Leibniz's Heuristic Derivation of the Product Rule and Quotient Rule

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The standard derivations of product rule and quotient rule are algebraic and involve adding zero in a clever way. These proofs are mathematically correct, but not pedagogically illuminating. We present Leibniz's heuristic derivation of product rule. A heuristic derivation of quotient rule, based on Leibniz's idea, is also given. While not mathematically rigourous, we believe an approach based on these ideas to be pedagogically superior.

Introduction

Let $x = 100$ and $y = 100{,}000$. Then $xy = 10$ million. Now let $x_1 = 97$ (3 per cent less than x) and $y_1 = 101{,}000$ (1 per cent more than y), and think about the following question:

Is xy greater than x_1y_1?

It turns out that $x_1y_1 = 9{,}797{,}000$, so xy is greater. In fact, xy is greater than x_1y_1 by 2.03 per cent.

That relative difference, 2.03 per cent, is a bit suspicious, and looks a lot like the sum of the relative difference in x (–3 per cent) and the relative difference in y (1 per cent).

As it turns out, that coincidence has an explanation, and that explanation gives us a heuristic proof of the product rule and of the quotient rule. The following is essentially Leibniz's derivation of product rule.

Product Rule

Given two numbers, x and y, change x by Δx and y by Δy. Now, the difference between the product of the new x and y and the old x and y is

$$\Delta(xy) = (x + \Delta x)\,(y + \Delta y) - xy,$$

and a routine calculation shows

$$\Delta(xy) = y\Delta x + x\Delta y + (\Delta x)(\Delta y),$$

and hence

$$\frac{\Delta(xy)}{xy} = \frac{\Delta x}{x} + \frac{\Delta y}{y} + \left(\frac{\Delta x}{x} \cdot \frac{\Delta y}{y}\right). \qquad (1)$$

Now if the relative changes in x and y are small, the third term in equation (1) is negligible and we have

$$\frac{\Delta(xy)}{xy} \approx \frac{\Delta x}{x} + \frac{\Delta y}{y}. \qquad (2)$$

In other words, *the relative change in a product is the sum of the relative changes in the factors.*

Using this heuristic, the product rule is almost immediate. For functions, the analogue of equation (2) is

$$\frac{(fg)'}{(fg)} = \frac{f'}{f} + \frac{g'}{g},$$

and multiplication by fg yields

$$(fg)' = f'g + fg',$$

which is the product rule.

Quotient Rule

Since the relative change in a product is the sum of the relative changes in its factors, it stands to reason that the relative change in a quotient is the difference of the relative change in the numerator and denominator. In fact, this is true. We leave it to the reader to verify that

$$\frac{\Delta(x/y)}{x/y} = \frac{y}{y+\Delta y} \cdot \frac{\Delta x}{x} - \frac{\Delta y}{y+\Delta y} .$$

Therefore, if the relative changes in x and y are small, one obtains

$$\frac{\Delta(x/y)}{x/y} \approx \frac{\Delta x}{x} - \frac{\Delta y}{y} . \tag{3}$$

The continuous analogue of equation (3) is

$$\frac{(f/g)'}{(f/g)} = \frac{f'}{f} - \frac{g'}{g} ,$$

which is equivalent to the quotient rule.

To be sure, our derivations (or should we say Leibniz's derivations) are not mathematically rigourous. We would argue the standard proofs presented in calculus classes are not entirely rigourous either, since they rely on an intuitive understanding of how one calculates limits, rather than definitions involving ϵ and δ. We believe Leibniz's idea has greater pedagogical value.