Instruction and Learning Through Formative Assessments

Teachers Can Use Rich Mathematical Tasks to Measure Students' Conceptual Understanding

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Assessment and instruction are interwoven in mathematically rich formative assessment tasks, so employing these tasks in the classrooms is an exciting and time-efficient opportunity. To provide a window into how these tasks work in the classroom, this article analyzes summaries of student work on such a task and considers several students' solution strategies to exhibit the usefulness of these tasks in assessment, learning and teaching in the classroom. This article also provides some guidance on implementing these tasks in the classroom.

The literature is replete with descriptions, uses and effects of rich mathematical tasks. These tasks draw on students' prior understanding; create conceptual connections among mathematical ideas; provide students with the opportunity to engage in activities that require them to attend to precision, use tools appropriately, model with math and critique the reasoning of others; provide interwoven assessment and learning experiences; direct students' attention to precise mathematical concepts rather than skills; engage students to creatively investigate and communicate concepts; and provide teachers with opportunities to assess student understanding, misunderstandings and gaps in knowledge (Arbaugh and Brown 2005; Boesen, Lithner and Palm 2010; CCSSI 2010; Henningsen and Stein 1997; Herbst 2003; Smith and Stein 1998).

It is commonly recognized that formative assessments provide opportunities for teachers to assess student understanding through "evidence of students' reasoning and misconceptions to use in adjusting instruction" (NCTM 2013, para 1). However, through well-designed formative assessment tasks, students can also learn the mathematics inherent in the task. Thus, formative assessments through mathematically rich tasks can have multifold effects of assessing student understanding and misunderstandings and discovering gaps in student understanding; providing information through which teachers can adjust instruction; offering student feedback to support their own learning; and being an engaging task through which the mathematics at hand can be encountered and learned (Black and Wiliam 2009; Clark 2011; Hobson 1997; Long, Clark and Corchran 2,000; NCTM 2013; Pryor and Crossouard 2008).

In concert, rich mathematical formative assessments possess a number of recognizable characteristics. They address conceptual understanding of precise mathematical concepts recognizable by both the teacher and the student; assess student understanding of particular mathematical concepts and also serve as springboards through which the associated concepts can be investigated and learned; can be generated to address any grade-appropriate mathematical concept; can be differentiated quite easily to address students of differing ability levels; often address Krutetskii's (1976) three processes of reversibility, flexibility and generalizability; and are solvable through multiple heuristics.

A Sample Task and Classroom Context

The task shown in Figure 1 was designed to pinpoint student conceptual understandings and misunderstandings regarding constructing and comparing function models (CCSS.Math.Content.HSF.LE.A.2) (CCSSI 2010). While seemingly straightforward and unambiguous, this rich task encompasses numerous notions associated with the concept of polynomials, including the definition of polynomial functions and their continuous nature; the role of the leading coefficient and the degree of a polynomial on the graph's extreme behaviours; the definitions of factors, linear factors and a factored polynomial; the graphical effects of roots of odd and even multiplicity; and the association of zeros, roots, factors and x-intercepts between the polynomial function and its graph.

Beginning with concepts from introductory algebra, this task intersects precalculus through the generalized solution

$$y = K(x - a)^{odd}(x - b)^{even}(x - c)^{odd}(x - d)^{even}$$

where $K \in R^+$ and $a, b, c, d \in R$.

This specific task addressed three of the basic processes identified in Krutetskii's (1976) model of mathematical abilities (that is, reversibility, flexibility and generalizability). It required that students reverse their thinking about polynomials and factoring in a direction counter to what they typically experienced during instruction; flexibly solve a problem in more than one way and understand more than one solution; and generalize from specific cases to make deductions from given or known facts.

Below is a truncated graph of a polynomial (All the behaviour near the x-axis is shown.) There is no scale for the y-axis. Write the equation of a polynomial function that would produce this truncated graph.

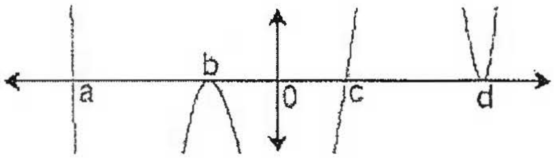


Figure 1. *This is an example of a mathematically rich formative assessment task.*

This mathematically rich formative assessment task was selected for a number of reasons. All students were from the same high school class under the same teacher and had previously investigated polynomial functions and graphical and algebraic representations in their high school precalculus class. They had all experienced identical content, instructional practices and extended problem-solving challenges. The task served as a means through which student knowledge, gaps and misunderstandings could be observed. The classroom teacher assessed the task as both challenging to most students and doable by all.

All students were given up to two hours to complete the three tasks; most took less time (10 to 90 minutes), as they were either able to solve the problem quickly or struggled to persevere through the problem-solving process. Students were primarily left alone to demonstrate what they knew and to learn through the activity while the researchers observed student work and assessed student understanding. The findings and summaries are addressed below in two parts: assessment and learning.

Assessment

The following are syntheses of narrative accounts of students' activity as they worked independently on the task. These summaries abbreviate much fuller transcripts; omitted materials were deemed as not furthering the findings. (See Bossé, Adu-Gyamfi and Chandler [2014] for a more detailed description of the associated study.)

Student 1 holds a course grade of C. The teacher believes that he will be able to do the task, albeit with a struggle. Trying to create a correct graph, Student 1 unsuccessfully uses trial and error, entering values and polynomial functions into the calculator. He does not know what "truncated graph" means and struggles, unnecessarily, to predict the behaviour of the graph above and below the x-axis. He claims that polynomials are in the form $x^2 + 2x + 3$ and does not understand "polynomial in factored form." Through trial and error, he unsuccessfully plugs numbers in for a, b, c and d into

$$y = ax^3 + bx^2 + cx + d.$$

He claims that a graph is the answer to a problem, not the beginning point. After he is shown (x – a), (x – b), (x – c) and (x – d) as factors, he is unsure how these are connected to the graph. When he is told that

$$y = (x - a)(x - b)^2(x - c)(x - d)^2$$

represents a possible solution, he tries to rewrite it in the form

$$px^n + qx^{n-1} + \ldots + rx + s$$

and shows no understanding that the leading coefficient has to be positive. Throughout, he is continually frustrated.

Although the teacher expected him to struggle some, she expected him to do better. She was surprised that he struggled with the vocabulary, linear factors and polynomials and that she had not seen this before.

The remainder of the work of Student 1 (beyond the summary provided) demonstrates that he perceives the polynomial function and graph as mostly disjointed and unconnected. He does not recognize zeros on the graph and does not understand the corresponding $(x - \_)$ binomial in the factored form of the polynomial or consider the far-left and far-right behaviour of the graph in respect to either the degree of the polynomial or the sign of the leading coefficient. He recognizes "polynomial" only in the form $y = px^3 + qx^2 + rx + s$. Altogether, this student has significant gaps in his knowledge that were revealed to the teacher through the implementation of this task. The teacher recognizes that much effort will be needed to bring him to satisfactory understanding and that most concepts will need to be readdressed in novel ways.

Student 2 holds a course grade of C. The teacher believes that this student will be quite successful with the task. Student 2 writes down expressions

$$(x + 2)(x + 1^2)(x - 1)(x - 2^2),$$

$$(x + 2)(x + 1)^2(x - 1)(x - 2)^2,$$

$$\text{and } -3x^3 + -2x^2 + 2x + 3,$$

superficially analyzes them, and then attempts to graph the function. She struggles as to whether a and b should be represented by $-a$ and $-b$. She repeatedly attempts to graph polynomials entered in general form and some in factored form. She recognizes that the roots are squared at a and c, but does not know how to represent that condition in factored form. She tries values for a, b, c and d in polynomials of the form $ax^3 + bx^2 + cx + d$. She remembers that a, b, c and d must be inside parentheses, but does not remember how to do this. She claims her confusion is because they are variable and not numbers. She struggles to determine if the linear factors should be $(x - a^2)$ or $(x - a)^2$ and decides on the example

$$(x + 2)(x + 1^2)(x - 1)(x - 2^2).$$

Her continued investigation (with numerous brief pauses) is full of inquisitiveness and problem solving, without any semblance of frustration.

The teacher is relatively pleased with the student's work but is surprised by her lack of understanding linear factors, positive and negative roots, and the position of the exponent.

Through this and additional work (beyond the transcripts provided), Student 2 recognizes a number of aspects of the graph itself, including the far-left and far-right behaviour of graphs of polynomial functions; the association of zeros, roots, and x-intercepts between the graph and the equation; and the nature and effects of roots of odd and even multiplicity. However, the specific nature of linear factors together with their multiplicities remains an obvious gap in her knowledge; she is unsure if the factors should be $(x - a) \cdot (x - b)$ or $(x + a) \cdot (x + b)$, and she is confused regarding whether the exponentiation should be inside or outside the parentheses. Notably, she attempts to map a, b, c and d from the graph to the equation without understanding the interconnection of zeros and intercepts on a graph and zeros and real roots of a function. While this student has significant gaps in her knowledge, they are less so than for Student 1, and the teacher comes to better understand precise concepts with which the student struggles. Now the teacher recognizes the particular concepts that need to be addressed to complete the student's understanding.

Student 3 has an A+ average in the course. The teacher expects that he will fully master all the concepts in these tasks. Almost immediately, Student 3 recognizes that the polynomial is of even degree (at least 6) with a positive leading coefficient. He claims that a, b, c and d represent the zeros of the function and writes

$$y = (x - a)(x - b)^2(x - c)(x - d)^2,$$

then rewrites the expression as

$$y = e\,(x - a)^{odd}(x - b)^{even}(x - c)^{odd}(x - d)^{even},$$

where $e > 0$.

Student 3 has a strong understanding of mathematical concepts embedded in this task. He fluently understands both representations and can communicate such without effort. The context of the problem immediately directs him to the structures that are most important in both representations. Through observing this student perform the task, the teacher recognizes that she has not sufficiently challenged the student in respect to his ability and current understanding. She decides to provide him additional mathematically rich tasks targeted to additional concepts.

Assessment Summary

As seen in some summaries, the teacher was surprised at the understanding, misunderstandings and knowledge gaps that she was able to observe through student work and communication on the task. Even though these students had passed her previous traditional assessments on this topic, she was surprised by the degree to which they struggled in general and on which concepts in particular. Specifically, she was pleased by the targeted way the task revealed individualized precise concept understanding among the students and prescribed similarly precise and differentiated instruction to help each and all be successful.

Learning

The following excerpt describes Student 2's progress.

Approximately 45 minutes later, Student 2 realizes that the polynomial has to be raised to an even power to produce the correct left and right behaviour, but she does not know how to use parentheses to accomplish this. She decides to graph

$$y = (x + -3)(x + -1)(x - 1)(x - 3)$$

and other such cases. Through protracted trials, she recognizes that

$$y = (x + 3)(x + 1)(x - 1)(x - 3)$$

implies

$$y = (x - a)(x - b)(x - c)(x - d).$$

After more investigation, she recognizes that the graph reveals some single and some double roots; struggles to know which are which; recognizes the need to distinguish these through $(x - b^2)$ or $(x - b)^2$; writes

$$(x + 2)(x + 1)^2(x - 1)(x - 2)^2;$$

and after more thought and experimentation, rewrites this into

$$(x - a)(x - b)^2(x - c)(x - d)^2$$

and finally to

$$(x - a)^{\text{odd}}(x - b)^{\text{even}}(x - c)^{\text{odd}}(x - d)^{\text{even}}.$$

The teacher is pleased that the student learned through only one task within one class period, since after days spent previously covering the associated mathematical topics in class the student had not gained sufficient understanding.

This student received no assistance from the teacher or the interviewer, but was given sufficient time to work through the investigation. Fortunately, since she had previously experienced time-intensive problem-solving tasks, she was able to persevere through this task. The progression from Student 2's previous transcripts to this transcript (over the total span of about 90 minutes) demonstrates a growth from misunderstandings and knowledge gaps to understanding most of the associated concepts. Moreover, the concepts learned are now strongly interconnected both within each representation and between the two representations, rather than being treated disjointedly. Altogether, the teacher was pleased at the rapidity, efficiency and thoroughness of the student's learning and credited this success to the nature of the mathematically rich task and the protracted time allowed for its investigation.

Learning Summary

While Student 1's extensive misunderstandings and knowledge gaps significantly slowed his learning of the concepts, extended transcripts reveal that he learned many of the mathematical concepts, but at a slower pace than Student 2. The teacher was pleased with the learning of Student 1, but she stated that she believed that if a simpler version of the task had been provided before this one, the students would probably have done better on both tasks. The teacher decided to create concept-similar tasks that would scaffold to this type of example for this student (for example, begin with quadratic functions). Additionally, the teacher decided that she would allow this student to work with another student on some future tasks to simultaneously scaffold his learning and diminish his frustration. Since Student 3 was already familiar with most of the mathematical content associated with the task, the transcripts show little gain in understanding. The teacher decided that she could create parallel tasks (using transcendental functions) to challenge this student and lead him to more advanced concepts.

Implications for Instruction

As demonstrated above, the mathematically rich formative assessment task served its dual purposes of assessing student understanding, misunderstandings and knowledge gaps while providing them with an effective learning experience. As students responded to the task, their understanding and connections of mathematical concepts deepened. Through these tasks, teachers can assess much more than whether or not students can answer questions or perform mathematical calculations; student conceptual understanding of numerous embedded notions can be assessed, and teachers can use that information to plan further instruction.

Students with greater gaps in understanding tend to learn much from rich mathematical tasks, albeit at a slower pace than others. Initially, they balk at these unusual tasks in which they are not given explicit direction on how to complete the task or what the correct response may be. However, as these tasks become more common, students will warm to them. For these students, it may be best to initially scaffold their experiences by using versions of tasks that are differentiated for their specific needs before employing more complex tasks. These students may need to complete a greater number of these tasks than may their classmates. Since these students are often more prone to be frustrated in problem solving and have difficulty persevering in such, care must be taken to not break their spirits. Thus, it is valuable to limit the duration of the tasks initially and increase the duration of tasks as is tolerable. Allowing students to work with others, rather than independently, may also help them avoid being overly frustrated.

Students who are comfortable with more advanced mathematics should be given tasks that also meet their needs. Most mathematically rich tasks are easily modified to be deeper and more challenging. These students often enjoy such tasks. Students can be given these tasks prior to instruction on particular topics; they can learn through these tasks, sometimes even independently of an instructor. Also, students can be asked to create and solve their own rich mathematical tasks, leading to tremendous learning experiences.

What to Expect in the Classroom

Mathematically rich formative assessment tasks may seem more difficult initially than traditional classroom instructional questions, particularly if they are seen as unusual or unfamiliar. These tasks address or assess

Selecting the Mathematical Tasks

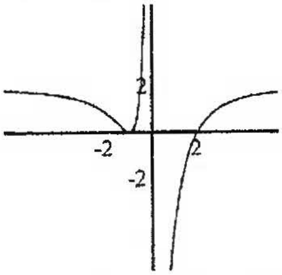
For any mathematical topic at any level, rich mathematics tasks are available. We provide additional examples applicable to high school. For each example, a variation differentiates the problem to be either more or less mathematically complex.

1. The following functions are equivalent, but in different algebraic forms. What information regarding the function is revealed or hidden in each of the forms?

$$f(x) = 2x^2 + 3x + 1$$
$$g(x) = x(2x + 3) + 1$$
$$h(x) = (2x + 1)(x + 1)$$

To make the task simpler: Provide options such as showing the function is a quadratic; showing its factors; revealing its roots; revealing its y-intercept; showing that it is concave up.

2. Explain why the accompanying function and graph are inconsistent.



$$f(x) = \frac{(x-2)^2(x + 1)}{x^2(x + 1)}$$

To make the task simpler: Use polynomial functions.

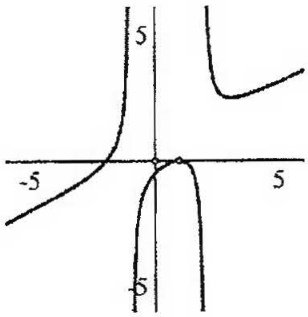
3. For $f(x) = 2x + 3$ and $g(x) = (x - 3)/2$, we find that $f(g(x)) = g(f(x))$. Is it usually the case that $f(g(x)) = g(f(x))$? Explain why or why not.

To make the task more complex: Explain necessary conditions for $f(g(x)) = g(f(x))$ to be true.

4. For ___x + ___ = ___x + ___, fill in the blanks such that the equation has one solution; no solution; an infinite set of solutions.

To make the task more complex: Create an equation including a quadratic and a linear function.

5. Without converting the graph below to an equation, explain everything you can about the graph and its respective function.



To make the task simpler: Use a polynomial function.

precise mathematical concepts and cause students to think more deeply about the mathematics at hand and the interconnectedness among mathematical concepts and representations. Although students may balk at these tasks at first, many students quickly come to enjoy the challenge and heartily participate in classroom discussions.

Classroom time must be planned for students to struggle with and learn through a mathematically rich task. Combining formative assessment and instruction focused on conceptual understanding can break the cycle of skills-based instruction, assessment, follow-up instruction and further assessment. Teachers must place some trust in students as learners and communicate high expectations to them. As students work through these tasks, their conceptual understanding can grow at an exceptional pace. When students show significant misunderstanding or knowledge gaps, teachers can intervene with instruction directed at particular concepts and scaffold understanding while not forfeiting time globally addressing concepts that students may have mastered.

There is a delicate balance between allowing students to persevere through the problem-solving task and providing them assistance before they become too frustrated and shut down. Classroom teachers must know their students well, adjust the task or the time allotted for the task appropriately for individual students and the class, and know when to intervene. We recommend that they allow learning to happen organically and not provide hints too quickly; jumping in to assist skews interpretations regarding what students know or learn.

Most of these tasks are excellent fodder for collaborative assessment and instruction. This practice elicits rich communication and dialogue among students, giving teachers greater access to student thinking and giving students access to greater learning. Teachers also enjoy students' robust mathematical dialogue.

The most obvious question for any teacher may now be, "But I have 30 students in my classroom! How can I possibly do this?" First, no one educational practice—even using mathematically rich tasks—is a panacea for all student learning. These tasks should supplement, and not completely replace, other instructional techniques. (See sidebar, Selecting the Mathematical Tasks.) Second, novel instructional techniques take time and practice to master. Third, when initially using these tasks, it may be beneficial to try them as either instructional or assessment tasks rather than integrating both.

We hope that this brief introduction to rich mathematical formative assessments will evoke interest in these tasks and encourage teachers to try them in their classrooms. The authors have used these tasks with great results. We hope others see their worth and enjoy them also.

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