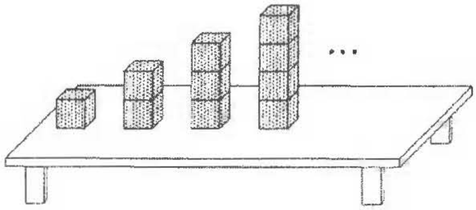
Calgary Junior High School Mathematics Competition 2016/17

Part 1

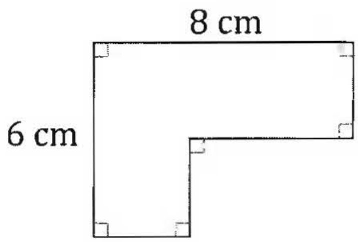
A1 If you place one die on a table. you can see five faces of it (the front, back, left, right and top). If you stack two dice on a table, then the number of visible faces is nine. In a stack of three dice. the number of visible faces is thirteen, and so on. How many dice do you need to stack on a table (in a single stack) so that the number of visible faces is 101?

A1
25



A2 What is the perimeter (in cm) of the following figure?

A2
28



A3 The integer 5 has the property that it is prime and one more than it (i.e.. 6) is twice a prime ($6 = 2 \times 3$). The next integer with this property is 13. since 13 is prime and one more than it (i.e., 14) is twice a prime ($14 = 2 \times 7$). What is the next integer after 13 with this property?

A3
37

A4 Srosh can jog at 10 km per hour in sunny weather and at 6 km per hour in rainy weather. She jogs 20 km in 3 hours. How much time (in hours) during her run was it raining?

A4
$2\frac{1}{2}$

A5 A number is multiplied by $\frac{3}{2}$ and $\frac{3}{2}$ is then added. The result is divided by $\frac{3}{2}$ and finally the original number is subtracted. What is the answer?

A5
1

A6 In the game of pickleball. the winner scores 9 points while the loser gets between 0 and 8 points (inclusive). Ruby plays 6 games and gets a total of 50 points. What is the smallest possible number of games she won?

A6
2

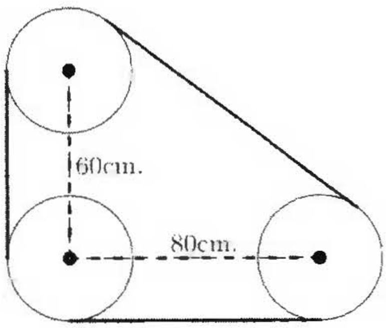
A7 Mary is 24 years old. She is twice as old as Ann was when Mary was as old as Ann is now. How old is Ann?

A7
18

A8 A belt runs tightly round three pulleys. each of diameter 40 cm The centre of the top pulley is 60 cm vertically above the centre of the second pulley. which is 80 cm horizontally from the centre of the rightmost one.

What is the total length in cm of the belt?

A8
$40(6+\pi)$ or 365.6



Solution. The straight portions of the belt have lengths 60 cm.. 80 cm.. and (by Pythagoras's theorem) 100cm. The curved portions comprise the circumference of one of the pulleys. length 40π cm. Total $240 + 40\pi = 40(6 + \pi) = 365.663706\ldots$ cm.

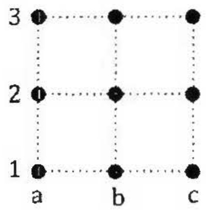
A9 Rahat has a jar with ten red balls. ten blue balls. and ten yellow balls. He picks one ball at random and puts it in his pocket. Then he picks another ball at random from the remaining 29 balls in the jar. What is the probability that the two balls Rahat selected have different colour?

A9
20/29

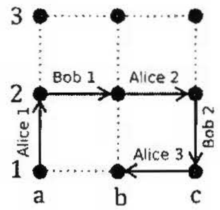
Part 2

B1 In the game Worm, Alice and Bob alternately connect pairs of adjacent dots on the shown grid with either a vertical line or a horizontal line. Subsequent segments must start where the previous one ended and end at a dot not used before, forming a *worm*. The player who cannot continue to build the worm (without it intersecting itself) loses.

For example, if Alice's first move is a1 a2, Bob may then continue with either a2 – a3 or a2 – b2. Suppose Bob plays a2 – b2, and Alice then plays b2 – c2, followed by Bob playing c2 – c1. Then Alice will win with the move c1 – b1 since Bob has no remaining moves to continue building the worm.



The Grid



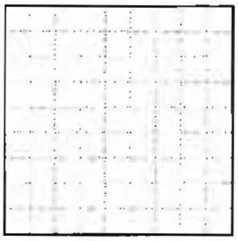
Sample Game: Alice wins

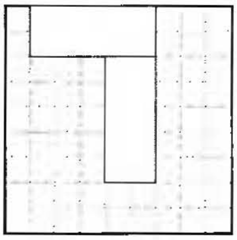
If Alice plays first, can she always win if she plays well enough? If so, how?

Solution. Alice can guarantee a win if she plays well enough. For example, Alice could first play a2 – a1, and Bob is then forced to play a1 – b1. Alice then plays b1 – c1 forcing Bob to play c1 – c2. Alice then plays c2 – c3 forcing Bob to play c3 – b3. Alice then wins with either b3 – b2 or b3 – a3.

Other solutions are possible but may require case work.

B2 We say that a 2 by 5 rectangle fits *nicely* into a 9 by 9 square if the rectangle occupies exactly ten of the little squares in the 9 by 9 square.

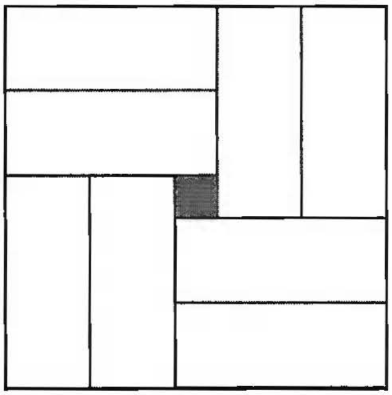




The diagram on the right shows the 9 by 9 square with two non-overlapping rectangles nicely placed in it.

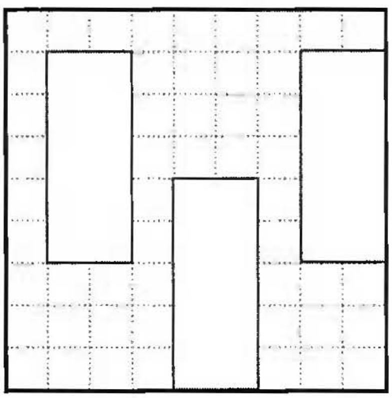
(a) How many 2 by 5 rectangles can you fit nicely into a 9 by 9 square without overlapping? The more rectangles you succeed in fitting into the square, the better your score will be.

Solution. The maximum number of rectangles that can nicely fit into a 9 by 9 square is eight. One such configuration is shown below.



(b) Show how to fit some 2 by 5 rectangles nicely into a 9 by 9 square so that no further 2 by 5 rectangles can be fit nicely into the 9 by 9 square. The *fewer* rectangles you use, the better your score will be.

Solution. The minimum number is three (one can justify why two is not possible by considering the cases of two horizontal rectangles, two vertical rectangles or one horizontal and one vertical rectangle, then analyzing the empty space left over). One solution demonstrating that three is possible is shown below.



B3 (a) Write 2017 as a sum of two squares of positive integers.

Solution. One solution is $2017 = 81 + 1936 = 9^2 + 44^2$ (in fact, it can be shown that this solution is unique).

In order to reduce trial and error, consider the following observations:

- Since 2017 is odd, one square must be odd, the other even.
- Odd squares end in 1, 5 or 9; even squares in 0, 4 or 6. Therefore the two squares must end in 1 and 6.
- An exploration of numbers then gives the answer. Alternatively, one can notice that odd squares are 1 more than a multiple of 8 and since 2017 is one more than a multiple of 16, the squares must be of the form $(4x)^2$ and $(8y \pm 1)^2$.
- Thus, $(4x)^2 + (8y \pm 1)^2 = 2017$ implying $x^2 + 4y^2 \pm y = 126$. Then x and y are of the same parity, both odd, or both even with y singly even.

Alternatively, one could compute a table of squares, subtract each from 2017 and check if the result is a square number.

n	n^2	$2017 - n^2$	check
1	1	2016	not a square
2	4	2013	not a square
3	9	2008	not a square
4	16	2001	not a square
5	25	1992	not a square
6	36	1981	not a square
7	49	1968	not a square
8	64	1953	not a square
9	81	1936	is a square

(b) Write 2017 as a difference of two squares of positive integers.

Solution. One solution is $2017 = 1009^2 - 1008^2$ (in fact, it can be shown that this solution is unique). One method to deduce this is as follows.

$$\begin{aligned} 2017 &= 2017 \times 1 \\ &= (1009 + 1008) \times (1009 - 1008) \\ &= 1009^2 - 1008^2 \end{aligned}$$

B4 Greg and Joey decide to race each other on an 800 metre track. Since Joey is faster than Greg, the two decided to give Greg a head start. In the first race, Greg was given a 20 metre head start, however, Joey still won and finished 2 seconds earlier than Greg. In the second race, Greg was given a 38 metre head start, and this time Greg won and finished 1 second ahead of Joey. Assuming both Greg and Joey ran at uniform speeds in both races, determine the speeds (in metres per second) of both runners.

Solution. The answer is that Greg runs at 6 metres per second and Joey runs at 6.25 metres per second.

Solution 1. Suppose Joey ran 800 metres in t seconds. Then Greg ran 780 metres in $t+2$ seconds and 762 metres in $t-1$ seconds. Since Greg ran at uniform speed in both races (by assumption), we have

$$\frac{780}{t+2} = \frac{762}{t-1}.$$

Cross-multiplying gives $780(t-1) = 762(t+2)$, thus, $t = 128$. This implies that Joey runs at $800/128 = 6.25$ metres per second, and Greg runs at $780/130 = 6$ metres per second.

Solution 2. Suppose Greg runs at x metres per second. Then Greg finished the first race in $780/x$ seconds and the second race in $762/x$ seconds. Joey finished the first race in $\frac{780}{x} - 2$ seconds and the second race in $\frac{762}{x} + 1$ seconds. By assumption, Joey ran at uniform speed in both races, and since he ran 800 metres in each race he must have finished both races in the same amount of time. Thus,

$$\frac{780}{x} - 2 = \frac{762}{x} + 1.$$

This implies, $780 = 762 + 3x$, hence, $x = 6$.

Thus, Greg finished the first race in 130 seconds and the second race in 127 seconds. This implies that it takes Joey 128 seconds to run 800 metres, that is, Joey runs at 6.25 metres per second.

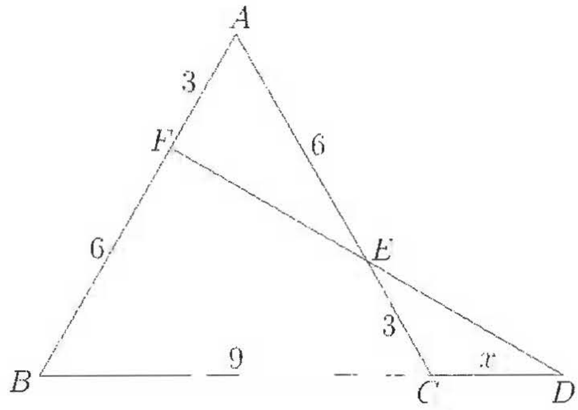
B5 Every day Tom puts on his socks, shoes, shirt, and pants. Of course he has to put his left sock on before his left shoe, and his right sock before his right shoe. He also must put on his pants before he puts on either shoe. Otherwise he can put these six articles on in any order. In how many orders can he do this?

Solution. Suppose that Tom puts his socks and shoes on in the order (sock, shoe, sock, shoe). There are only two ways to do this, namely Tom starts off with either his left sock or his right sock, and then he has no choice for the other three items. Then he must put his pants on either before he puts on the first sock or immediately after, so he has two choices for when he puts on his pants. This gives $2 \times 2 = 4$ ways to put on everything but his shirt in this case.

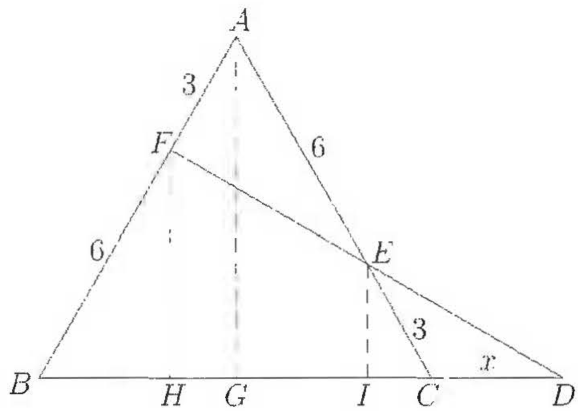
Suppose instead that Tom puts his socks and shoes on in the order (sock, sock, shoe, shoe). He again has two choices for which sock he puts on first, and this time he also has two choices for which shoe he puts on first, so he has $2 \times 2 = 4$ ways to put on his socks and shoes in this case. He can put on his pants either before the first sock, or between the two socks, or immediately after the second sock, so he has 3 choices for when to put on his pants. Thus he has $4 \times 3 = 12$ ways to put on everything but his shirt in this case.

Thus Tom has $4 + 12 = 16$ ways to put on everything but his shirt. He can put on his shirt at any time, so he has 6 choices for that (before the first sock, after the last shoe, or anywhere in between). So the total number of ways he can put on all six items is $16 \times 6 = 96$.

B6 A straight line is drawn across the equilateral triangle ABC of side-length 9, cutting the sides AB and AC at points F and E, as shown. What is the length of CD?



Solution. Let G, H and I be the feet of the perpendiculars from A, F and E, respectively onto BC.



Then $BG = \frac{1}{2}BC = \frac{9}{2}$. Triangles BHF and BGA are similar triangles, thus,

$$\frac{BH}{BG} = \frac{BF}{BA} \quad \rightarrow \quad \frac{BH}{9/2} = \frac{6}{9}$$

implying $BH = 3$. Finally, $ID = HI = \frac{9}{2}$, thus, $BD = 12$ implying $CD = 3$.