\ cm^2$

Part 3

14. Mary divides by 5 each number from 1 to 2017, inclusive. She then adds together all the remainders she gets. Find the sum Mary obtains.

Solution:

When we divide the number 1, 2, 3, ... , 2017 by 5, the remainders have a repeating pattern: 1, 2, 3, 4, 0, 1, 2, 3, 4, 0, ... , 1, 2.

The pattern 1, 2, 3, 4, 0 repeats 403 times and ends with 1, 2. The sum is $403(10) + 1 + 2 = 4033$.

15. How many 4 digit palindromes are divisible by 7?

Solution:

A 4 digit palindrome has the form $abba = a(1001) + b(110)$. Since $7|1001$, we need $7|110b$. This is possible when $b = 0$ or 7. Since there is no restriction on a except $a \neq 0$, we have 9 choices for a and 2 choices for b. In total, there are $9 \times 2 = 18$ such numbers.

16. Nickels, dimes and quarters are to be used to make exactly $1.00. At least one of each type of coin must be used. In how many different ways can this be done if an even number of coins must be used?

Solution:

Using a table of values to organize the number of coins, we have 6 ways to make $1.00 using even number of coins.

25¢	10¢	5¢
1	2	11
1	4	7
1	6	3
2	2	6
2	4	2
3	2	1

17. A girl and a boy play the game Rock, Paper, Scissors ten times, where rock beats scissors, scissors beat paper and papers beat rock. The boy uses rock three times, scissors six times and paper once. The girl uses rock twice, scissors four times and paper four times. None of the ten games is a tie. How many games has the boy won?

Solution:

Scissors are used ten times altogether. Since there are no tied games, exactly one player uses scissors in each game. In the six games where the boy uses scissors, the girl wins two of them when she uses rock, and lose the other four games. In the four games where the girl uses scissors, the boy wins three of them when he uses rock, and lose the other one. Hence the boy wins seven games.

18. Of all the whole numbers N from 1 to 2017 inclusive, how many have the property that there exists a number M such that the sum of M and N is equal to the sum of the reciprocal of M and the reciprocal of N?

Solution:

Let *m, n* be the two numbers. We have

$$m + n = \frac{1}{m} + \frac{1}{n}$$
$$m + n = \frac{n + m}{mn}$$
$$1 = \frac{1}{mn}$$
$$mn = 1$$

Hence *m* and *n* are reciproal of one another. Of the numbers from 1 to 2017, there are 2017 reciprocals. Therefore, there are 2017 values for M.

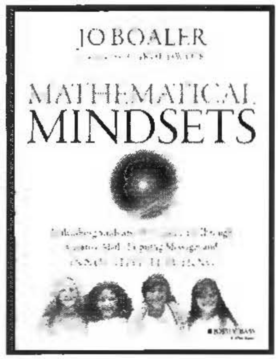
19. Find a positive integer whose ones digit is 5, and when it is multiplied by 4, the 5 becomes the first digit while all other digits shift one place to the right.

Solution:

Create a pure repeating decimal x where the positive integer is the repeating block. Then $40x$ is the same as x except with an extra 5 in front of its decimal point. Hence $x = \frac{5}{39} = 0.\overline{128205}$. Thus the desired positive integer is 128205.

Alternative solution:

Divide 5 by 4. The quotient is 1.25. Divide 51 by 4. The quotient is 12.75. Divide 512 by 4. The quotient is 128. The division is exact but the ones digit is not 5. So we continue. Divide 5120 by 4. The quotient is 1282.5. Divide 51282 by 4. The quotient is 12820.5. Divide 512820 by 4. The quotient is 128205. This is the positive integer we seek.



Mathematical Mindsets
by Jo Boaler
Jossey-Bass, 2016

Reviewed by Ashley Durbeniuk and Terry Freeman

Mathematical Mindsets, by Jo Boaler, brings to light the five Cs of learning: curiosity, collaboration, connections, challenge and creativity. It allows students to see that math is not just a black and white subject. Multiple pathways can get learners to their final destination. In our experience, allowing students to use their own creativity in math gives them the satisfaction of connecting their life to mathematical concepts. It also gives them a chance to succeed in a subject that they may have previously failed. An example of this is the linear relations task that introduced the idea of linear relations in Grade 9 math. It encouraged students to think outside the box, see their own patterns and express their learning in multiple ways. Students were given the opportunity to collaborate with one another and share their ideas with a table group, where students of all levels of understanding were able to be experts in their own right. As a class, they discussed the multiple pathways.

The book discusses many nontraditional approaches to the learning of math. Boaler proposes "Positive Norms to Encourage in Math Class." These norms include (1) everyone can learn math to the highest levels, (2) mistakes are valuable, (3) questions are really important, (4) math is about creativity and making sense, (5) math is about connections and communicating, (6) depth is more important than speed and (6) math class is about learning not performing. Jo Boaler provides the research behind each of these norms. *Mathematical Mindsets* is ripe with tangible examples. Taking on the debate over Mad Minutes, the book references an article by Boaler entitled "Fluency without Fear." The article discusses the stress associated with timed fact tests. She proposes cooperative, nontimed activities like Close to One Hundred. This game is a favourite of many students. It strengthens numerical fluency while working with a partner in a nonthreatening environment.

The lens of the 5 Cs encourages teachers and students to interact with math in meaningful, real-world situations. Watching Grade 4 students work on a challenge requiring collaboration, creativity, connecting and curiosity to discover the area and perimeter of an alien ship produced an unexpected mathematical discovery—how to discover the area of a triangle. This activity was what Jo Boaler calls "Low Floor—High Ceiling" tasks. Such tasks engage all students in meaningful ways. Grade 1 students were challenged to discover the cost of a pizza party for their class, grade and school. These young mathematicians were totally engaged for 90 minutes. "The Power of Mistakes" was celebrated. Favourite mistakes were celebrated and moved thinking forward. The beauty of each of these "Rich Mathematical Tasks" is that they were carefully crafted so that each student would have success. More important, the work and effort of each team member contributes to the final outcome.

Mathematical Mindsets is not a theoretical discourse on what could be. Rather, it is thoughtfully written so that teachers of all grade and ability levels can affect positive change in their practice. More important, these shifts in practice engage the students and they see the beauty of math, how powerful struggle is and how a growth mindset can be fostered. Join the revolution and read this book.

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Mathematics of Planet Earth

http://mpe.dimacs.rutgers.edu/

Lorelei Boschman

Recently, I came across an open source mathematical website that grabbed my attention. It is worth exploring for either demonstrative or investigative purposes with mathematics students. Below is information directly from the website. Can you think of ways to integrate some of this excellent mathematical and scientific work into your curriculum?

Mathematics of Planet Earth (MPE) is an initiative of mathematical science organizations worldwide designed to highlight the ways in which the mathematical sciences can be useful in tackling our world's problems. The exhibition Mathematics of Planet Earth consists of modules submitted by the community. It started with a competition in 2012; winning modules from that competition were presented at the official opening event of the first MPE exhibition in Paris in March 2013.

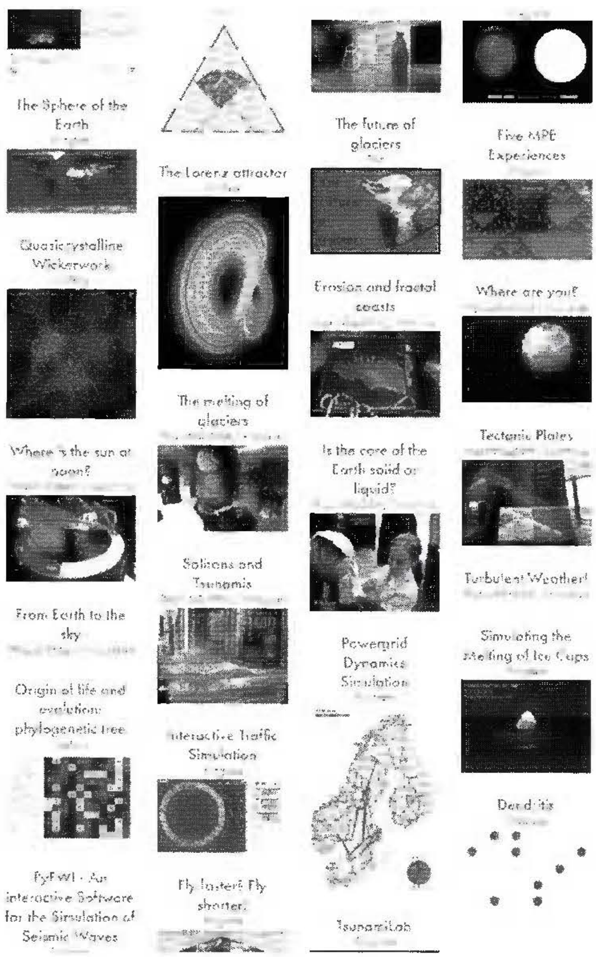
The modules of the Mathematics of Planet Earth Open Source exhibition can be reproduced and adapted by science museums and schools around the world (https://imaginary.org/exhibition/mathematics-of-planet-earth) (scroll down to the bottom to see the "exhibits").

Users worldwide from science museums to schools can reproduce and utilize the modules. The exhibition has a virtual part as well as several material parts. Copies of the material parts can be recreated or travel around the world, and the virtual modules are available on the basis of creative commons licenses.

In one way or another, all exhibits are demonstrating the crucial role mathematics plays in planetary issues. The modules cover a wide variety of topics such as astronomy, fluid dynamics, the mathematics of volcanoes or glaciers and problems in cartography.

The virtual modules displayed in the exhibition come from an international competition organized by the initiative MPE, IMU, ICMI and IMAGINARY in 2013 and 2017. They are of four types: interactive modules, films, posters and instructions to realize a physical module. The three winners of the first competition received their prize at UNESCO during the MPE Day in March 2013; the three winners of the second competition received their prize at the MPE exhibition at Imperial College London in October 2017. The exhibition is still under development. New ideas and modules are welcome. See the MPE project (https://imaginary.org/content/new-mpe-exhibits) for more information.

For more information on the Mathematics of Planet Earth initiative, please visit http://mpe.dimacs.rutgers.edu/.



The Alberta Teachers' Association

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