Backing Up and Moving Forward in Fractional Understanding

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A well-crafted opening problem can provide preassessment of students' fraction knowledge and assist teachers in determining next steps for instruction.

After watching a demonstration lesson that exposed students' misunderstandings regarding division of fractions, a teacher shared this sentiment:

> When I read a standard, I think about what that standard says I have to teach and I find a way to teach it. I don't think about how far I need to back up. (Pamela, a Grade 5 math teacher)

In a discussion of the lesson among colleagues, two key ideas surfaced. First, standards such as those presented in the Common Core State Standards for Mathematics (CCSSM) (CCSSI 2010) "identify the end goal of a unit of instruction that encompasses more than a skill that may be taught in one or two lessons" (Barlow and Harmon 2012, 500). Second, carefully crafted word problems provide a means for identifying students' misconceptions (Barlow 2010) and guide the teacher in knowing how far to back up along the path of the learning trajectory. This process of backing up begins with using responses to a word problem to identify categories of students' understandings in relation to the expectations of the standard and using this information to make instructional decisions. In some instances, students will provide evidence of meeting or exceeding lesson expectations; instructional decisions, therefore, will need to advance their thinking. Instructional decisions for other students, who are working toward lesson expectations, should help them connect prior knowledge to new concepts. Students who are lacking fundamental understandings require instruction aimed at filling gaps in prior knowledge. The purpose of this article is to demonstrate this backing-up process—by examining categories of student work taken from a carefully crafted problem—and suggesting instructional decisions.

The Backing-Up Process

Use students' responses to a carefully crafted word problem to identify categories of understandings and to make instructional decisions.

1. When students exceed expectations, instructional decisions should advance their thinking toward an identified end goal. Build students' understanding, perhaps by using a different problem context or by using numbers that are more complicated.

2. When students meet lesson expectations, they are ready to begin exploring the new concept. Guide group discussions to attend to key aspects of the context to allow students to move deeply into the concept.

3. When students are working toward meeting lesson expectations, instructional decisions should help them connect their prior knowledge to new concepts. Supply supportive tasks that will prepare students.

4. When students lack fundamental understandings, aim instruction at filling in gaps in students' prior knowledge before expecting them to work toward the lesson expectations.

The Measuring Scoops Problem

To begin thinking about the backing-up process, we present a problem used in a professional development project that elicited students' fraction understandings. The first author created the Measuring Scoops problem using a problem-creation framework (Barlow 2010) with a goal of engaging students in interpreting the remainder of a division problem that involved repeated subtraction of a fractional quantity. The problem, which follows, is significant in terms of CCSSM content standard 6.NS.1 (CCSSI 2010).

> Chef Frederick is mixing ingredients to bake a dessert. His recipe calls for 2 1/2 cups of sugar. The only measuring scoop that Chef Frederick has measures 1/3 cup. How many measuring scoops of sugar will Chef Frederick need?

In thinking about this problem, several key features emerged that we considered important in terms of its ability to meet our instructional goal:

- Students are likely to be familiar with measuring scoops and will relate to the context of the problem.
- Measuring scoops represent different fractional amounts and can support students in counting with a fractional amount as the "unit."
- By using 2 1/2 and 1/3, students can represent the problem in a variety of ways, including drawings and manipulatives.
- The remainder of 1/2 can be identified visually, supporting students in making sense of the remainder. More specifically, they can see that the remainder is half of what they are counting.

To solve this problem, we anticipated students representing 2 1/2 cups with pictures or pattern blocks. Recognizing that they need to know how many scoops of size 1/3 cup are in 2 1/2 cups, students would divide their cups into thirds (representing the scoops) and then count the scoops or thirds. We expected a rich discussion regarding the remaining partial scoop's value. Is the remainder 1/2 or 1/6? We anticipated encouraging students to think about what they were counting to support making sense of this.

In our professional development project, we use demonstration lessons as a means for enhancing participants' knowledge of content and instructional strategies. During a demonstration lesson, one project team member teaches a lesson while the project participants observe. For this demonstration lesson, the first author implemented the Measuring Scoops problem in a participant's fifth-grade class of 20 students. About 50 project participants observed the lesson. Although the problem aligns with a standard from the sixth-grade curriculum, we felt it was appropriate for the fifth-grade class, given that the lesson occurred near the end of the school year. In addition, we were interested in the ideas brought by students who had not yet been taught the standard, which would likely not have been the case had we been in a sixth-grade classroom.

Although the Measuring Scoops problem was designed to support students in interpreting the remainder of a division problem involving fractions, the student work it generated supplied vital insights into students' understandings. This analysis of student work led Pamela to express the sentiment shared at the beginning of this article. Next, we will share this student work and demonstrate how the problem supported project staff and participants in thinking about the backing-up process.

Examining Student Work

Considering the purpose of the backing-up process, we deliberately made the choice to engage students in solving the Measuring Scoops problem even though they had no prior instruction on interpreting the remainder in fraction division. This allowed us to preassess students' understandings on the topic and gauge their readiness to learn, which is the intent of the backing-up process. Although previous student experiences included working with models as well as the algorithm for dividing fractions, we did not expect to have students who would meet the expectation of interpreting the remainder in fraction division. Doing so at this time would result in students exceeding our expectations for this lesson. Ideally, we expected students to make sense of the context of the problem, generate appropriate representations of the quantities involved, and select a reasonable approach to solve the problem. In reviewing students' responses to the problem, we found it useful to group the students' work into four categories:

1. Exceeding lesson expectations
2. Meeting lesson expectations
3. Working toward lesson expectations
4. Lacking fundamental understanding

We begin with an example of students who exceeded the lesson expectations and then move through the remaining categories.

Exceeding Lesson Expectations

Although students had not received instruction on the topic, we unexpectedly had a few students who were able to correctly interpret the remainder in the Measuring Scoops problem (see Figure 1). These students correctly modelled 2 1/2, separated it into 1/3 pieces (the scoops) and correctly counted 7 1/2 scoops. By correctly interpreting the remainder in this way, students exceeded our expectations for the lesson. We hypothesized, however, that the problem context supported these students with interpreting the remainder. Therefore, a teacher might offer these students the opportunity to interpret the remainder in a division problem in a different context, perhaps with less simple numbers. In this way, students would be able to engage in reasoning and recognizing patterns and thus build understanding.

Meeting Lesson Expectations

In general, students who meet the lesson expectations are ready to begin thinking about the new content contained in the standard (that is, the interpretation of a remainder). For the Measuring Scoops problem, students who meet the lesson expectations should demonstrate their ability to model fractions and use the fraction models to solve a division problem. Students began by drawing models for 2 1/2 and 1/3 (see Figure 2). Next, they drew 2 1/2 again but this time divided the wholes into thirds and labelled each third. Although they did not label the remainder with 1/6 in the model, we see on the right side of the poster that they used 1/6 in their check as well as in their solution statement. In thinking about this remainder piece, however, students did not attend to the problem context or the unit being counted (that is, thirds or scoops). As a result, they did not present evidence of meeting the expectations of the standard. They are ready, though, to begin thinking about interpreting the remainder. A teacher might use this example to facilitate a whole-class discussion regarding the meaning of the remainder for the Measuring Scoops problem. Such questions as the following might be useful in guiding this discussion.

- How can we deal with the fact that Chef Frederick has only a 1/3 scoop if he needs 1/6 of a cup of sugar?
- How can you report your solution in terms of one unit?
- What are you counting?

Working Toward Lesson Expectations

In some instances, students who are working toward lesson expectations will provide evidence of possessing foundational understandings but an inability to connect these to the problem context. Such students are not ready to think about the new mathematics contained in the standard but rather need support to be ready to learn it. For the Measuring Scoops problem, students must be able to model division of a whole number by a unit fraction as well as division of another fraction by the unit fraction. Figure 3 presents an example of

Figure 1. *Students had not been instructed on interpreting remainders. Nevertheless, some met lesson expectations, exceeding what the authors anticipated.*

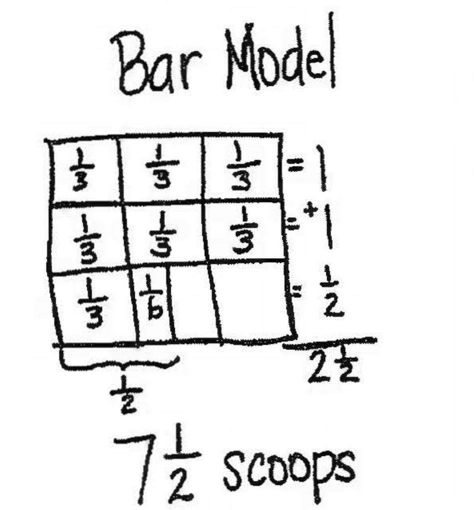


Figure 2. *Although this work shows students' alternative interpretation of remainders and their readiness to interpret remainders, failure to attend to the problem context and the unit being counted show a lack of evidence of meeting expectations of the standard.*

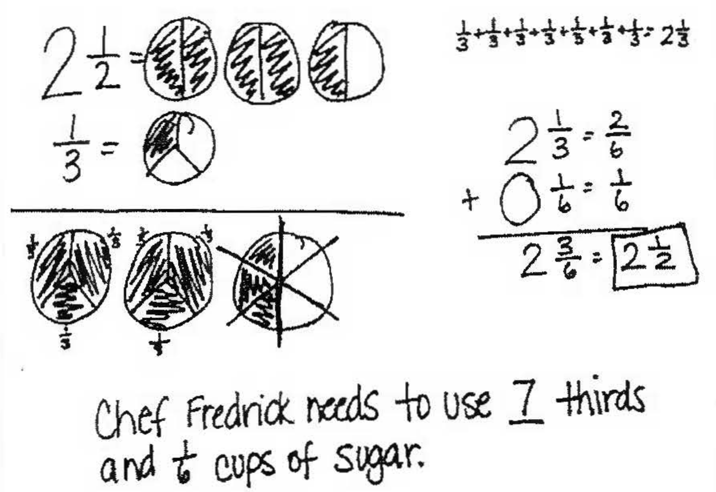


Figure 3. *For the Measuring Scoops problem, students must be able to model division of a whole number by a unit fraction as well as division of another fraction by the unit fraction.*

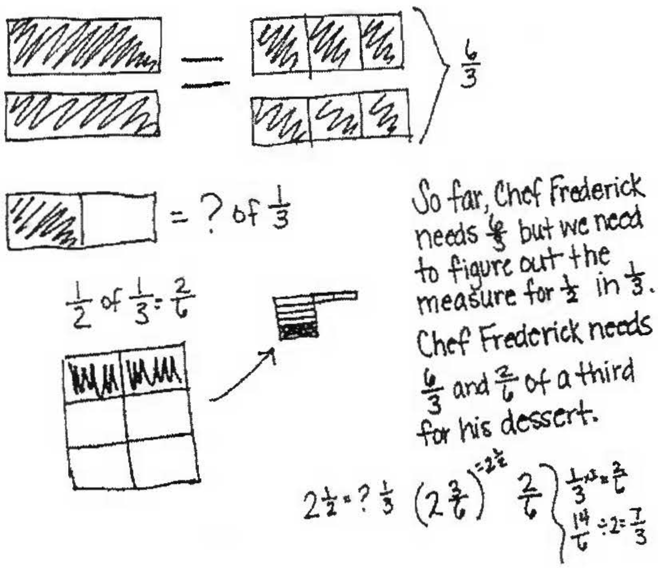
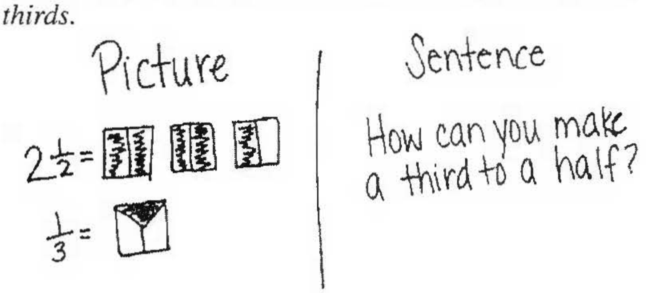


Figure 4. *This student work fails to accurately model thirds.*



this; students demonstrated 2 divided by 1/3 but were unable to extend this to modelling 1/2 divided by 1/3. The statement on the right side of the poster reads, "So far, Chef Frederick needs 6/3, but we need to figure out the measure for 1/2 in 1/3 (thirds)." This statement indicates that students were attempting to think about how many one-thirds are in one-half. However, as they attempted to find their solutions, students appeared to have gotten lost in their computations and models.

A teacher might ask questions concerning what the students were attempting to count during their fraction-by-fraction division or questions leading to a different model by which students might make sense of the problem. Follow-up tasks involving fraction-by-fraction division on appropriately marked grids may help these students progress in their thinking (Battista 2012) and eventually become ready to attend to interpretation of the remainder in fraction division.

Lacking Fundamental Understanding

Most classrooms will inevitably have students who lack fundamental understandings, which prevents them from being able to meaningfully engage in the intended topic. The ability to accurately model fractions is fundamental to modelling and solving division problems. On the left side of Figure 4, students have correctly modelled 2 1/2 and incorrectly modelled 1/3. Interestingly, the sentence on the right asks, "How can you make a third to a half?" indicating that they recognize the goal of the problem (that is, determining how many thirds are in 2 1/2). Their inability to model thirds, however, seems to be a stumbling block for beginning the solution process.

For these students, returning to basic understanding of fractions is essential. The introduction or reintroduction of manipulatives, such as pattern blocks, and returning to visual modelling of fractions can allow students entry into this problem (Battista 2012). However, expecting students to make sense of fraction operations, such as those represented in the Measuring Scoops problem, is unreasonable without first addressing these fundamental gaps.

Anticipating Roadblocks in the Backing-Up Process

The goal for using the Measuring Scoops problem was to preassess students' readiness for interpreting the remainder in fraction division by eliciting and understanding students' thinking. When examining student work in this way, though, a "roadblock" may sometimes be encountered if the work does not clearly align with one of the previously described categories. In these instances, additional questioning of the students is needed to better understand their readiness for interpreting the remainder. To help the reader anticipate potential roadblocks, we describe two examples in which students' work provided inconclusive evidence about students' understandings or misunderstandings related to the division of fractions, in general. In both cases, students produced work that held the potential for modelling division of fractions, but to draw conclusions regarding their understanding of fraction division would require too many assumptions on our part.

Anticipated Roadblock One

In some instances, students get lost in their work and lose sight of the problem goal. We see this in Figure 5. Here, students began by representing the problem with a bar model twice. They correctly drew and labelled thirds as well as sixths. In the process, though, they seem to have forgotten that they were counting thirds (for scoops). Instead, they began "putting the thirds back together" and announced that their answer was 2 1/2. In reviewing this work, it was problematic for us to determine what these students understood about fraction division and the remainder, making it difficult to categorize the work.

Anticipated Roadblock Two

A second roadblock involves students generating algorithmic statements using the numbers in the problem without considering the problem context. In Figure 6, students appear to have performed numerous calculations with the numbers that have been extracted from the problem. They began by subtracting 2 1/2 minus 1/3 in multiple ways. Then students began repeatedly adding thirds, arriving at 2 1/3, for which they then drew a bar model.

Figure 5. *Sometimes students lose sight of the problem goal. The authors had difficulty determining from students' work below what they understood about fraction division and remainders.*

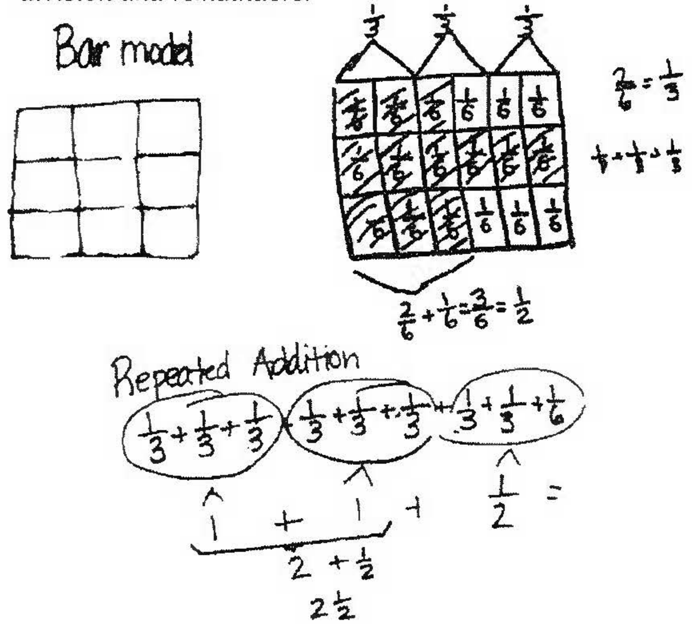
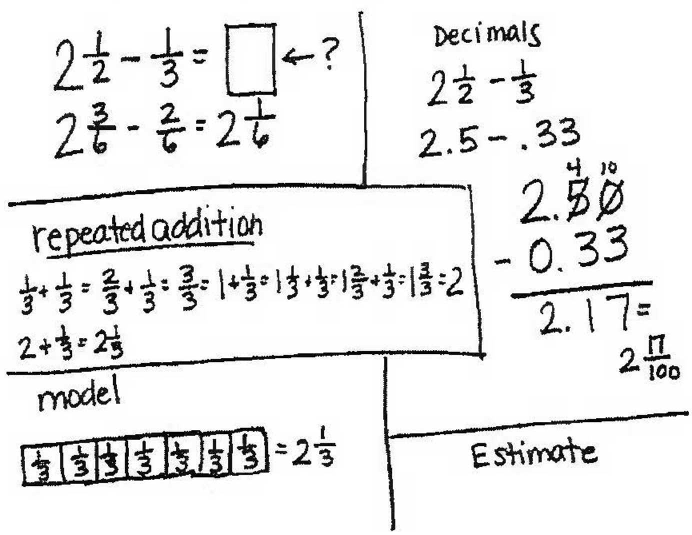


Figure 6. *This work focuses on calculations with extracted numbers but shows no evidence that students considered the problem context.*



Although their calculations appear to be correct, they have not provided evidence of ability to model division with fractions and to begin thinking about the meaning of the remainder. We could even hypothesize that these students did not recognize the problem as one involving division. However, the work alone does not clearly indicate an appropriate categorization.

Backing Up as Formative Assessment

This use of student work as formative assessment and as a driving force for instruction supports both the Teaching and Learning Principle and the Assessment Principle described in *Principles to Actions: Ensuring Mathematical Success for All* (NCTM 2014). Assessing student work in this fashion allowed project participants to see the reality of students' understandings of fraction-related concepts and to think about instructional strategies that would support student learning within each category of student work (that is, exceeding lesson expectations, meeting lesson expectations, working toward lesson expectations and lacking fundamental understandings). We began by posing a problem beyond students' current knowledge that allowed for multiple solution methods, provided opportunities to connect to prior knowledge and promoted productive struggle. By doing so, we

> embrace[d] a view of students' struggles as opportunities for delving more deeply into understanding the mathematical structure of problems and relationships among mathematical ideas, instead of simply seeking correct solutions. (NCTM 2014, 48)

As project staff and participants considered students' work within each category, we acknowledged that the final goal of understanding for all students could be accomplished only through incremental movement. Determining the instruction and intervention needed to facilitate this movement is one of our primary roles as mathematics teachers. This assessment-driven process for making instructional decisions is crucial in advancing our students' understanding of fractions. By starting with a carefully crafted problem, we were able to identify student understandings and misconceptions and make instructional choices by which we could guide students to our goal.

Common Core Connections

3.NF.1

5.NF.7

6.NS.1

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