delta-k

Journal of the Mathematics Council of the Alberta Teachers' Association

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Mathematical Connections

Guidelines for Manuscripts

delta-K is a professional journal for mathematics teachers in Alberta. It is published twice a year to

- promote the professional development of mathematics educators, and
- stimulate thinking, explore new ideas and offer various viewpoints.

Submissions are requested that have a classroom as well as a scholarly focus. They may include

- personal explorations of significant classroom experiences;
- descriptions of innovative classroom and school practices;
- reviews or evaluations of instructional and curricular methods, programs or materials;
- discussions of trends, issues or policies;
- a specific focus on technology in the classroom; or
- a focus on the curriculum, professional and assessment standards of the NCTM.

Suggestions for Writers

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2. If a manuscript is accepted for publication, its author(s) will agree to transfer copyright to the Mathematics Council of the Alberta Teachers' Association for the republication, representation and distribution of the original and derivative material.
3. Peer-reviewed articles are normally 8–10 pages in length.
4. All manuscripts should be typewritten, double-spaced and properly referenced. All pages should be numbered.
5. The author's name and full address should be provided on a separate page. If an article has more than one author, the contact author must be clearly identified. Authors should avoid all other references that may reveal their identities to the reviewers.
6. All manuscripts should be submitted electronically, using Microsoft Word format.
7. Pictures or illustrations should be clearly labelled and placed where you want them to appear in the article. A caption and photo credit should accompany each photograph.
8. References and citations should be formatted consistently using *The Chicago Manual of Style's* author-date system or the American Psychological Association (APA) style manual.
9. If any student sample work is included, please provide a release letter from the student's parent/guardian allowing publication in the journal. The editor will provide this letter on request.
10. Letters to the editor, description of teaching practices or reviews of curriculum materials are welcome.
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MCATA Mission Statement

Providing leadership to encourage the continuing enhancement of teaching, learning and understanding mathematics.

delta-k

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From the Editor's Desk

Lorelei Boschman

I was reminded recently as I prepared for one of my undergraduate math courses about the beauty and exquisiteness in mathematics. The early cultures astounded us with their measurements, designs and depth of inquiry into mathematics. The golden ratio, found in nature, biology and even our bodies as well as embodied in art, music and architecture is absolutely stunning. We see beautiful geometric designs, tessellations, fractals and 3-D images that illustrate this mathematical beauty. Snowflakes, crystals of salt forming perfect cubes, tiling and Fermat equation generated surfaces dazzle us with this beauty. We see eloquent mathematical proofs and detailed solutions to problems. The beauty of logic, motion and statistics challenge us daily. To allow students to envision some of these aspects of math is wonderful. How can we teachers present some of these aspects to students throughout the curriculum? Can we help them to see the beauty that mathematics can create and that mathematics is?

A review by the *Economist* (1999) about Keith Devlin's book *The Language of Mathematics: Making the Invisible Visible* (1998), states: "Devlin succeeds both in giving us a glimpse of the internal beauty of [mathematics] and in demonstrating its usefulness in the external world." Devlin has expanded on a definition of math that one must truly stop and consider: mathematics is the identification and study of patterns (originally from W H Freeman's Scientific American series *Mathematics: The Science of Patterns*). When we consider the strands of mathematics, and especially the strands we currently focus on in today's curriculum, we can see that patterns exist in all of them. In fact, patterns are prevalent in all of them. The beauty of these patterns is something I truly hope all students, and people in general, can experience throughout their lives.

In this journal you can explore articles on group problem solving, university acceptance of high school mathematics and creativity. Ideas for lessons to use right away are provided in the Teaching Ideas section. Also included in this section is an article written by the Alberta Regional Professional Development Consortia (ARPDC) through the Elementary Mathematics Professional Learning (EMPL) initiative which discusses the equal sign. (There are a tremendous amount of resources to explore and use for all math levels at http://learning.arpdc.ab.ca. Click on the ARPDC Learning Portal and then click Elementary Mathematics Professional Learning Resources link.) Make sure to check out the Math Competitions section in this journal as well with great questions and solutions provided. As always, we've included conversation starters, problem-solving moments, a book review and some website highlights. We hope to create thought and discussion on mathematics for you personally as well as for your mathematics setting.

As an ending note, a recently released report entitled *Mathematics Review: Report to Premier and Minister* (December 2016) was compiled by the Mathematics Curriculum Review Working Group with various panels of educators' input. "The scope of the committee's work was to provide insight into the transition from K–12 mathematics to post-secondary mathematics courses from their perspective" (p 4). This report is well worth the time to read to get a truly current perspective on mathematics education. The link is https://education.alberta.ca/media/3402136/final_mathematics-curriculum-review_05dec16pdf.pdf.

Thank you for your continued thought, effort and dedication to providing excellent math learning opportunities for students. You provide their beginning glimpses and experiences in mathematics and take students through to their secondary development of mathematics principles and thought. You have a valuable role to play!

Reference

Economist. 1999. Review of *The Language of Mathematics: Making the Invisible Visible*, by Keith Devlin. www.economist.com/node/325505 (accessed May 5, 2017).

History of Mathematics and the Forgotten Century

Glen Van Brummelen

The history of mathematics is being reinvented. Over the past few decades, we have started to realize how delicate a matter it is to portray historical mathematics without distorting it with our modern viewpoints, especially if the subject is centuries old. For instance, we are now careful to avoid expressing, say,

> Elements 11.4: "If a straight line be cut at random, the square on the whole is equal to the squares on the segments and twice the rectangle contained by the segments."

as its algebraic equivalent, $(a + b)^2 = a^2 + b^2 + 2ab$. This representation changes the impact the theorem would have had on Euclid's audience, in this case obscuring its applications to irrational magnitudes and conic sections. This new sensitivity is a good thing. History isn't about translating ancient accomplishments into modern equivalents; it's about understanding how other communities and cultures thought differently from ours.

But we still have a long way to go. There is more to representing history than our choice of language. For instance, we still often select what mathematical topics in history to study by their interest and accessibility to us. As a result, the broader mathematical community considers the story of European mathematics to be not far off from the story of the development of today's university mathematics curriculum. We talk a great deal about the origins of calculus (not a bad thing, on its own!) and the associated pivotal transition of mathematics from its Euclidean life in the Platonic realm to interactions with the physical world. We also hear of the beginnings of analytic geometry, and more recent topics such as 19th-century analysis, the rise of abstractions resulting in abstract algebra and so forth. To some extent, by what we choose to discuss, we are still talking about us, not them.

Putting this to the test: quickly, name a 16th-century European mathematician. Names like Newton, Fermat, Descartes and Leibniz roll off our tongues easily . . . but they were all active in the 17th century. If pressed, some of us might come up with Girolamo Cardano, who solved the cubic equation, or François Viète, who is associated with the establishment of symbolic algebra. But that's about it. Was the 16th century really so sparse?

In fact, there is a flourishing community of scholars who focus on this period, but their efforts have not entered easily into the popular mathematical imagination. What has filtered through, in addition to Cardano and Viète, are several contributions to the emergence of algebra and the beginning of various notations, Rafael Bombelli's early discovery of complex numbers and the invention of decimal fractions. Again, these are topics related to the modern mathematics curriculum. Behind this, the literature contains crucial developments, many yet to be discovered. Some of the most important of these should lead us to reconsider whether or not it was calculus that brought European mathematics in contact with the physical world.

My own recent readings in my research area, the history of trigonometry, have brought these points home. Of course, I may well fall victim to my own critique that we tend to focus on historical topics selected by our own interests! However, most of this story is not well known. Spherical trigonometry and early approaches to mathematical astronomy, alas, are not on everyone's lips these days. This is a clear instance where shifting tides in today's school mathematics have obscured for us significant historical events in mathematics.

The fundamental work in trigonometry of the 16th century was Regiomontanus's *De triangulis omnimodis*, written in 1464 but not published until 1533. As the title indicates, it provided solutions to all types of triangles, both plane and spherical. His purpose, as with all such writings at the time, was to provide effective tools for mathematical astronomy. In fact, he referred to his book as "the foot of the ladder to the stars." Unlike surveying, the sciences, or other applications, astronomy was considered to be a fit subject for higher mathematics. When mathematics was needed for earthly matters, more elementary tools from "practical geometry" were used.

The middle of the century saw the appearance of the six nowstandard trigonometric functions in Georg Rheticus's 1551 *Canon doctrinae triangulorum*, and in this same work, a hint of the discovery of the 10 standard identities for right-angled spherical triangles. These results appeared explicitly in Viète's first mathematical work, *Canon mathematicus seu ad triangular* (1579), where, incidentally, we first see his propensity toward symbolic representations. But still, authors' eyes were fixed firmly on the goal of astronomy.

This began to change just a couple of years later. In 1581 Maurice Bressieu hesitantly included an appendix to his *Metrices astrvnomicae* that showed how to use trigonometry to find the altitude of a castle. Just two years later, Thomas Fincke's influential *Geometria rotundi* included an entire chapter devoted to the use of trigonometry in surveying. Mathematicians' enthusiasm for these new benefits of their work continued to accelerate; Bartholomew Pitiscus's 1600 *Trigonometriae* (the first appearance of the word) lists prominently on its title page geodesy, altimetry and geography, along with the more conventional astronomy and sundials.

Around the same time, Edmund Gunter and others were building instruments, such as his quadrant and his scale, to solve problems that could be used in navigation and other practical arts. These tools became popular, but they were not immediately accepted by the mathematical establishment. Interviewed by Henry Savile for the first post of Savilian chair of geometry at Oxford, Gunter demonstrated the amazing powers of his instruments. It is reported that Savile responded, "Do you call this reading of geometry? This is showing of tricks, man!"

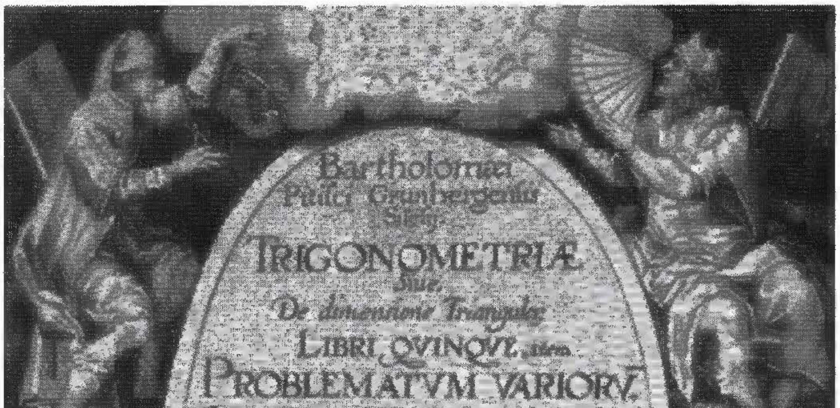

Figure 1. *Title page, Bartholomew Pitiscus,* Trigonometriae, *rev ed (1600), Rare Book and Manuscript Library, Columbia University in the City of New York. See www.maa.org/press/periodicals/convergence/mathematical-treasures-bartholomew-pitiscuss-trigonometry.*

Broader acceptance of mathematical methods received a major boost with John Napier's introduction of logarithms in his 1614 *Miriffci logarithmorum canonis descriptio*. Napier's purpose in this work was to streamline calculations especially in spherical trigonometry, which frequently requires the multiplication of irrational trigonometric quantities. Laplace later said that Napier, "by shortening the labours, doubled the life of the astronomer." But the biggest impact of logarithms was not heavenly, but earthly. It accelerated the acceptance of mathematics by practitioners. Authors like John Norwood started writing manuals demonstrating the use of the combined trigonometry and logarithms to facilitate calculations for topics such as military architecture. The world, truly, was becoming mathematized.

It was in this context that Galileo's famous quotation from *The Assayer* (1623), that the universe is written in the language of mathematics, was written. The inventions of analytic geometry, and later the calculus, were just around the corner. But the integration of mathematics into the physical world was well on its way before these innovations came along. With this episode, along with others, we might enrich our understanding of the history of mathematics by following a few paths that are now overgrown with weeds, but were once major thoroughfares.

Glen Van Brummelen is founding faculty member and coordinator of mathematics at Quest University (Squamish, BC). He is author of The Mathematics of the Heavens and the Earth *(Princeton 2009) and* Heavenly Mathematics *(Princeton 2013), and has served twice as CSHPM president. In January, he received the MAA's Haimo Award for distinguished teaching.*

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MIRIFICI

Logarithmorum
Canonis deſcriptio,

Ejuſque uſus, in utraque
Trigonometria; ut etiam in
omni Logiſtica Mathematica,
Ampliſsimi, Facillimi, &
expeditiſsimi explicatio.

Authore ac Inventore,
IOANNE NEPERO,
Barone Merchiſtonii,
&c. Scoto.

EDINBVRGI,
Ex officinâ ANDREÆ HART
Bibliopolæ, cIↄ. DC. XIV.

Figure 2. *Title page, John Napier,* Mirifici logarithmorum canonis description *(Edinburgh, 1614)*

Math PAT Changes Are Regressive and Counterproductive

Jonathan Teghtmeyer

When I was in the classroom, I taught mostly high school mathematics. I see great importance in mental mathematics, including the ability to easily recall math facts. Without these fundamental skills, students will continue to struggle in future years.

However, the biggest challenge I experienced in teaching mathematics was trying to overcome the deeply entrenched math phobia that affected so many students.

Consequently, I was very disappointed to learn that Alberta Education is introducing a 15-question, timed, no-calculator number section to the Grade 6 provincial achievement test (PAT).

Now, I should note that ministry officials are quick to argue that this new part A is not timed because teachers are provided the opportunity to extend the 15 minutes allocated to it. But I'm not convinced. Extra time allowed for part A comes from any extra time allowed for part B and is limited to a combined total of 30 minutes. Students who struggle on part A will be left with limited time to complete part B, and they will be rushed by the sight of students who have handed in part A and received their calculator to start on part B.

This undoubtedly will create higher stress for many students when it is neither helpful nor necessary. It simply exacerbates the test and math anxiety that exists for far too many students. A pressure test of basic number facts like this is regressive and counterproductive toward achieving the goal of ensuring that students have strong mental math and numeracy skills.

The department is fully aware of how these testing changes will reverse engineer instructional practice in schools—frankly, it's their intent. But I'm worried it will drive bad practices in schools, like the widespread reintroduction of "mad minutes," which do nothing to challenge strong students while simultaneously doing nothing to help the students who need it the most.

Some will confuse my concern with a desire to remove rigour or stress from education. Nothing could be further from the truth. I prefer authentic rigour and authentic stress that actually equip students with skills to endure and manage stress healthily.

Authentic stress is created by presenting students with problems and challenges where it's not immediately clear how to get the answer. Teachers ensure that the students have the knowledge, skills and resources required to get the answer but also let them struggle with how to get to it. This struggle and stress is actually quite healthy because students build confidence through discovering they have the skills and capacity to succeed. For students who don't have the required skills, forcing them to sit through a manufactured stress test only teaches them helplessness and reinforces any self-perception of failure.

But my biggest concern is how this testing change was made as a knee-jerk concession to back-to-basics crusaders who have manufactured a math crisis out of a moderate 6 per cent decline over 12 years in Programme for International Student Assessment (PISA) scores—a decline that still leaves Alberta tied for the tenth-best jurisdiction in the world. These vocal crusaders want Alberta to be more like the countries overtaking us, countries where rigid, regimental drill-and-kill instruction is combined with ridiculous amounts of after-school boot-camp-style tutoring.

If this perspective is winning the ear of Alberta Education on testing, then it will prove quite problematic as we enter new curriculum design. Our new curriculum should aim to achieve an Alberta-made vision of public education instead of trying to meet the goals established by an international economic cabal like the Organisation for Economic Co-operation and Development (OECD) (which administers the PISA tests). Our curriculum should strive to educate students for 2030, not 1950.

Then again, perhaps I'm overreacting. Minor tweaks to PATs will become irrelevant once the tests have been eliminated, as promised.

I welcome your comments—contact me at jonathan.teghtmeyer@ata.ab.ca.

Jonathan Teghtmeyer is the editor-in-chief of the ATA News.

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The Pirates and the Diamond

Peter Liljedahl

A band of 10 pirates is going to disband. They have divided up all of their gold, but there remains one giant diamond that cannot be divided. To decide who gets it, the captain puts all of the pirates (including himself) in a circle. Then he points at one person to begin. This person steps out of the circle, takes his gold and leaves. The person on his left stays in the circle, but the next person steps out. This continues with every second pirate leaving until there is only one left.

- Who should the captain point at if he wants to make sure he gets to keep the diamond for himself?
- What if there were 11 pirates? What if there were 12 pirates? What if there were 27 pirates and so on?

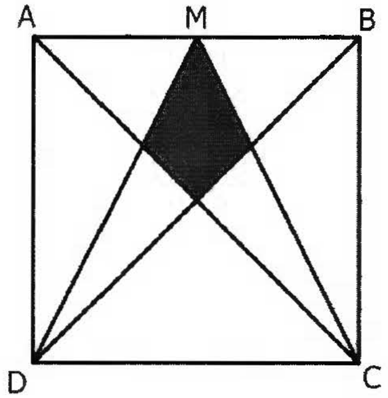
Kite in a Square

NRICH Mathematics

ABCD is a square. M is the midpoint of the side AB. By constructing the lines AC, MC, BD and MD, the shaded quadrilateral is formed.

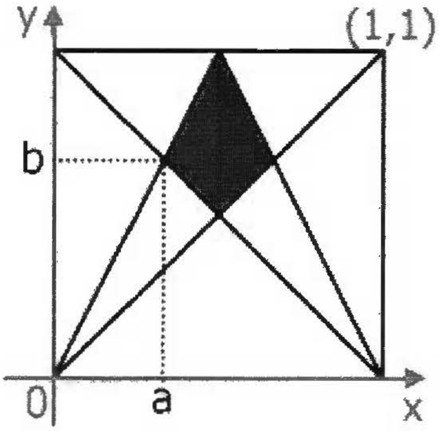
What fraction of the total area is shaded?

Below are three different methods for finding the shaded area. Unfortunately, the statements have been muddled up. Can you put them in the correct order?



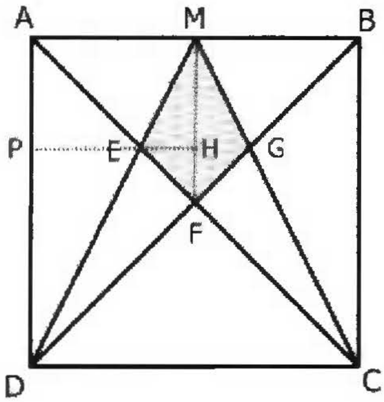
Coordinates

1. The shaded area is made up of two congruent triangles, one of which has vertices (1/3, 2/3), (1/2, 1/2), (1/2, 1).
2. The line joining (0, 0) to (1/2, 1) has equation $y = 2x$.
3. Area of the triangle = 1/2 (1/2 × 1/6) = 1/24.
4. The line joining (0, 1) to (1, 0) has equation $y = 1 - x$.
5. Therefore the shaded area is 2 × 1/24 = 1/12.
6. The point (a, b) is at the intersection of the lines $y = 2x$ and $y = 1 - x$.
7. Consider a unit square drawn on a coordinate grid.
8. The perpendicular height of the triangle is 1/2 – 1/3 = 1/6.
9. So $a = 1/3$, $b = 2/3$.
10. The line joining (0, 0) to (1, 1) has equation $y = x$.



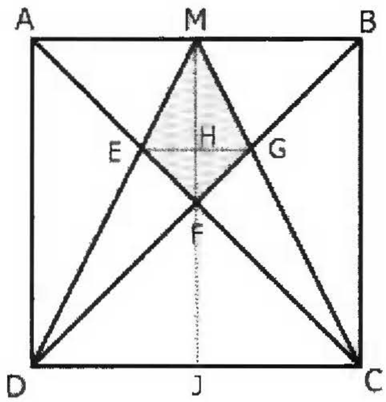
Similar Figures

1. As line AC intersects line MD at point E, the two opposite angles $\angle MEF$ and $\angle AED$ are equal.
2. The line MF is half the length of AD.
3. Line AD is parallel to line MF, so $\angle EDA$ and $\angle EMF$ are equal, and $\angle EAD$ and $\angle EFM$ are equal (alternate angles).
4. Therefore, ΔAED and ΔFEM are similar.
5. Therefore, the line EH is half the length of PE.
6. Let ABCD be a unit square.
7. Therefore, the shaded area MEFG = 1/24 × 2 = 1/12 sq units.
8. PH has length 1/2 units, so PE has length 1/3 units and EH has length 1/6 units.
9. ΔMEF has area 1/2 (1/2 × 1/6) = 1/24 sq units.



Pythagoras

1. The area of ΔDMC = 2 sq units. The area of ΔDFC=1 sq unit. Thus the combined area of ΔDFE, ΔCFG and shaded area $MEFG$ is 1 sq unit.
2. $(EH)^2 + (HF)^2 = (EF)^2$
   $EH = HF$
   $(EH)^2 = 1/2\ (EF)^2$
   EH = EF/2√
3. Areas of ΔDFE, ΔCFG and shaded area $MEFG$ are equal, so each must have an area of 1/3 sq units.
4. Area of ΔMEF = 1/2 (1 × EH) = 1/2 (EF/2√)
5. By Pythagoras, DF has length 2√.
6. The total area of the square is 4 sq units, so the shaded area is 1/12 the area of the whole square.
7. Area of ΔDFE = DF × EF/2 = 2√ × EF/2 = EF/2√
8. So the shaded area $MEFG$ is equal to the area of ΔDFE.
9. Assume that the sides of the square are each 2 units long. Thus, DJ and FJ are each 1 unit long.



What Problem Are They Posing? Viewing Group Problem Solving Through an Enactivist Lens

Nat Banting

Instructional practices that facilitate deep mathematical understanding and elicit student thinking were thrust into the limelight by the National Council of Teachers of Mathematics (NCTM) by two and a half decades' worth of influential curriculum documents (1989, 2000, 2014). Closer to home, the Western and Northern Canadian Protocol (2008) called for similar pedagogical approaches in the mathematics classroom, including the incorporation of meaningful student discussion as a channel for developing mathematical understanding. Group problem solving presents itself as a key structure for the classroom teacher attempting to fulfill these curricular mandates. For this reason, group work in the mathematics classroom warrants examination from various theoretical frameworks, each of which brings different implications for employing the structure in the mathematics classroom. In this article, group problem solving is viewed through an enactivist lens, which stresses the evolutionary nature of problem solving as the learners and the problem mutually define one another. In other words, a problem does not live outside of the solvers; it is the interaction between knowers and task that shapes the nature of the problem. A mathematical problem, then, takes on a plural character as the solution to a provided task may involve several instances of problem drift, where a new problem becomes the focus of attention because it emerges as a relevant inquiry in the course of action with the evolving mathematical environment. A portrait of group action is provided to illustrate the evolutionary process of coming to know, and the implications of problem drift for teacher action are discussed.

Problem Solving for the Social Constructivist

This shift in the interpretation of a problem-solving activity is better explained when compared to the learning theory through which group problem solving is usually supported: social constructivism. Though there are many branches of social constructivism (Ernest 2010), I use the work of Lev Vygotzky (1978) to structure the discussion because of the importance he placed on language development and his construct of the zone of proximal development (ZPD) as a vehicle for its development, both of which are often discussed in introductory psychology courses and as part of teacher training. According to this theory, learning occurs when students internalize subjective meanings that fit into their social (and mathematical) worlds. In order to pull learning forward, interactions must fall within the learners' ZPD, a theoretical space that consists of tasks a novice could not carry out without an expert's guidance. The process of intentionally supporting learners to go beyond what they could do individually is called scaffolding. Small-group cooperative work, then, activates students as scaffolds. In other words, it allows students to encounter those with more sophisticated knowledge and thus provides scaffolding toward an understanding of the mathematical problem. It is the job of the students, through the use of socially negotiated language, to fit their understanding to invariants in the environment. It is the job of the mathematics teacher to provide a mathematical environment, in the form of a task to be completed, and to mitigate the learners' personal meaning of that environment. Teaching, then, is the process of diagnosing where in the solution process the learners are and deciding what assistance will allow them to move forward.

Problem Solving for the Enactivist

Enactivism and social constructivism are not disjoint (Reid 1996); both learning theories treat social interaction as the impetus for learning. However, a key difference is that constructivism is concerned with issues of fit, whereby learners use experience in an environment to construct sense from it. Enactivism, on the other hand, is "not so much about the invariants within the environment, but about the coordination of the knower and the environment" (Proulx and Simmt 2013, 66). According to enactivist theory, environment and cognition evolve alongside one another. In the case of group problem solving in the mathematics class, the environment consists of both the given task and the other learners working with the task. The actions of participants negotiate their way through a world that is not fixed or pregiven, but rather one "that is continually shaped by the types of actions in which we engage" (Varela, Thompson and Rosch 1991). The nature of the mathematical task that the learners are operating with and the mathematical knowledge used to arrive at coherence are constantly influencing each other. Problem solving is thus a process of "dynamic co-emergence of knowing agent-and-known world" (Davis 1995, 8). Acting within a mathematical environment changes the nature of the environment. This allows the characteristics of the task to become problematic for the learners, and, thus, the environment to trigger further action.

Problem solvers enter a mathematical environment with patterns of action established through their past interactions with environments. In other words, learners bring their history of viable, mathematical action with them when they begin to solve a problem. Interaction with the new environment begins here. The feedback provided by interacting with the structure of the new environment shifts the nature of the environment. Because the group now has more information about the nature of the task, there is further impetus for action, which is reciprocated by a shifting of the mathematical task. This is the image of evolutionary coming to know, the process of bringing forth a world of mathematical significance (Kieren and Simmt 2009; Proulx and Simmt 2013) where problem and solvers concurrently define one another. The task is not fixed; it is defined by the solvers. The actions of the solvers are not fixed; they are proscribed by the task. Problem solving, then, involves a fundamental circularity between knowers and their environment as they mutually specify one another (Davis 1996). According to enactivist theory, learners are not appropriating an individual, subjective meaning through the mechanism of social interaction, as proponents of social constructivism would contend; rather, through their interaction with others and environment, learners are bringing forth—enacting—mathematical meaning together.

In order to solve problems, problem solvers have the ability to pose "relevant issues that need to be addressed at each moment. These issues are not pregiven, but are enacted from a background of action" (Varela, Thompson and Rosch 1991, 145). In other words, the mathematical problem does not reside in the task itself; the structure of the task only triggers action. Through an enactivist lens, "prompts are given, not problems. Problems become problems when knowers engage with them, when they pose them as problems to solve" (Proulx and Simmt 2013, 70). That is, the prompt is not thought to contain the mathematics to be internalized. Instead, mathematical knowledge "emerge[s] from the knower's interaction with the prompt, through posing what is relevant in the moment" (Proulx and Simmt 2013, 69). Problem solving, then, is the process of coordinating intelligent action in an ever-changing process of problem posing. In order to do this, learners pose the problem that they believe will focus their mathematical action toward a coordination with the task's requirements. These relevant problems evolve as the group acts with the task, and I have termed this shift in the relevant problem that focuses students' action *problem drift*. Viewed through the lens of enactivism, learning is the process whereby problem solvers redefine their action in relation to the shifts in the mathematical environment; it is the process of bringing forth mathematical significance. Teaching, then, is the process of providing information, orienting attention and coordinating the possible in the mathematical environment (Towers and Proulx 2013). It requires attuning to the problem of relevance and coordinating further encounters with the environment to continue the problem-solving (posing) process.

A Classroom Portrait of Problem Drift

To illustrate the evolutionary process of knowledge and the notion of problem drift, an excerpt from a group of Grade 11 students working with the Squares Task (Appendix A) is analyzed. The Squares Task was a two-part task that asked Piper, Ben and Carter to first count the total number of squares that did not contain a blacked-out portion on an eight-by-eight grid and to determine the position of the blacked-out portion that would maximize the number of such possible squares. The excerpt begins as the group addressed the second question of the task. The use of an ellipsis represents a gap in transcription; all names are pseudonyms.

Piper: Wouldn't the smartest thing to do be to put it in a corner so you can get the largest amount of area?

Ben: Yeah, because then you'll get the. . . .

Carter: But we aren't talking about the biggest squares, we're talking about the most squares.

. . .

Piper: Regardless, there's going to be 55, one-by-one squares.

Carter: Yeah. Some things can't change.

Ben: OK, well like, the most that we got was two-by-two, right? Do you think if we moved the square anywhere on the piece it would change the amount of two-by-twos?

Carter: Two-by-twos wouldn't change.

Piper: Three-by-threes wouldn't change either.

Carter: Because it still takes the same amount of area.

. . .

Ben: If we just blanked out a corner, what would change?

Carter: Two-by-twos would be the same because you're still having the same area.

Piper: Yup.

Carter: Three-by-three is going to be the same, but you do get four-by-fours and five-by-fives.

Piper: Exactly, so the corner is the best!

This interaction between Piper, Ben and Carter maps an example of problem drift as their action couples with the environment. Not included in the episode, for sake of brevity, is the solution strategy for question one of the squares task (see Appendix A). The group used a combination of multiplication and subtraction operations to count the number of one-by-one squares and then decided to use a system of dots (Figure 1) to count the two-by-twos. After all, the prompt asked them how many, and the counting strategy paired well.

As the group began to address question two of the squares task, there was a converging of past histories. Piper suggested that moving the square as far out of the way as possible would be most effective, and Carter was quick to remind her that the goal of the task was not to create the most space, but to create the most squares; this action revealed hesitancy toward using the mathematical idea of area to solve the problem. For Piper, the relevant problem might be framed as, "What is the largest possible square we could fit on the grid?", but Carter's action signalled that the relevant problem for him was, "How can we create the largest number of squares?" The structure of the problem offered both of these possibilities, but Piper and Carter's mathematical pasts led them in different initial directions. A social constructivist might interpret Piper and Carter as having constructed different understandings of the nature of the task, but an enactivist sees the onset of two very different worlds of mathematical significance—one interacting with concepts of size and area, and the other with systematized enumeration.

I, as the provider of the task, anticipated the relevant problem to become, "Where do you place the blacked-out portion to create the most squares?", but Piper's comment that the number of one-by-one squares would not change regardless of the placement caused the group to attend to a new problem that they deemed important to the resolution of the task; namely, "Which sizes of squares will be unaffected by movement of the blacked-out portion?" In other words, Piper's comments produced problem drift. This was immediately validated by the other group members. Ben extended this line of action by wondering if Piper's pattern would remain true for the two-by-two squares. At the end of this episode, the action of all three students centred on the problem that was deemed as relevant to move toward a solution. The group was not solving the problem I encountered ("Where do you place the blacked-out portion to create the most squares?"); rather, they were coordinating their actions to solve the new, drifted problem, "What is the largest possible square we can create that doesn't include the blacked-out portion?" This was something quite similar to the proposition made by Piper at the onset of action, but quite different than the counting of squares that they engaged in throughout the middle portion of their time with the task—to verify that the number of squares was unaffected by the movement of the blacked-out section. For the enactivist, the mathematics emerged through interaction with the prompt, and the prompt was reciprocally redefined by the group's action on it. The process of problem solving (posing) evolved within and alongside the mathematical environment. Piper, Ben and Carter all became convinced of the conjecture that if larger squares are possible, then more squares are possible, and their solution can be seen shaded in the bottom left corner of the grid in Figure 1.

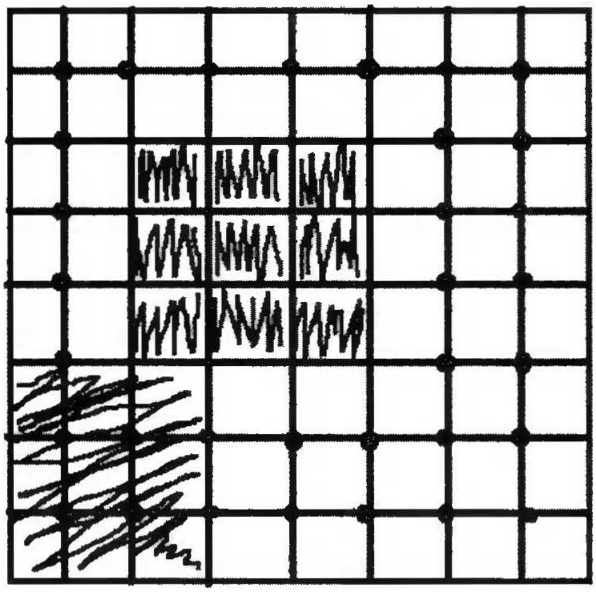


Figure 1. *The group's final workspace. When counting the two-by-two squares, each dot represented the centre of a two-by-two square. Their final solution was shaded in the bottom-left corner.*

The task afforded many possible actions; counting squares presented itself as viable for question one. Question two triggered the group toward a much different collective understanding, one where the notion of area emerged as crucial to the task. At the end of the episode, the group arrived at the understanding that maximizing the number of possible squares is the same as maximizing the area of the largest possible square. Their action led them to play with the notions of area and quantity in a geometric and arithmetic amalgam—a much different world of mathematical significance than was originally available to all three.

Implications for Teaching

How, then, should teachers act when viewing group problem solving through the lens of enactivism? The recognition of problem drift affects how teachers choose, prepare and enact problem solving (posing) activities in the classroom. First, the understanding that mathematics is an actively evolving phenomenon implies that classroom structure should provide spaces for student action. Problem-solving tasks should be designed to make students active producers (rather than strictly consumers) of mathematics, and provide arenas for worlds of significance to manifest. Teachers are called to include tasks with multiple avenues for conceptualization (like the arithmetic and geometric notions emerging through the Squares Task) and allow for student choice, conjecture, discussion, disagreement, justification and refutation. These visible (and audible) mathematical interchanges widen the sphere of mathematical action.

Second, problem drift requires teachers to be intentional in how they anticipate student responses to problem-solving (posing) activities. The goal is not to engineer a situation where students will act in particular ways; rather, it is to anticipate which features of the task will grab students' attention and to decide what intervention may trigger further student action. However, no matter how much teachers anticipate student interactions with the mathematical environment, the evolutionary character of knowing resists predictability. It is crucial that teachers, as they become fully involved in the knowing action in their classroom, remain flexible when student action breaches the sphere of anticipated strategy. To allow room for problem drift, lesson planning needs to give way to lesson preparing.

Third, while enacting problem-solving tasks, teachers must attune themselves to problem drift—what the group has posed as a suitable way forward. Attuning to a group's mathematical knowing begins with the identification of problem drift in the moments of teaching. This is more than the recognition that groups may find several solution pathways through a task or hold different constructions of the problem—it is the recognition that groups may actually be solving several different problems altogether. It means that the orienting question of teaching moves away from "How did they solve the problem?" and toward "What problem are they posing?" Viewing group problem solving (posing) through an enactivist lens allows the teacher to see the mathematical as plural, as emerging in the moment. We can no longer speak of pathways within a single problem, but must rather think of each problem as unique, emerging through the group's action with the task. The teacher is then tasked with anticipating possibilities, recognizing where the problem has drifted in classroom activity and acting with the groups to further the mathematical action.

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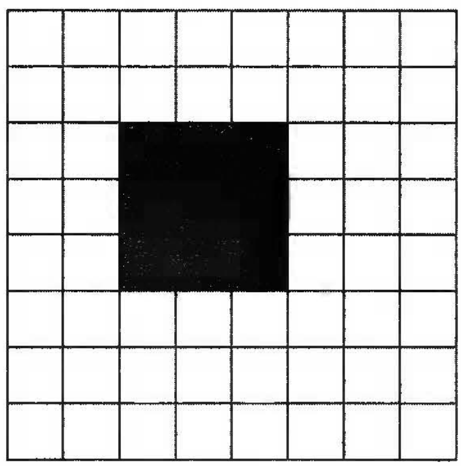
Appendix A



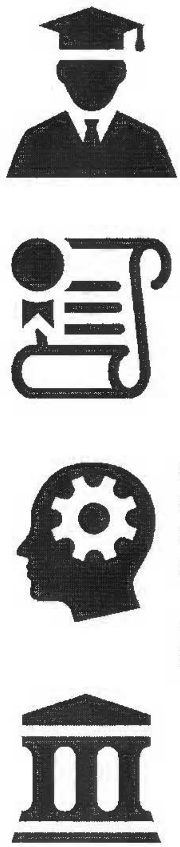
The Squares Task

1. How many squares (of any size) can be drawn on the above eight-by-eight grid that do not include any area from the blacked-out portion?
2. Where would you place the blacked-out portion to maximize the number of possible squares?

University Acceptance of High School Mathematics in Alberta

Richelle Marynowski and Landry Forand

The transition between secondary and postsecondary mathematics is a complex process which involves an array of problems and issues that students have to overcome. Some of the issues that students experience transitioning to postsecondary mathematics include "type of mathematics taught, conceptual understanding, procedural knowledge required to advance through material, and changes in the amount of advanced mathematical thinking needed" (Hong et al 2009, 878). Hong goes on to say that research on this topic indicates that the mathematical under-preparedness of students entering university is an issue. Additionally, Kajander and Lovric (2005) stated that "there is evidence of similar gaps in other disciplines in science and beyond, it seems that the transition in mathematics is by far the most serious and most problematic" (p 149). With this in mind, understanding the relationship between secondary mathematics and university mathematics is key to ensure the smoothest possible transition for students.



In addition, completion of algebra-intensive high school mathematics courses has been shown to improve the likelihood of students later completing a bachelor's degree at postsecondary. Adelman (1999), through the U S Department of Education, used data from a national longitudinal study that spanned from 1980 to 1993 to illustrate that of all the courses within the high school curricula, finishing high school mathematics courses had the strongest effect on future bachelor's degree completion. Trusty (2003) performed a similar study with 5,257 participants taken from the National Education Longitudinal Study finding that the greater number of math courses (trigonometry, pre-calculus, calculus and algebra 2) a student took, the greater the likelihood of that student completing a bachelor's degree. Furthermore, finishing even one algebraic math course more than doubled the likelihood of a student completing a future degree. The studies from Adelman and Trusty clearly illustrate that providing students with opportunities to take algebraic math classes will benefit them in future degree completion; thus, ensuring that students have the appropriate experiences with mathematics in high school is important to consider.

With both the transition experience of students and the content of high school mathematics courses impacting student degree completion in mind, we investigated the Alberta high school mathematics program of studies and its acceptance at certain universities in Alberta. Our main goal was to illustrate which high school mathematics courses in Alberta are being accepted at which universities in Alberta for which programs. With new curriculum currently being developed, it would be important for both secondary and postsecondary education to collaborate and maintain educational pathways full of options that do not limit student potential to be successful in postsecondary. Within the new curriculum, ensuring that students are able to access multiple pathways to get experience with algebraic mathematics would be beneficial for all students.

In 2010, Alberta's high school mathematics program was in the process of changing. New Grade 10 courses were being implemented in 2010 with Grade 11 following in 2011 and Grade 12 in 2012. The mathematics program that was introduced in 2010 was developed as part of the Western and North Canadian Protocol (WNCP) which included four provinces (Alberta, British Columbia, Manitoba and Saskatchewan) and three territories (Nunavut, Northwest Territories and Yukon). The WNCP's vision was to provide high-quality K–12 education for all students (System Improvement Group 2006). Specifically, the mathematics program was designed to provide students with the mathematical skills and competencies necessary to make a smooth transition from secondary mathematics studies to postsecondary programs and the world of work.

These new courses and course sequences replaced the Mathematics Pure and Applied course sequences that had been in place since 2000 (Alberta Learning

2002 a, b). One of the critiques of the Applied course sequence was that it was universally not accepted as a gateway to university entrance due to its lack of algebraic focus (System Improvement Group 2006). Students who were interested in pursuing bachelor's degrees were required to take Pure Mathematics 30, thus relegating the Applied course sequence to those who were seen as not going into university studies. The System Improvement Group report (2006) suggested three high school pathways that should satisfy stated admission requirements for the appropriate postsecondary programs or workforce requirements as follows:

1. Ensure that Pathway 1 satisfies the entrance requirements of the calculus-based (and similar) post-secondary programs with the fewest possible outcomes.
2. Ensure that Pathway 2 satisfies the entrance requirements of most of the remainder of the non-calculus-based programs.
3. Ensure that Pathway 3 satisfies the entrance requirements of the trades and agriculture, and with a few additions, meets the needs of business and industry for positions for which they recruit from high schools. (p 57)

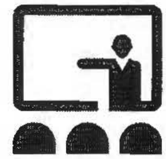
While the other WNCP provinces accepted the three pathways as articulated above, Alberta altered the content of Pathway 2 in order to satisfy additional algebraic requirements as articulated by Alberta's postsecondary institutions to ensure acceptance of Pathway 2 for university entrance. As the focus of this paper is illustrating the Alberta university acceptance of mathematics courses in Alberta, we will focus on the first two pathways.

The course descriptions from the Alberta Program of Studies for the "-1" and "-2" course sequences that were introduced in 2010 are as follows:

> "-1" Course Sequence: This course sequence is designed to provide students with the mathematical understandings and critical-thinking skills identified for entry into postsecondary programs that require the study of calculus. Topics include algebra and number; measurement; relations and functions; trigonometry; and permutations, combinations and binomial theorem. (Alberta Education 2008, 10)
>
> "-2" Course Sequence: This course sequence is designed to provide students with the mathematical understandings and critical-thinking skills identified for post-secondary studies in programs that do not require the study of calculus. Topics include geometry, measurement, number and logic, logical reasoning, relations and functions, statistics, and probability. (Alberta Education, 2008, 10)

Both the "-1" and "-2" sequence descriptions advocate for entry into postsecondary programs. The "-1" is for programs that entail calculus, and the "-2" for studies that do not require calculus. Even though the high school mathematics course sequences had changed to better align with university expectations, teachers were unsure if universities would accept the "-2" sequence as an entrance requirement for those programs that do not require calculus. This paper illustrates the acceptance of the "-1" and "-2" sequences at five universities in Alberta in order to highlight potential opportunities for students based on secondary mathematics courses.

To illustrate the current acceptance of Alberta high school mathematics courses, we investigated the entrance requirements of five Alberta universities: Mount Royal University, MacEwan University, the University of Calgary, the University of Alberta and the University of Lethbridge. These universities were chosen as they represent five of the six largest universities in Alberta. The data sources included the university's websites, course outlines and academic calendars from summer 2016. We obtained entrance requirements for all programs at the institution as well as the listed prerequisites for entry level mathematics courses. In our investigation we also included which university programs and individual mathematics courses listed Math 31 as a prerequisite. Math 31 is the high school level calculus course in Alberta that did not go through a redesign with the WNCP course development thus it has been unchanged in Alberta since its introduction in 1995 (Alberta Education 1995).



In Figure 1, the faculties and programs that require mathematics in order for students to be accepted are illustrated. Only programs that require mathematics as a prerequisite were listed along with the specific course that was identified as the prerequisite. If a program is not listed, it did not state a high school mathematics course as a prerequisite.

At the University of Lethbridge, there are 12 programs in the Faculty of Arts and Science and the Faculty of Management that require Math 30-1 for admission. Of these programs, five of them do not need calculus to progress through the degree. With

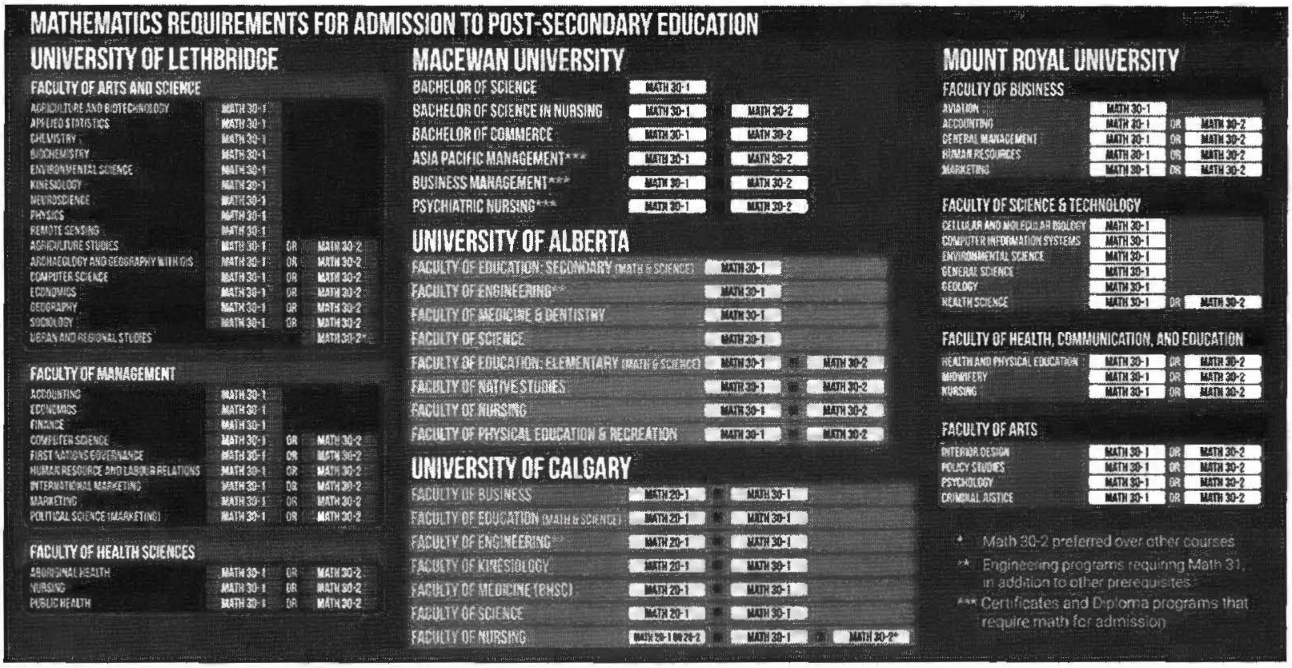


Figure 1. *Illustration of the mathematics requirements for entrance to specific programs at five universities in Alberta (Marynowski and Forand 2016).*

respect to the University of Calgary, the Faculty of Nursing is the only faculty that not only accepts Math 30-2 but also gives preference to Math 30-2 over the other mathematics courses. Of all the faculties at the University of Alberta, four of them accept Math 30-2 as a prerequisite.

In addition to mapping out specific program mathematics requirements, the prerequisites for individual first-year university mathematics courses were identified. Figure 2 illustrates how many first-year university mathematics courses require which high school math course as a prerequisite. The intent of the mapping was to visually display the predominant acceptance of Math 30-1 as a prerequisite for university mathematics courses and to illustrate that Math 30-2 sequence is also accepted as a university mathematics course prerequisite, only on a smaller scale. Only the university mathematics courses that required a high school mathematics course as a prerequisite were included in the graph.

A total of 40 first-year mathematics courses are offered by the five universities shown above. Of those courses, Math 30-1 is indicated as a prerequisite for every introductory university mathematics course. Math 30-2 is accepted as a prerequisite for 11 of the 40 courses while Math 31 (calculus) is a prerequisite for 8 courses. Of the 40 university mathematics courses, only 15 of them are identified as calculus courses, implying that Math 30-2 could lead to 25 university courses based on the Alberta Education description for the "-2" sequence. The acceptance of the Math 30-2 course as a university prerequisite is

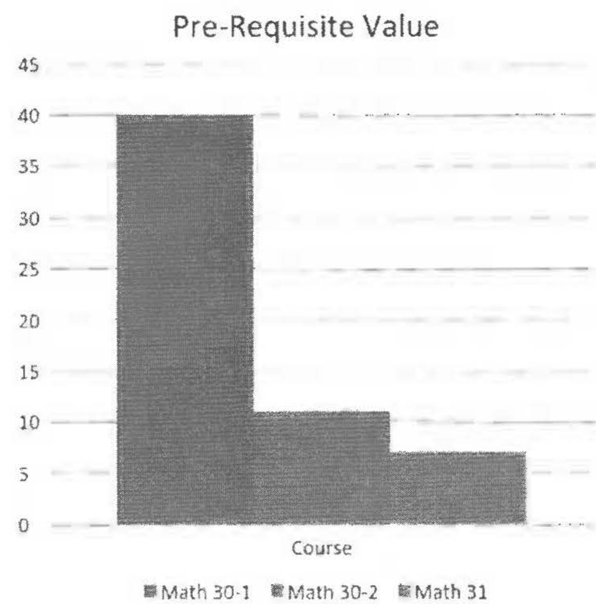


Figure 2. *First-year university mathematics course prerequisites.*

a positive development from the Applied Mathematics sequence that existed prior to 2010.

The intent of the current math curriculum in Alberta was to make both Math 30-1 and Math 30-2 acceptable prerequisites in Alberta. As shown above, the "-1" sequence is much more prominent and does not limit the student's choice when looking into postsecondary options. Math 30-1, in comparison to the previous mathematics course Pure Mathematics 30, has fewer outcomes and concentrates on the preparation of students to be successful in calculus. Content like statistics and probability are no longer included in the "-1" sequence with the focus of the course sequence being algebraic manipulations and algebraic representations of functions in preparation for students to study calculus. Math 30-2 in the meantime includes statistics, logical reasoning and set theory which are content that is applicable for noncalculus courses.

According to Alberta Education (2008, 10), "each course sequence is designed to provide students with the mathematical understandings, rigour and critical-thinking skills that have been identified for specific postsecondary programs of study." The intention for the "-2" course sequence was not to be easier than "-1" but to provide opportunities to engage with different content. Ideally, students choose which mathematics course sequence to take depending on the topics covered and postsecondary aspirations (Alberta Education 2008) rather than on the difficulty of the course. However, the legacy that Applied Mathematics left in Alberta as a course that was perceived by teachers and students to be less rigourous than Pure Mathematics might be impacting the implementation of the intended rigour of the "-2" sequence.

Despite both the "-1" and "-2" sequences being designed for students entering postsecondary, Math 30-2 is not accepted as widely as it might be. However, knowing specifically why the "-2" course is not accepted for noncalculus courses at these five universities in Alberta cannot be determined from this data. Further exploration is needed to identify what, if anything, is prohibiting the "-2" course sequence from being more widely accepted and to gather information from colleges and technical institutes in Alberta with respect to what secondary mathematics courses serve as prerequisites for their programs and courses. The intent of this article was primarily to highlight improvements in the acceptance of the "-2" sequence over the Applied sequence and secondarily to inform secondary educators as to the university programs that accept the existing mathematics courses so that they can inform their students of the options available to them.

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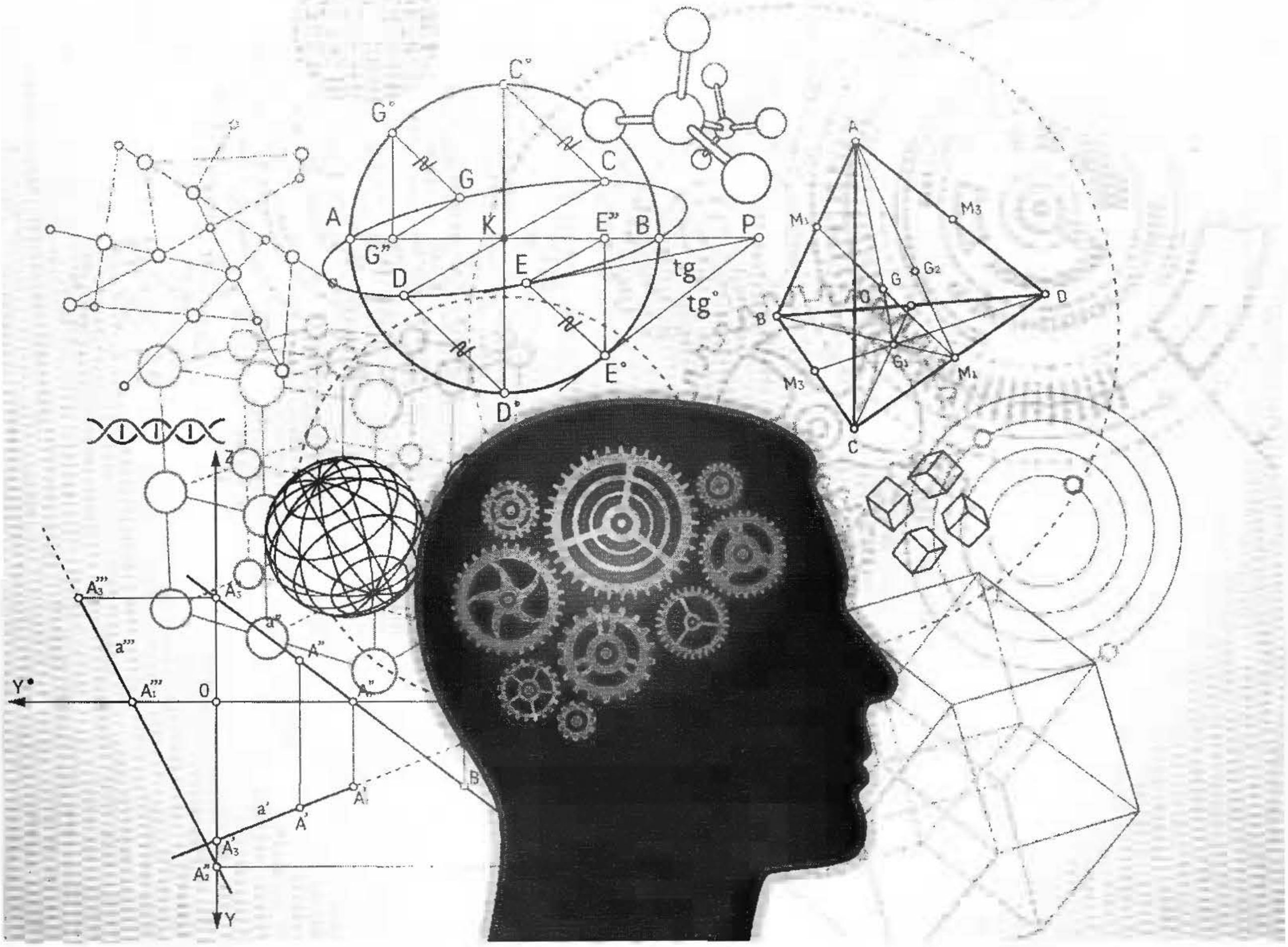
Creativity in Mathematics Classroom Settings: Recognizing the Collective Level

Ayman Aljarrah

Introduction

Researchers see creativity as an essential life skill and recommend that it should be fostered by the education system (Burnard and White 2008; Craft 2000; Torrance 1988). For example, Burnard and White (2008) suggested that creativity is needed to meet the multiple needs of life in the 21st century, which calls for enhanced skills of adaptation, flexibility, initiative and the ability to use knowledge in different ways. A glance at the literature on both creativity and "education reform efforts" asserts that creativity in the classroom is not an added frill to be taken or left; on the contrary, it is an important thinking and acting skill that should be fostered. It is "now considered good for economies, good for society, good for communities and good for education" (Burnard and White 2008, 669). Friesen and Jardine (2009) argued that in today's globalized context, everyone needs robust, rigorous thinking abilities and skills—one of which is creativity—that not only the labour market increasingly calls for but also life in all its manifestations.

Sawyer has done extensive work in the field of creativity, in particular identifying creativity as a collaborative emergent phenomenon (for example, Sawyer 1999, 2001, 2003, 2011). According to him, creativity is an emergent phenomenon that results "from the collective activity of social groups....



Although collaborative emergence results from the interactions of individuals, these phenomena cannot be understood by simply analysing the members of the group individually" (Sawyer 1999, 449).

As my concern here is classroom settings, it is important to point out that Vygotsky (2004) recognized the importance of the development of creativity in the process of constructing a human collective. For him, creativity should be conceived as essentially collective, and it is the pedagogue's responsibility "to create collaborative, imaginative, and ethical classroom communities that could empower and motivate teachers and students" (Knapp 2006, 108). These suggestions are congruent to both Davis's (2005) argument that "the classroom community can and should be understood as a learner—not a collection of learners, but a collective learner" (p 87), and Martin and Towers's (2003) suggestion to consider levels other than the individual at which mathematical understanding may emerge in classroom settings, namely, the collective.

As much as this glance at the literature explains the importance of creativity in education, it also includes implicit and explicit suggestions to go beyond the individualist view of creativity. These suggestions are supported by findings of studies which tried to combine collectivity and creativity. For example, a study of collective creativity in the workplace by Hargadon and Bechky (2006) considered collective creativity to emerge when the social interactions between individuals yielded new interpretations that could not be generated by an individual working alone. Moreover, Sanders (2001) argued that collective creativity can be very powerful and lead to more culturally relevant results than does individual creativity. In relation to collective creativity in mathematics, a study by Levenson (2011) found that working as a collective may actually encourage students to persevere and try new ideas. In addition, Sarmiento and Stahl (2008) found that creativity is often rooted in social interactions and that innovative creations should often be attributed to collectivities as a feature of their group cognition.

Creativity and Education

Huebner (1967) asked "how does a person learn to be creative?" According to him, "the very question itself demands a definition of the word creative" (p 134). Huebner put forward the possibility that creativity is not learned but is an aspect of human nature. Huebner argued that there is much theological thought that supports this idea. Therefore, "it would be more appropriate to ask what prevents creativity than to ask how one learns to be creative" (p 134). It may be possible that creativity is not confined to special people or to particular arts-based activities, nor is it undisciplined play. Craft (2000) described it as "a state of mind in which all our intelligences are working together [involving] seeing, thinking and innovating" (p 38), and the NACCCE report (1999) defines it as "imaginative activity fashioned so as to produce outcomes that are both original and of value" (p 29).

A brief tracking of the origins and uses of the word *creativity* in different cultures indicates that this word reflects a kind of biological fruitfulness, which means to bring something new into being. This definition is why most scholars in the field of creativity suggest newness and fruitfulness as two criteria for judging creativity. The richness of the word creativity, which can be seen through its multiple synonyms (for example, innovation, imagination, inspiration, novelty, originality, resourcefulness and so on), requires a kind of description that can reflect such richness.

In the field of mathematics education, Sinclair, Freitas and Ferrara (2013) used a sociocultural approach to frame creativity in a mathematics classroom. Their approach "emphasizes the social and material nature of creative acts" (p 239), and it does not conceive of creativity as a property or competency of children, but as emergent from their actions and doings. According to Sinclair, Freitas and Ferrara (2013), creative acts occur in the confluence of material agency, the people in the classroom agencies and the agency of the mathematical discipline, and they "collectively engender … a new space, which enable[s] new forms of arguments to emerge" (p 251). Such acts introduce or catalyze the new, they are unusual, unexpected or unscripted, and they cannot be exhausted by existent meaning.

Collectivity and Education

Gathering together, as a collective of all our diversities, stories and perspectives, lays the ground for effective problem solving, which also requires creative collaboration. Although diversity may increase the difficulty of collaborating, it also can make our experiences richer, worthier and more memorable. It "increases the creativity and wisdom of solutions" (Gray 1989, 13), and it "increases acceptance and support for creative ideas" (Isaksen 1994, 2). According to van Osch and Avital (2010), collectivity "refers to the collective and collaborative engagement of a group of people (i.e., a community) with shared interests or goals in meaningful actions" (p 5). It is a kind of a learning collective in which the focus on "the activities and insights of the collective [does not] mean to erase or to minimize the activities or insights of individuals" (Davis and Simmt 2003, 147). On the contrary, working collectively can "make space for, and support the development of, individual student's ideas" (p 147), and offer opportunities for all of the participants to be more creative (Davis, Sumara and Luce-Kapler 2008). Davis (2012) assumed that individual and collective knowing are inseparable, interrelated and interwoven.

It might be important to start to think how we can—those who gather together in the classroom; teachers and students—start "thinking the world together" (Jardine, Clifford and Friesen 2003). The starting point in this great, imaginative and exciting adventure lies, as Pratt (2006) explained it, in "the willingness of the teacher to be re-positioned, not as knower but as a significant participant" (p 93). Such a participatory approach to teaching, triggers the emergence of "collective, momentary, situated knowledge" (p 93), and this is how the knowledge is collectively created. Pratt (2006) stated that "experiences, interpretations, learning, teaching, epistemologies, all of these are dynamic negotiations that occur in-between, neither yours nor mine, yet both of ours" (p 94).

Some informing studies embraced the collective process in mathematics education (for example, Martin and Towers 2003, 2009; Martin, Towers and Pirie 2006). For example, Martin and Towers (2003) suggested that students' collaborative work and "improvisational performances" in mathematics trigger the emergence and the evolving of collective mathematical understanding. According to them, collective mathematical understanding "is a phenomenon that emerges and exists in collective action and interaction" (p 251). Martin, Towers and Pirie (2006) described collective mathematical understanding as an emergent and gradually growing phenomenon, which cannot be traced to the individual learners, but emerges from their coactions as a collective. By coactions, Martin, Towers and Pirie (2006) refer to specific kinds of mathematical actions that are carried out by the members of a group, and that, at the same time, are "dependent and contingent upon the actions of the others in the group. [They] can only be meaningfully interpreted in light of, and with careful reference to, the interdependent actions of the others in the group" (p 156).

Martin, Towers and Pirie (2006) described doing and understanding mathematics as "a creative process, and thus believe that because mathematical understanding can grow at both the individual and the collective level (and will be different in the two contexts) it is necessary to consider it at both levels" (p 176). Through coacting, the mathematical ideas and actions "stemming from an individual learner, become taken up, built on, developed, reworked, and elaborated by others, and thus emerge as shared understandings for and across the group, rather than remaining located within any one individual" (p 157).

Collective Creativity in Classroom Settings

A dominant and an unresolved challenge in studying creativity in classroom settings is to find a well-established definition that is widely accepted and applicable in such settings. According to Torrance (1988), although there have been many attempts to define creativity, it still defies a precise definition. According to him, it seems unseen, nonverbal and unconscious, but it also involves every sense and extrasensory perception. Despite such claims about creativity, when we want to study creativity, and/or educate for creativity, it seems unavoidable to approximate a description as a framework. While trying to do this, it is important to keep in mind that creativity in real life exists in many different forms (Tardif and Sternberg 1988). Therefore, I believe that it will be more appropriate to describe creativity in classroom settings based on the actions and doings of the classroom community while they are working on worthwhile problematic situations, ones that require a learner or a group of learners "to develop a more productive way of thinking about [them]" (Lesh and Zawojewski 2007, 782).

Based on a brief review of the literature about creativity in different contexts and at different levels, I found that although scholars in pedagogy, mathematics education and teacher education have generated a solid literature base promoting learning for individual creativity, the fostering of individual creativity and characterizing mathematical creativity (Leikin 2009; Silver 1997; Sriraman 2009), only a few of the current approaches to creativity are suited to the distributed and collective enterprise of the classroom (Levenson 2011; Sinclair, de Freitas and Ferrara 2013). This does not mean that earlier accounts are wrong or unfruitful; on the contrary they provide food for thought concerning creativity in mathematics education. They may, however, be incomplete, given that they mostly restrict themselves to one path, vision, description or experience of creativity. Because of such incompleteness "people seem to be talking past each other" (Klein 2013, 108).

Based on both an interpretive review of the literature on creativity and collectivity and many problem-solving sessions with groups of learners, I perceive collective creative acts as the actions, coactions and interactions of a group of curious learners, while they are working on an engaging problematic situation. Such acts, which may include (1) overcoming obstacles, (2) divergent thinking, (3) assembling things in new ways, (4) route-finding, (5) expanding possibilities, (6) collaborative emergence and (7) originating, trigger the new and the crucial to emerge and evolve (Martin and Towers 2003, 2009, 2011; Martin, Towers and Pirie 2006; Sinclair, de Freitas and Ferrara 2013).

I based my description of collective mathematical creativity on three elements: (1) an assumption that creativity is not a property or competency of children, but rather is an emergent from their collaborative actions and doings (Martin, Towers and Pirie 2006; Sawyer 2003; Sinclair, de Freitas and Ferrara 2013), (2) the origins and uses of the word creativity that reflect a kind of biological fruitfulness, which means to bring something new and crucial into being and (3) as suggested in the seven metaphors above, that can be used to describe the experience of creativity as it emerges in classroom contexts (Klein 2013). This is an attempt to add to our understanding of this phenomenon, and consequently to transform our practice as educators by thinking about how to create and offer genuine classroom opportunities for students to exercise creativity; opportunities that have the potential to transform the classroom into a space of expanding possibilities.

My suggested description of collective mathematical creativity indicates that the starting point to trigger collective creativity in mathematics learning environments is to create and offer genuine classroom opportunities for students to practise collective creativity: opportunities that encourage students to do what real mathematicians do. According to the NCTM (2000), to enrich students' mathematical experiences, deepen their knowledge, and enhance their opportunities and options for shaping their futures, we need to promote their understanding and applying of mathematics, and to engage them in what Davis (1996) named the mathematical, which he used to refer to "inquiry which has allowed our mathematics to emerge. It involves a noticing of sameness, pattern and regularity amid one's explorations. It involves comparing, ordering, creating, and naming" (p 93). And, it involves a dialogical conversation about, and "an active and intersubjective questioning of the world" (p 94).

As I noted before, doing and understanding mathematics are usually described as creative acts (Martin, Towers and Pirie 2006). To do mathematics, according to King (1992), means to produce mathematics that is new and significant. Herein, creativity is not the final end product that results from students' interactions and coactions while they are working on a mathematical task; rather creativity is located in the coactions and interactions themselves that result in what might be considered as new and significant to, at least, the local classroom community.

Martin, Towers and Pirie (2006) offered some suggestions regarding tasks that have the potential to prompt mathematical doing and understanding. For example, they should be open-ended tasks that allow for a variety of responses and invite a variety of paths, and they should be at an appropriate mathematical level. In addition, such tasks should encourage students to use different mathematical processes (problem solving, reasoning, communicating, connecting and representing) to deepen their mathematical understanding and apply their mathematical knowledge (NCTM 2000). In other words, mathematical tasks should be rich, approachable and encourage mathematical inquiry (Davis 1996).

I think it is important to include an example of a task that may encourage students' mathematical sensibilities and mathematics to emerge, interact and evolve. The task was used in a recent research study of collective creativity in elementary mathematics classroom settings. The participants were two mathematics teachers in a Canadian school setting, and their sixth-grade students in the academic year 2015/16. The task was introduced to a group of three sixth grade students (S1, S2 and S3) in an interview setting with the author. The task states that "three children, Alex, Zac and John, shared a chocolate bar. Explain in as many ways as you can how those children may divide the chocolate bar into three pieces such that Alex will get twice what John got, and John's part is no more than one-fourth of the original bar and no less than one-tenth of it." Herein, I am including accounts drawn from a 25-minute problem-solving session with the three students. These accounts are included here to exemplify just two of the seven metaphors of creativity (namely, overcoming obstacles and expanding possibilities). In addition, a brief description of each of the seven metaphors is included.

Overcoming Obstacles

This metaphor suggests that the spark of creativity glimmers when we are addressed by a worthwhile problematic situation. Consequently, many scholars in the field of mathematics education describe problem solving as a form of creativity (Mann 2006; Silver 1997; Sriraman 2009). According to Silver (1997), problem-solving and problem-posing tasks can be used to foster creativity. Such tasks include less structured, open-ended problems that permit the generation of multiple goals and multiple solutions. The first obstacle that the participant group confronted was "where to start, and how to proceed."

"But it is [Pause 2 seconds] John barely gets any," S1 commented after S3 finished reading the problem. "Wait, but how much does Zac get?" S3 asked. His question initiated kind of collective overcoming obstacles activities. "Let's just give him a third," S1 suggested. "It is just whatever's left," S2 responded to S3's question and S1's suggestion. After a brief conversation S1 suggested to "Draw the chocolate bar," and on a shared piece of paper, he drew a rectangle and split it into four equal-sized pieces (quarters) to represent the chocolate bar. The three students engaged in a conversation while they were working collectively on their shared representation of the chocolate bar. S1 summed up the group's suggestions by stating that "Oh, yeah, it would work, yeah because if John," S2 interrupted and completed S1's statement "gets 25 per cent, Alex gets 50 per cent, then there is 25 per cent left from the bar, we just give that to Zac." S3 noted that "I guess we just have to work with, Alex and John because Zac doesn't matter."

The group agreed on S3's comment. To this end, the group was collectively engaged in overcoming obstacles activities. They tried to understand the problem and to consider the conditions of it. They listen respectfully to each other, and respond thoughtfully to the wonderings and suggestions that emerge through the conversation.

Divergent Thinking

According to Webster's online dictionary, divergent thinking is creative thinking that may follow many lines of thought and tends to generate original solutions to problems. There are four key components of divergent thinking which can be considered components of creativity; these are fluency, flexibility, originality and elaboration. Herein, our group's divergent thinking started with S2's question "OK, so how many other ways can we do this?"

Assembling Things in New Ways

Creativity includes using what we have creatively, which, in turn, may require finding connections, combining ideas and information, and assembling things in new ways. Klein (2013) argued that our discoveries and our solutions to different problems are all based on the idea of combining and recombining pieces of information to produce new ideas or to understand anew. Within the same paradigm, insight may eventually be gained by engaging with several events to discover a pattern or other relationship.

Route-Finding

Koestler (1964) argued that "the creative act is not an act of creation in the sense of the Old Testament. It does not create something out of nothing: it uncovers, selects, re-shuffles, combines and synthesizes already exciting facts, ideas, faculties, skills" (p 120). This vision of creativity is very close to Craft's (2003) "little c creativity," which may be understood as navigating new pathways, manoeuvring, charting a new path, discovering, uncovering or tracing. Students in the participant group were curiously engaged in processes of negotiating, selecting, combining and synthesizing different ideas and information to find their routes around the problem.

Expanding Possibilities

To be creative, according to Norris (2012), means "to be in a state of openness to the unknown, a place of possibilities, a place that a playful environment fosters" (p 300). Craft (2000) argued that one of the engines for little-c creativity (everyday creativity) is possibility, that is, using imagination, asking questions and playing. Craft described "possibility thinking" as "refusing to be stumped by circumstances, but being imaginative in order to find a way around a problem or in order to make sense of a puzzle" (p 3).

The group's starting point for building on, and expanding of their different suggestions and ideas was S1's wondering, "We cannot have three ninths?" S2 commented, "No, we cannot have three-ninths because then it won't be split into three." S1 interrupted S2's comment and completed it by stating that "because Alex would have six, John has three, and there is nothing left for Zac." S3 wondered, "Zac and Alex don't have to be equal, right?" S1 replied, "No, but." Accordingly, S3 interrupted S1 and noted that "so, Zac can have a tiny little piece [Pause 2 seconds] as long as Alex is twice as much as John." S1 completed S3's comment by stating that "as long as Alex is twice as much as John, and John is no more than one-fourth and no less than one-tenth, Zac can get as much as he wants or little as he needs." After a brief conversation, S1 argued that "I don't think it can go forever. We cannot go more than one-fourth and we cannot go less than one-tenth." But S3 didn't agree with him, and he believed that "technically, if you just kept on zooming in, slicing like into three, then zooming in to the last section depressing into three that goes on forever then, it goes on forever." Later, S1 agreed with him and suggested that "you can also do the opposite way by expanding [Pause 2 seconds], well, no, expanding will work too but it would stop, but this zooming in will go on forever." S2 agreed with them and summarized their different basic options: "OK, so we have our ninths, and we have our eighths, now sevenths, sixths and fifths, yeah, these are our options for that."

Collaborative Emergence

Imagination and play can be considered improvisational practices, because they involve uncertainty and unpredictability and because they are unscripted. Through the practice of improvisation, creativity may also be a collaborative emergence. Sawyer (1999) conceived of creativity as an emergent phenomenon that results "from the collective activity of social groups. Although collaborative emergence results from the interactions of individuals, these phenomena cannot be understood by simply analysing the members of the group individually" (p 449).

The previous accounts show us that the same characteristics that Martin and Towers (2009) aligned to improvisational coactions are applied to the mathematical inquiry of the participant group. These are:

1. No one person driving: "as the students work together and collaborate, no one student is able to lead the group to a solution" (p 13).
2. An interweaving of partial fragments of suggestions and representations: "the group, through offering fragments of [different possibilities especially concerning John's and Alex's parts], coact to make and develop confidence in a new [emerging possibilities]" (p 14).
3. Listening to the group mind: During students' coacting and interacting, there were "several places where innovations are offered (often in fairly incoherent fragments) and where, by listening to the group mind, the group is able to pick up on ideas and interweave the fragments to build a collective [ideas and solutions]" (p 15).
4. Collectively building on the better idea: "when an image is challenged and an innovation offered, a coacting group must collectively determine whether the innovation is to be accepted into the emerging performance. They achieve this by listening to the group mind" (p 15).

According to Sawyer (2003), improvisation "exaggerates the key characteristics of all group creativity: process, unpredictability, intersubjectivity, complex communication, and emergence" (p 5). And this exaggeration is completely demonstrated by the previous accounts of the participant group. The group coactions and interactions triggered and sustained the emergence of "new ideas, suggestions, connections, paths, processes, etc."

Originating, or Making Something New

The word creativity, both in its origins and in most of its varied uses, reflects a kind of newness, originality or novelty. In addition, the new thing that is brought into being is seen as something valuable, fruitful, effective, appropriate and so on. For the purpose of describing creativity in classroom settings, both Baer (1997) and Starko (2009) suggested that a product or idea is original to the degree that it is original to the creator, and it is appropriate if it meets some goal, purpose or criteria within a sociocultural context.

Students in the participant group demonstrated evidence of newness and appropriateness of their actions and doings. Apparently, students were engaged in the task for its own sake. I didn't offer them any kind of extrinsic motivations to participate or to engage in doing the tasks. The previous brief accounts from the problem-solving session with my participant group show many indications of new and effective things emerging during the flow. Flow is a notion used by Csikszentmihalyi (1990) to refer to the state of being completely involved in an activity for its own sake.

Concluding Remarks

These brief accounts (few minute conversations) show us the richness, the collective and the emergent nature of students' conversation. Here, I would like to advise the reader that there was no intervention from anyone other than the three students during the session. The problem-solving session with this group of students can best be described as a free yet constrained mathematical inquiry. The task was open-ended with two constraints. Students' thoughtful, and sometimes playful, arguments, talks and negotiations show us the conversational and dialogical nature of mathematical inquiry (Davis 1996). Observing such a group of students while they were working on some mathematical tasks afforded me invaluable opportunities to understand what it really means to do and understand mathematics. Students were free to make decisions, work on the task, experiment, move, talk, mingle, play, and accept or reject. Adding some constraints to the task didn't hinder the task; rather they made it more interesting, challenging and intrinsically engaging.

Despite the good work in the field of mathematical creativity, it remains unclear how it might look in a classroom setting. Herein, I presented a description of mathematical creative acts based on seven metaphors. In addition, I introduced a brief description of each metaphor. Two metaphors were exemplified by some observations from a problem-solving session with a group of sixth grade students. This paper is an attempt to describe mathematical creativity as it may emerge in mathematics learning environments. The metaphors can be considered design principles to support teachers' efforts in creating and offering genuine classroom opportunities for their students to exercise creativity—opportunities that have the potential to transform the classroom into a space of expanding possibility.

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When Is the World Going to Be Full?

David Martin

This project was designed to fit in the Alberta Math 30-2 curriculum. Feel free to alter as you see fit.

Pose the question: When is the world going to be full?

Give students five minutes to talk in groups, then discuss and share possible hypotheses.

Next, ask for information that might be needed to answer this question more accurately. Here are some other possible questions:

1. How many people can fit on the planet?
2. How many people can the planet sustain? (Different question from the first.)
3. What is our current growth rate?
4. What is our current population?

Some of these questions will be easy to solve, while others might be more difficult. I would ask students to research the answers to their questions in groups. Some questions will have an answer all will agree with, such as "What is our current population?", while other questions might have a range of answers depending on the website found, such as "How many people can the planet sustain?" Create answers that the whole class can agree with or common answers in different groups that the whole group can agree with.

Body

Creating an Extrapolation of Our Population

Our current growth rate can be searched; however, this number most likely has no meaning or even understanding of how it was derived. As a class, possible growth rates will be explored. On the next page is a chart of recent population numbers on our planet. (You might want to use different or more years).

Now the question becomes, "What sort of data is this?" This is when I would pause and teach exponential, sinusoidal, linear, quadratic, cubic and logarithmic functions. Using this data, I would create an equation of each type of function. Below is a graph of the functions displayed together. (I used 1970 as year 0. This occurred as I needed more data points after deciding this. I also added a logistic curve to show what would happen if population growth becomes 0.)

Here are some questions to ask during discussion time:

- Any general thoughts?
- Any similarities?
- Could you create a possible scenario in which each graph would be accurate?

The critical learning of each function could be embedded into this; for example, you could ask:

- What are the amplitude, period and median values for the sine function? What does that mean in this context?
- What are the zeros of the cubic function? What does this mean in this context?
- What is the growth rate of the exponential function? What does this mean in this context?

You may want to remove any functions in which the class agrees might be inaccurate, such as cubic, sinusoidal and linear.

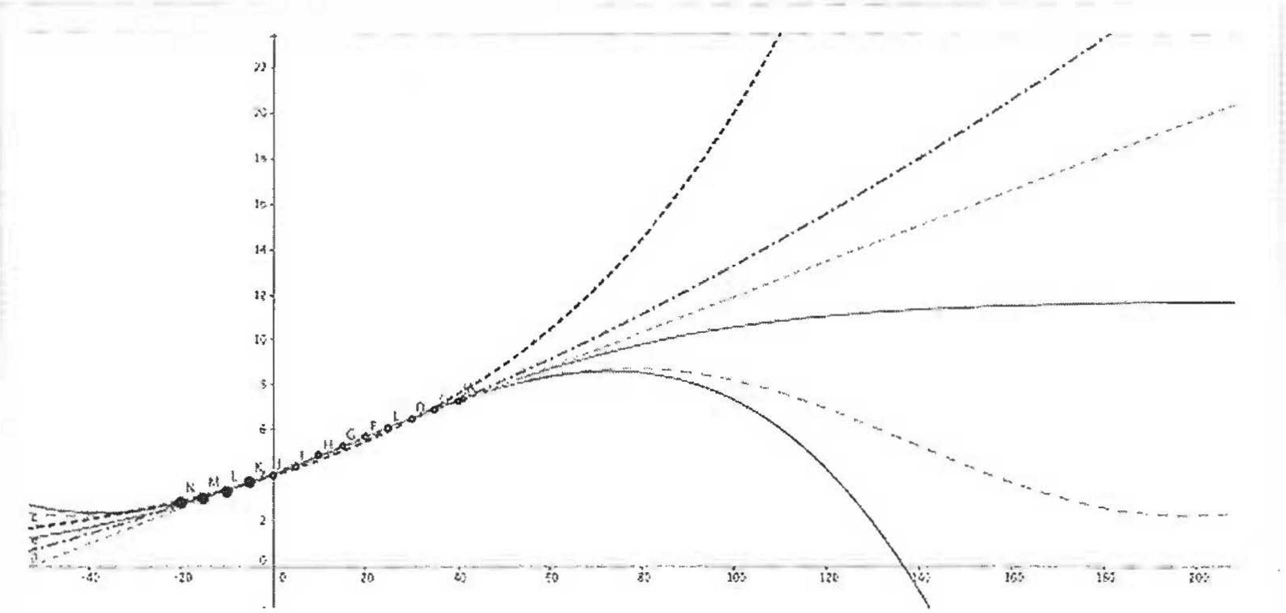
Using the remaining graphs determine when the earth would be full, using the number the class researched earlier. The answers might be earlier or later than the class first hypothesized.

Year	Population (Billion)
2016	7.4
2015	7.3
2010	6.9
2005	6.5
2000	6.1
1995	5.7
1990	5.3
1985	4.9
1980	4.4
1975	4.0
1970	3.7
1965	3.3
1960	3
1955	2.8

Conclusion

Discuss "Should population growth be addressed? Why or why not?"

Do not rush this project. Stop throughout to teach certain skills or concepts. Also allow students to research on their own and pose their own questions throughout.



Trigonometry Workshop for High School

Rosalind Carson

For the Love of Triangles

This approach to visualizing trigonometry is based on how the great mathematicians learned to measure chords in circles and use these measurements to accurately map the celestial objects in the sky. It is my experience (in formal classrooms and tutoring) in teaching Grades 10, 11 and 12 that students have an easier time understanding and applying the trigonometry concepts when given the opportunity to see and explore the origins of this mathematics. This is how I begin with the students: "Tell me all that you know about triangles."

The story begins with the man responsible for today's theme, "For the Love of Triangles." His name was Thales (640–546 BCE or 636–546 BCE). In written history, he is the first recorded mathematician. Of course there were earlier ones—Babylonian math was recorded on stone tablets 1,200 years before Thales, but there are no individual names. Thales was Greek and lived in Miletus, which was a prosperous colony on the coast of Asia Minor, a thriving trading port. Of the many stories told of Thales, his quest to understand things himself was most prevalent. A famous saying from these tales is "in order to seek wisdom, you must know thyself." He was prized for his logic and mathematical contributions by later Greek generations and was honoured with the title of being one of Seven Wise Men of Antiquity. Thales was the only mathematician to have this honour.

In one of the many tales of how Thales used his knowledge to be wise, he became a rich man. A poor friend was complaining about how the poor man always stays poor. Thales disagreed strongly and told the man that if one sets one's mind to a task, he can become rich. Thales told his friend to give him six months to prove his point. At this time, there was a drought and many poor olive farmers. Thales put his observations to use, talked to old farmers and predicted a good year. Due to current economics, many farmers were willing to sell their olive presses at very low prices. When the rain came, the farmers had to rent the presses from Thales in order to handle the crop, and Thales prospered. It is told that Thales was not in this for the money. Eventually he sold the presses back to the farmers and moved on to other pursuits.

Discoveries Made by Thales

Below are the mathematical contributions that Thales made:

- A circle is bisected by its diameter.
- Angles at the base of an isosceles triangle are equal.
- When two straight lines cut each other (intersect), the vertically opposite angles are equal.
- The inscribed angle (triangle) in a semicircle is (has) a right angle.
- The sides of similar triangles are proportional.
- Two triangles are congruent if they have two angles and a respective side congruent (ASA).

Thales was the first mathematician to have a specific mathematical discovery named after him, Thales's Theorem: The triangle inscribed in a semicircle is a right triangle.

Similar Triangles Activity

Thales travelled and studied in Babylon and in Egypt. In Babylonia, he learned astronomical measurement techniques. In Egypt, he learned about the geometric surveying techniques, re-establishing the land boundaries after every flood of the Nile River. One of the stories states that when admiring the Great Pyramid of Giza, he asked the locals what the pyramid's height was. He was surprised that people did not know the height of such a great structure, so he quickly calculated it (to his spectator's surprise) by using similar triangles. He used indirect measurement and these simple tools: his shadow, his staff, knowledge of similar triangles and the sun. How did he do it?

This is where I leave the students to think about how Thales performed this calculation.

Thales' achievement was a small step for trigonometry, the science of triangles, and a giant leap for mankind. By deducing a measurement logically from the intrinsic properties of a shape he was thinking differently from the Egyptians, who had shown remarkable skills in practical activities like pyramid-building but whose mathematical knowledge was essentially limited to rules of thumb and triangles that actually existed. Thales' calculation involved a triangle that was an abstraction of reality, made by the sun's rays. His insights marked the beginning of Greek rational thought, which we generally consider the foundation of Western mathematics, philosophy and science. (Bellos 2014, 59)

Thales also knew that any triangle can be split into two right triangles. Did you know that? In fact, Greeks were obsessed with right triangles because of this fact. The Greeks learned the word *hypotenuse* from the Egyptians. In Egypt, a land surveyor would hire three slaves to stretch the special rope knotted in the designated lengths of what we know as Pythagorean Triples, typically the 3, 4, 5 triangle. This surveyor was called harpedonopta, literally meaning rope stretcher. The Greek word *hypotenuse* means stretched against.

Three centuries later, Eratosthenes used similar triangles and logic to get a good abstract measurement for the circumference of the earth.

Pythagoras

According to various resources, Pythagoras was born between 560 and 580 BCE on the island of Samos in the Aegean Sea. He travelled most of his life as far as India, learning from the Babylonians and Egyptians. It has been said that Thales was one of his teachers. The Babylonian point of study is an interesting one because of the fact that 1,200 years earlier the Babylonians knew the relationship of the sides in a right triangle (Pythagorean triples) because stone tablets have been found written in Cuneiform (base 60) with lists of these triples.

If you want to create right triangles (side lengths that actually make right triangles), use the following:

Let p and q be whole numbers; where $p > q > 0$, p and q have no common divisor (save 1), p and q are both not odd, then the following produce Pythagorean Triples:

$x = p^2 - q^2$; $y = 2pq$; $z = p^2 + q^2$

Google Plimpton 322 Babylonian Stone Tablet to get many results on how to read and work in base 60, which is the reason why we use 60 as a base measurement for degrees, time and so on.

Pythagoras returned to Samos and started his school teachings at the age of 50. The society (that it became) was strict. There were many levels of teachings. Entry level was for learners and above who contemplated and discussed mathematics, trying to explain the universe with numbers—rational numbers since they only could work with whole numbers and factors—that is, any number that can be multiplied and divided, hence their version of fractions. Those who followed his teachings did not eat meat or beans, did not drink wine nor wear wool or touch a white rooster! Pythagoras died after refusing to run through a bean field when an angry mob was chasing him. The mob caught up and killed him.

Here is another story about root 2. The Pythagoreans explored mathematics to describe nature by writing in the sand or using pebbles. When they explored a square of side length one, it was said that the diagonal was impossible to determine, as a ratio (of the diagonal to side length) of whole numbers could not be written to represent that diagonal. Hippasus tried to convince Pythagoras that there must be another type of number, an irrational number. This was so absurd that they threw him overboard to his end. This was the first sign of irrational behaviour.

As the Pythagoreans studied harmony with the universe, they studied astronomy, music, numbers (number theory) and geometry. They believed that all matter was composed of four elements: fire, water, air and earth. These elements were closely associated with the numbers assigned. All four together were the holy tetractys whose sum $1 + 2 + 3 + 4 = 10$, the number for the universe. Two represented female, three was male, five was marriage, four was justice (a square deal) and six was the creation of the universe, and is a perfect number (perfect numbers are those that are the sum of their proper factors). The pentagram was held sacred for all the hidden pentagrams within the structure, and all the ratios of sides that represent the Golden Ration (not that they knew a number for that because the Golden Ration is an irrational number!) but that they knew this ratio is aesthetically pleasing, as it was used in all current and ancient art and architecture.

Pythagoras (or one of his students) created the word *mathematics*, which means science.

Handout: Pythagorean Theorem Discovery—Chinese way.

Show picture on page 22 in* Euclid's Window, *by Leonard Mlodinow or* Bedside Book of Geometry, *by Mike Askew and Sheila Ebbutt.

Invention of Trigonometry: Hipparchus of Rhodes

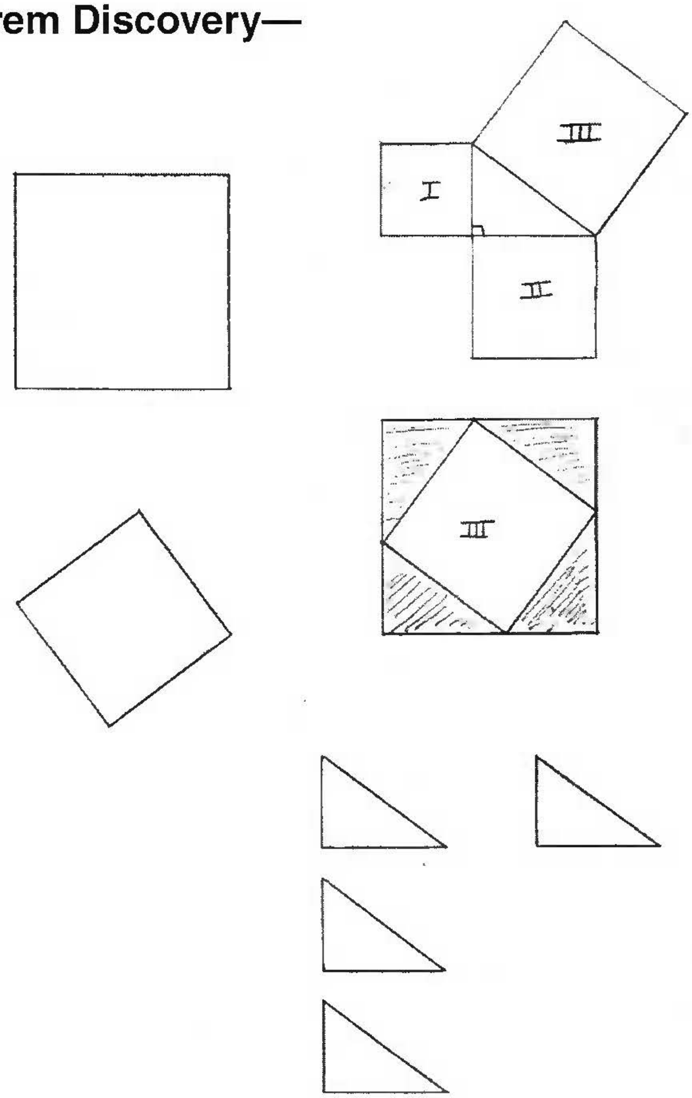
Hipparchus of Rhodes (Greece, 170–126 BCE) created trigonometry (not called that until 1600s) because the main focus was the mapping and movement of the celestial objects. Trigonometry was used for astronomy—plain and simple. It was not used for surveying until the 1600s (first noted in a book in 1595). He used similar triangles and the chord length in a circle to measure the movement of stars or distance between celestial objects.

There are many different accounts of why we use 360 degrees for a full rotation of a circle. Basically, the Babylonian method has stood the test of time! They used a base 60 number system. The word *degrees* refers to the measurement of movement. Dividing the arc length of a circle by 60 leaves 6 equilateral triangles. Angle measurements were arc length measurements and a degree was divided into 60 smaller units each called *pars minuta prima*, meaning the first minute part. Those were then divided into 60 units, each called *pars minuta secunda*, meaning second minute part and translated into seconds, hence the basis of time keeping. The Chinese originally divided the circle into 365 ¼ parts making the sun movement one degree per day—nice! But that made other calculations difficult, so we stuck with the Babylonians.

Hipparchus created huge tables of values of degrees and chord lengths that were readily used by all astronomers. In the second century, Ptolemy compiled a table of chords for a circle with radius = 60 units and every half degree angle. These tables were invaluable to western astronomers.

Astronomy was expanding in India because they (like the Babylonians) had a place value number system allowing large and small calculations. Indian place value was base 10. They took the knowledge of chords further and created tables of half-chord lengths. By using half-chords, there was the need to utilize the exciting and popular relationship of the sides of a right triangle (Babylonian

Pythagorean Theorem Discovery—Chinese Way



Theorem—oops!—Pythagorean Theorem). The Arabians learned from the Indians and translated the Sanskrit word *jya-ardha* (string-half) to *jiba* in Arabian, which sounded the same but was meaningless. Since *jiba* looked like an Arab word *jaib* (meaning bosom or bay), that was the word recorded for the half-chord length. Only when Fibonacci studied in Arabian countries, learning and hence promoting the use of Arabian numerals, zero and the line that separates the numerator and denominator in fractions, did the Europeans finally adopted the half-chord knowledge. The Latin translation for *jaib* was sinus (meaning the fold of cloth of a toga over the woman's chest), which became sine.

It was Georg Joachim Rheticus (1550) who pushed the idea (that stuck) of focusing the vision of trigonometry to be just a close look at right triangles. Rheticus also established *sine*, *cosine* and *tangent* as the names for the ratio of the sides of triangles that we now memorize and apply.

Show photo on page 90 of* A Mathematical Mystery Tour, *by A K Dewdney, to visualize the 3-D sense of mapping stars and how we then transform that image onto a 2-D plane = the coordinate plane. Explore the length of the chord to measure the location of the stars.

We can easily see the sine, cosine and tangent of an angle when the stretched side (hypotenuse) = 1.

Now let us look at similar triangles: (draw all examples and how that transcends into five sections of workbook or textbook topics).

Let us look at triangles without right angles. Remember, all triangles can be split into two right triangles. Let us do that in the triangle we drew! Voilà! Law of Cosines.

Draw another triangle that is not a right triangle. Split it into two right triangles. Label with sines. Voilà! Law of Sines!

Mapping using trigonometry: You just learned how to do all the calculations your phone does to find the nearest coffee shop! You are a genius! (Pages 74–76 of* The Grapes of Math, *by Alex Bellos.)

Questions to ponder:

1. Kevin argues that there is no need for knowing trigonometry. He states that all you have to do is use a ruler to find the side lengths. Do you agree or disagree? Explain.
2. The sine of an angle in a right triangle is quite large. What might the two acute angles be? Explain.
3. The cosine of an angle in the right triangle is approximately 0.707. What might the dimensions of the triangle be?
4. Jason said, "The sine of angle A in a right triangle cannot be larger than the cosine of angle B in the same triangle." Do you agree? Explain.
5. The cosine of angle A (think of side length of triangle) is a lot more than the sine of that angle. What do you know about angle A (in a right triangle)?

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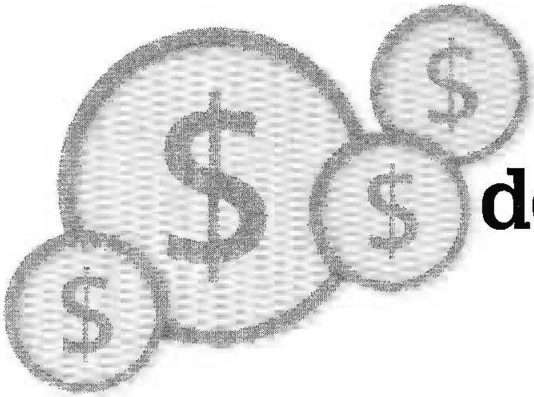
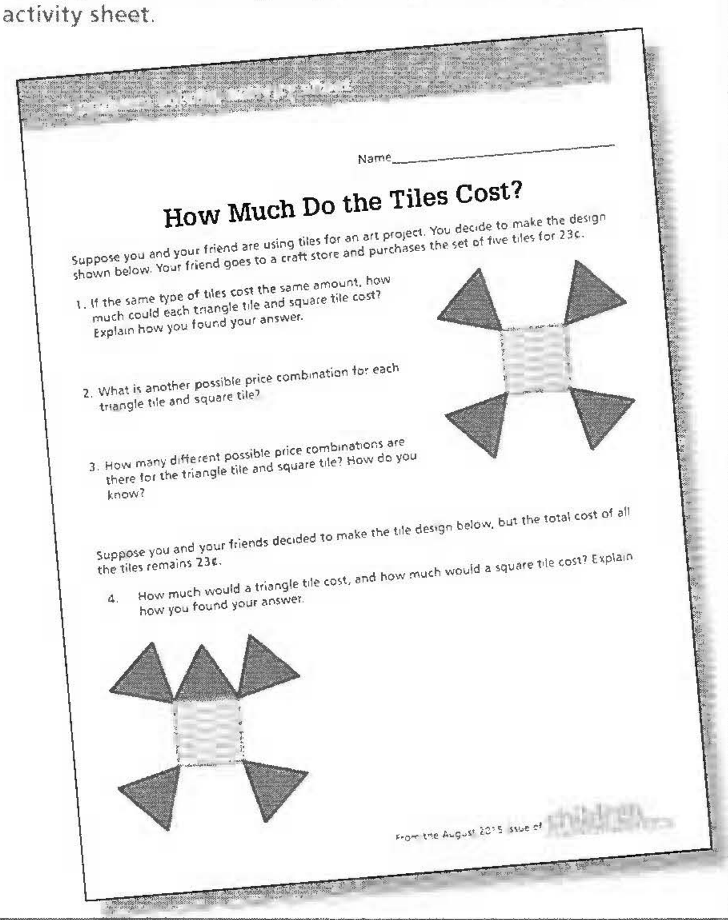
Sarah Quebec Fuentes



How much do the tiles cost?

The August 2015 problem scenario presents students with a design composed of square and triangular tiles as well as the total cost of the tiles needed to make the design. Students must determine the cost for one square tile and one triangular tile. Go to http//:www.nctm.org/tcm, All Issues to access the full-size activity sheet.



Name__________

How Much Do the Tiles Cost?

Suppose you and your friend are using tiles for an art project. You decide to make the design shown below. Your friend goes to a craft store and purchases the set of five tiles for 23¢.

1. If the same type of tiles cost the same amount, how much could each triangle tile and square tile cost? Explain how you found your answer.
2. What is another possible price combination for each triangle tile and square tile?
3. How many different possible price combinations are there for the triangle tile and square tile? How do you know?

Suppose you and your friends decided to make the tile design below, but the total cost of all the tiles remains 23¢.

4. How much would a triangle tile cost, and how much would a square tile cost? Explain how you found your answer.

From the August 2015 issue of

Algebraic reasoning should be incorporated throughout the K–grade 12 curriculum. The How Much Do the Tiles Cost? problem is a high-level task that fosters algebraic reasoning and problem solving, is accessible to students across the elementary school grades, and can be solved in multiple ways. Rachael Quebec implemented the problem in her first-grade class at Joseph Estabrook Elementary School in Lexington, Massachusetts, providing a window into students' algebraic reasoning in the early elementary grades.

Introducing the problem

Quebec regularly uses a workshop model. While she works with a small group of students, the others freely move to and participate in various activities that preview, develop, reinforce, and extend the current focal mathematics learning goals. During the week before implementing the How Much Do the Tiles Cost? problem, Quebec previewed the task by presenting the 10¢ Designs activity to her class. Students received prices for various tiles and had to determine the total cost of several tile designs (see **fig. 1**). Because the total cost of each tile design was 10¢, as Quebec introduced the tiles task, she simultaneously reinforced the early number-sense ideas of part-part-whole and benchmarks of ten.

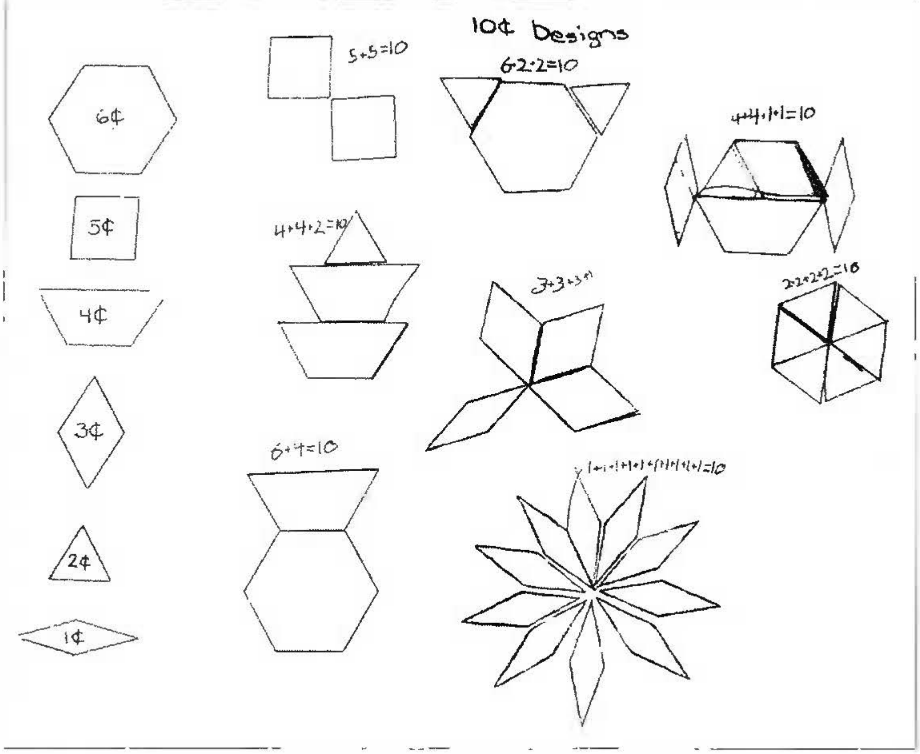
To launch the How Much Do the Tiles Cost? problem, Quebec met her first graders on the carpet and shared the tile design (from the activity sheet) with this story:

> Suppose you and your friend are using tiles for an art project. Your friend goes to a craft store and purchases the set of five tiles for 23¢. If the same type of tiles cost the same amount, how much could each triangle tile and square tile cost?

Quebec reminded her students about the 10¢ Designs activity and emphasized that instead of being given the tile prices and

FIGURE 1

During the previous week, this first-grade teacher had used the 10¢ Designs activity to reinforce the early number-sense concepts of part-part-whole and benchmarks of ten, simultaneously preparing her class for the How Much Do the Tiles Cost? task.



determining the total cost of a design, they were being given the total cost of the tile design and must determine the cost of individual tiles. Before dismissing students to work on the problem, Quebec reviewed her group-work expectations, such as listening actively, evaluating others' ideas, and explaining one's reasoning. Students returned to their table groups of four with a copy of the activity sheet to record their solution strategies.

Working on the problem

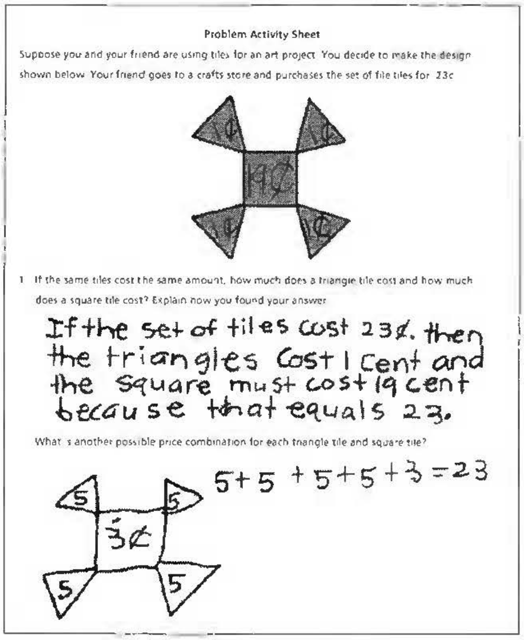
While groups tackled the task, Quebec walked around the classroom, monitoring students' strategies for solving the problem. To maintain the problem's high level of cognitive demand, she made sure not to tell students how to solve the problem. Instead, Quebec asked questions that built on the ways that students were already thinking about the problem. For example, one group equated the tile's number of sides with its cost (i.e., square tiles cost 4¢, and triangle tiles cost 3¢) and determined that the total cost of the design was 4 + 3 + 3 + 3 + 3 = 16¢. Quebec prompted these children to reread the problem and evaluate their answer. After the group realized that the total cost of the tiles must be 23¢, Quebec encouraged students to consider alternative strategies for solving the problem. Most groups initially generated one possible price combination totaling 23¢. Quebec circulated among the groups, asking questions to prompt students to find additional price combinations:

- Is there another possible combination of costs for the triangular and square tiles? Why or why not?
- How could you determine another combination of costs for the triangular and square tiles?

FIGURE 2

Student groups represented their solutions in various ways:

(a) Some groups labeled the tile design and wrote the corresponding number sentence.


Problem Activity Sheet

Suppose you and your friend are using tiles for an art project. You decide to make the design shown below. Your friend goes to a crafts store and purchases the set of tile tiles for 23¢

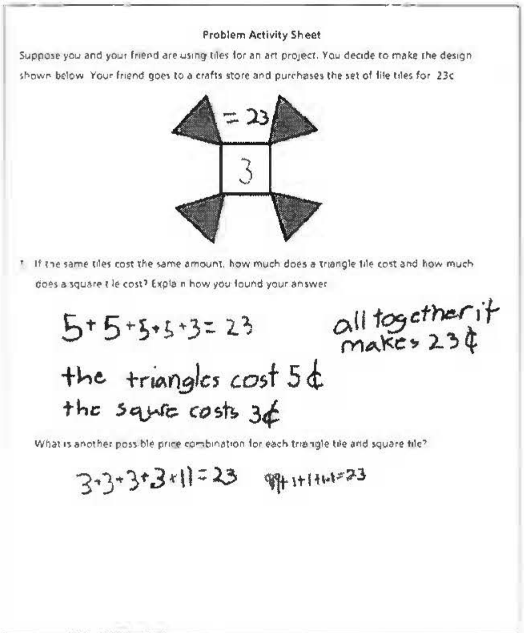
1. If the same tiles cost the same amount, how much does a triangle tile cost and how much does a square tile cost? Explain how you found your answer.

If the set of tiles cost 23¢, then the triangles cost 1 cent and the square must cost 19 cent because that equals 23.

What is another possible price combination for each triangle tile and square tile?

5+5 +5+5+3=23

(c) Some children simply wrote a number sentence only.


Problem Activity Sheet

Suppose you and your friend are using tiles for an art project. You decide to make the design shown below. Your friend goes to a crafts store and purchases the set of tile tiles for 23¢

1. If the same tiles cost the same amount, how much does a triangle tile cost and how much does a square tile cost? Explain how you found your answer.

5+5+5+5+3= 23 all together it makes 23¢

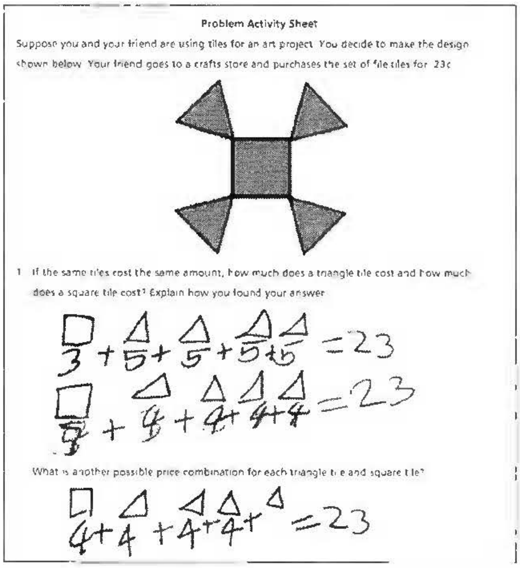
the triangles cost 5¢

the square costs 3¢

What is another possible price combination for each triangle tile and square tile?

3+3+3+3+11=23

(b) Other students marked each number in a number sentence with its corresponding shape.


Problem Activity Sheet

Suppose you and your friend are using tiles for an art project. You decide to make the design shown below. Your friend goes to a crafts store and purchases the set of tile tiles for 23¢

1. If the same tiles cost the same amount, how much does a triangle tile cost and how much does a square tile cost? Explain how you found your answer.

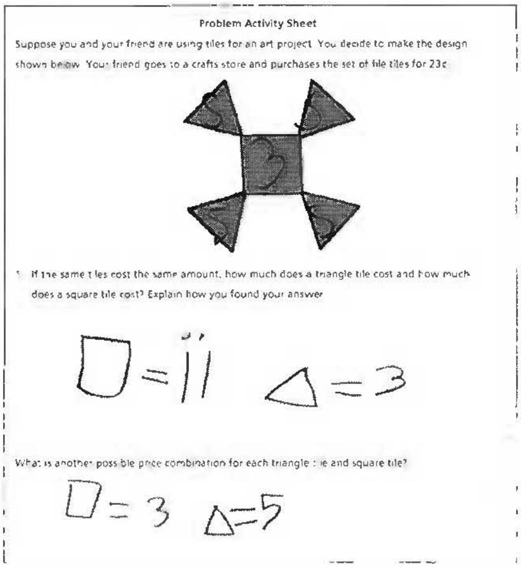
3+5+5+5+5 =23

7+4+4+4+4=23

What is another possible price combination for each triangle tile and square tile?

4+4 +4+4+4 =23

(d) These first graders equated the two shapes with their cost.


Problem Activity Sheet

Suppose you and your friend are using tiles for an art project. You decide to make the design shown below. Your friend goes to a crafts store and purchases the set of tile tiles for 23¢

1. If the same tiles cost the same amount, how much does a triangle tile cost and how much does a square tile cost? Explain how you found your answer.

□=11 △=3

What is another possible price combination for each triangle tile and square tile?

□= 3 △=5

While monitoring her students at work, Quebec noted student approaches to the problem, their various price combinations, and representations of their solutions:

- Many groups assigned a price to the triangle tile, found the total cost of the four triangle tiles, and then counted on to 23 to determine the cost of the square tile.
- Each group found two or three of the five possible cost combinations.
- All groups found the price combination of 5¢ for a triangle tile and 3¢ for a square tile. Using benchmarks of five and ten, the students easily determined the total cost for the triangle tiles (5 + 5 + 5 + 5 = 20¢).
- None of the groups identified the price combination of 2¢ for a triangle tile and 15¢ for a square tile.

The groups represented their solutions in various ways: labeling the tile design and writing the corresponding number sentence (see **fig. 2a**), marking each number in a number sentence with its corresponding shape (see **fig. 2b**), writing a number sentence only (see **fig. 2c**), and equating the two shapes with their cost (see **fig. 2d**). Quebec used these observations to help structure the whole-class closing discussion about the problem.

Sharing and summarizing the problem

Students returned to the carpet, where Quebec asked them to share their problem-solving strategies and any patterns they noticed in the price combinations. One student conjectured that as the cost of the triangle increased, the cost of the square decreased. Quebec recorded this conjecture and collected students' various price combinations in a table on chart paper (see **fig. 3**). She introduced the table as a way to organize and represent the price combinations. As a class, students completed the table, starting with the last row for the 5¢ triangle tile price combination (i.e., the most commonly found combination) and adding the rows for the 4¢, 3¢, and 1¢ triangle tile price combinations. After realizing that the cost of the square tile for the 2¢ triangle tile was missing, students determined this cost and added it to the table. Referencing

FIGURE 3

Although none of the small groups had recorded the cost of the square tile when the triangle tile was 2¢, students noticed it was missing when the teacher recorded a table of possible price combinations during their whole-class discussion.

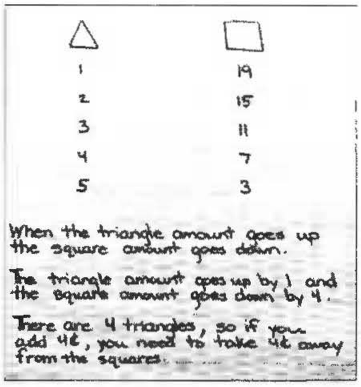
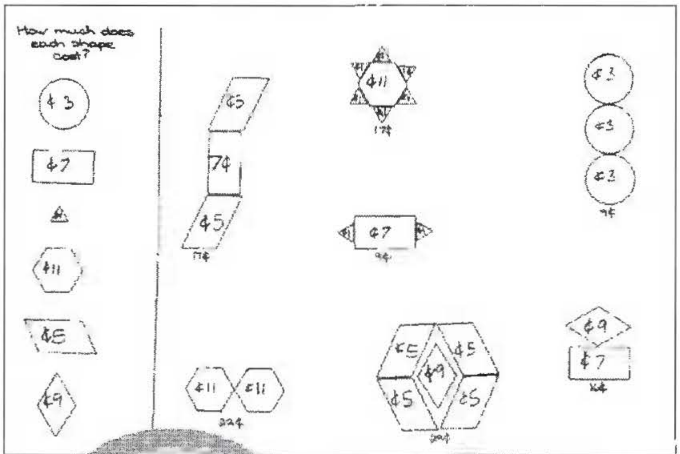
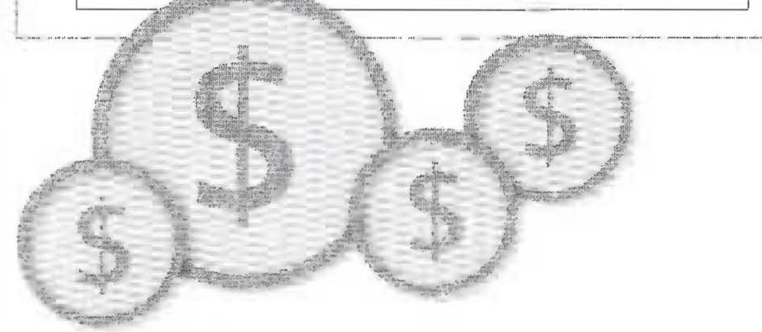


FIGURE 4

Subsequent to the How Much Do the Tiles Cost? task, Quebec incorporated the Shapes for Sale! activity into her math workshop. Using information they were given for tile design costs, students determined the cost of each tile shape.

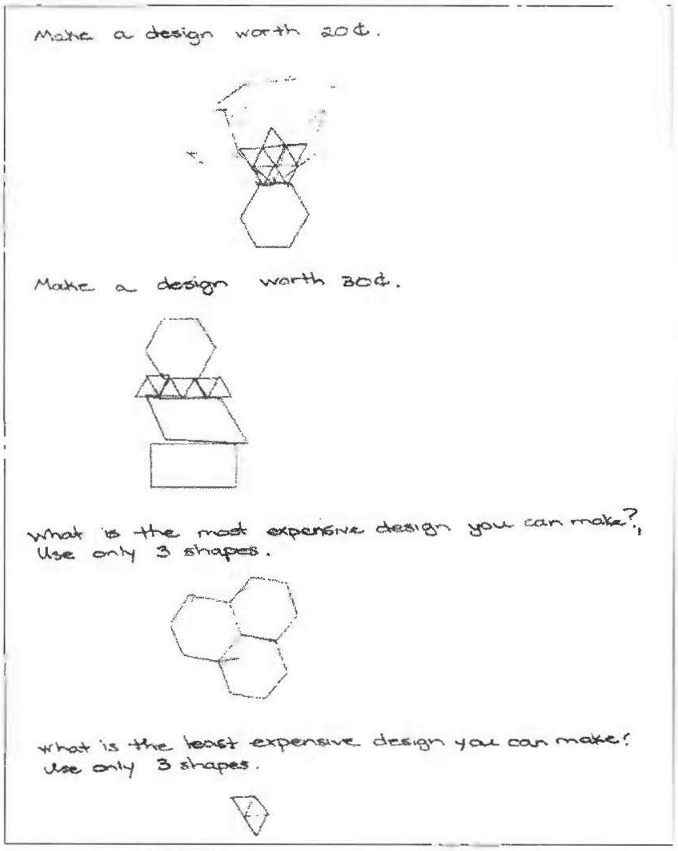


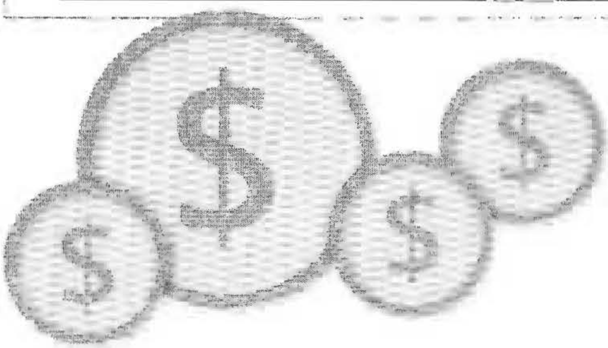


"Students explicitly articulated and justified how a change in the cost of a triangle tile affects the cost of a square tile."

FIGURE 5

After students had determined the cost of each tile in the Shapes for Sale! activity, Quebec presented another task, Make a Design, during which students created designs to meet different criteria.





the table of all possible price combinations, students were able to explicitly articulate and justify how a change in the cost of a triangle tile affects the cost of a square tile (see the second and third observations in **fig. 3**).

Extending the problem

For the next several days, Quebec incorporated two related activities into her mathematics workshop. The first gave students the cost of various tile designs. Using this information, students determined the cost of each tile shape (see **fig. 4**). The second activity built on the first. On the basis of the determined costs of each tile, students created tile designs meeting different criteria (see **fig. 5**).

Quebec commented that the How Much Do the Tiles Cost? problem gave her students the chance to develop their algebraic reasoning through engaging with a problem that presented a realistic scenario, did not have an obvious and immediate answer, and had multiple solutions. Additional experiences with similar high-level tasks will give her students opportunities to enhance their ability to identify relationships between variables and expand their repertoire of representations.

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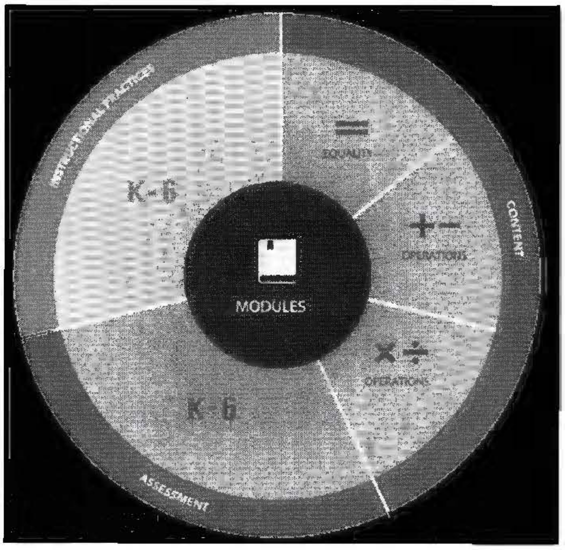
Understanding the Equal Sign Matters at Every Grade

Geri Lorway

As a K–6 mathematics teacher, have you logged on to the Elementary Mathematics Professional Learning website at http://learning.arpdc.ab.ca/? This site provides you with a variety of resources, materials, professional learning and coaching ideas arranged around three big topics:

- Curricular Content (subdivided into Equality, Additive Thinking and Multiplicative Thinking)
- Instructional Practices
- Assessment

These free materials are being developed to explicitly meet the expectations of the Alberta program of studies. This article highlights a small snippet of materials drawn from the Content Module: Equality. The examples provide focus on key foundational pieces that students are exposed to in the early grades. We encourage you to visit the Elementary Mathematics Professional Learning (EMPLO) website to develop a deeper understanding of this topic.



What Is Equality?

In mathematics, *equal* refers to a relationship not an action or an operation. Equal is used to describe a comparison. We compare amounts, values and measurement like length, volume, capacity and so on. If a unit is not identified, the comparison is how much or how many in each set or how much each set is worth. For example, 1 = 10 would be not equal whereas 1 cm = 10 mm is equal because it has units.

The equal sign indicates that the items being compared represent the same amount, quantity, value or measure. The equal sign does not mean here comes the answer. The equal sign should be read as "is the same amount as" or "represents the same amount or value as." Once students understand the meaning of the equal sign and expressions, they understand that either side of the equation can be read first. The values are not affected. For example, $3 + 4 = 7$ can be read as $7 = 3 + 4$, and $3x - 2 = 9$ can be read as $9 = 3x - 2$. Once the meaning of the equal sign has been introduced, teachers in subsequent grades must help students extend and transfer that meaning to new applications, for example $7 \times 8 = 8 \times 7$; $48 = 6 \times 8$; $\frac{24}{8} = \frac{12}{4}$.

Researchers agree that by introducing students to the relational meaning of the equal sign as in 9 = 9 or 5 = 5 before they are expected to write equations, teachers can reduce the potential for students developing misconceptions (Capraro et al 2007, Watson, Jones and Pratt 2013). Teachers and textbooks often only represent equations in the form of $6 + 2 = 8$. The left side has the operations while the right side has the answer. To help students develop a more complete understanding, teachers should consider how often they expose students to equations written as $8 = 6 + 2$. In this case, the left side has the answer, and the right side has the operation. A simple idea to think about: Every time students read or write an equation, ask them to state it in another format. For example, a student says $3 + 2 = 5$. How many ways could you record that equation without changing the meaning? $2 + 3 = 5$; $5 = 3 + 2$; $5 = 2 + 3$. Look at all the practice!

Why Is Equality Important?

The understanding of equality as a relationship forms the basis for all number properties. The understanding and application of number properties is a significant factor in students building efficient strategies for computation and forms a foundation for success in algebra.

However, if you asked students to explain what the equal sign means, what would they say? Current research confirms that students who demonstrate a correct understanding of the equal sign show the greatest achievement in mathematics (Capraro et al 2007). When asked to explain the equal sign, the majority of North American students, across K–12, are apt to say, "It means here comes the answer." This makes it difficult, if not impossible for learners to accept equations in the form $7 = 7$; $8 = 3 + 5$; $3 + 4 = 2 + 5$; or $x = x$.

Why Begin with Equality?

It is important that students first understand relationships between numbers before they are asked to operate with them. Operations emerge as a thinking and communication tool to help us in our quest to determine and prove equality and inequality. The equal sign naturally arises through a need for the formal presentation of determining equality.

During a research study conducted by Falkner, Levi and Carpenter (1999), students were asked to make the following statement true: $8 + 4 = \_\_ + 5$. Student responses varied and many were simply incorrect.

1. Some created a run-on equation: $8 + 4 = 12 + 5 = 17$. This is incorrect thinking and notation because $8 + 4$ does not equal $12 + 5$. The convention is that there should be one equal sign per equation. One exception would be an example such as $1 + 5 = 2 + 4 = 3 + 3$. In this case, all three expressions are equivalent.
2. Some wrote $8 + 4 = 12$ and then crossed out the + 5, claiming the author made a mistake in the print. Imagine a student in junior high applying this error to a question such as $3x + 4 = 2x + 9$!
3. Some changed the symbols to $8 + 4 + 12 + 5 = 29$, claiming the author forgot to finish the equation.
4. Some understood equality as a relationship and realized that $8 + 4 = 7 + 5$.

Undoing a misconception is way more work than teaching it well the first time.

What's a Teacher to Do?

In kindergarten and Grade 1, set up opportunities for students to compare two sets as equal as students work on basic recognition of quantity.

In Figure 1, the teacher places the blocks down at the same time in each set. Students identify how many in each set. The teacher replaces the blocks with the numerals and places the equal sign between the numerals. The last step is to read the equation out loud. Two equals two. Two is the same amount as two. The left side equals the same value as the right side. The left set has the same amount as the right set. Practice with this idea should include maintaining the same quantity but varying the items in one of the sets, for example, 2 blocks = 2 bingo chips.

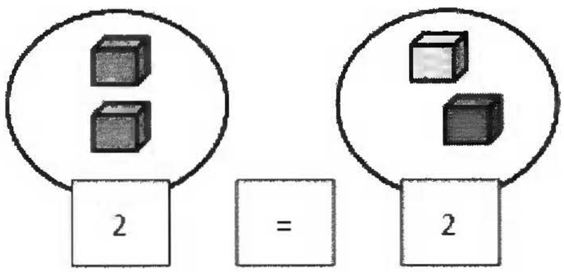


Figure 1.

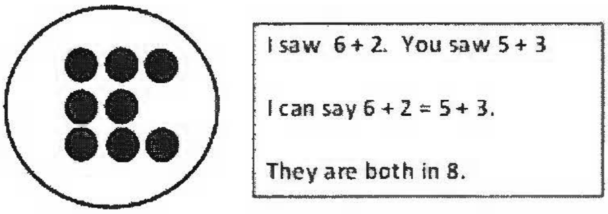


Figure 2.

As students in Grades 1 to 3 develop fluency with numbers to 20, their practice can include a variety of equations that describe equality such as those in Figure 2. Practice should include explaining and writing equations in a variety of ways. Activities like this help students recognize that an expression can equal an expression.

In Figure 3 students are comparing two equal sets rather than a single set as in Figure 2. In this example, students were presented with the model and asked to create the equations.

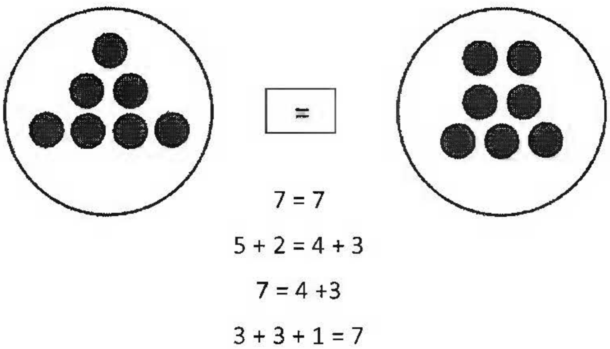


Figure 3.

Ask students to help you use materials to prove whether or not equations are true. In Figure 4, students were presented with the equation (an equality) and asked to prove it with a model.

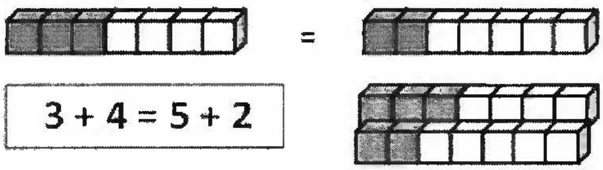


Figure 4. *How do these students use models to prove the equation?*

The understanding of equality as a relationship forms the basis for all number properties. From Grades 2 to 6, the understanding and application of number properties is a significant factor in students building efficient strategies for computation. For example, building multiplication facts as area models is one way to demonstrate $3 \times 4 = 4 \times 3$. In Figure 5, students can manipulate their models to explore the idea of the commutative property as an equality.

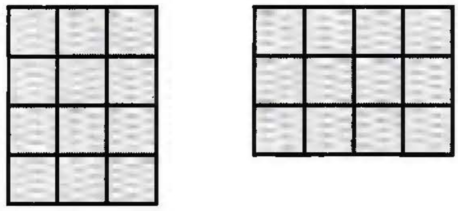


Figure 5. *The array for 3×4 also represents 4×3. They cover the same area. $3 \times 4 = 4 \times 3$*

Another significant number property is the distributive property which is introduced in Grade 4. In Figure 6, students can manipulate an area model to explore the idea of the distributive property as another demonstration of an equality.

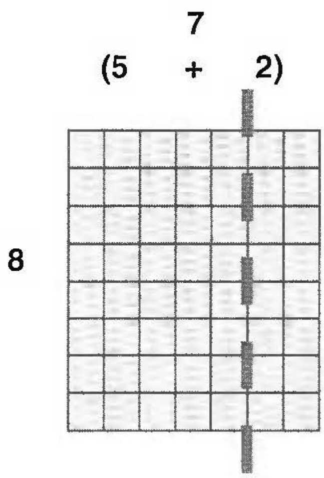


Figure 6. *The array for 7×8 is used to demonstrate $7 \times 8 = (5 + 2) \times 8$*

The Balance Metaphor

At Grade 1, the curriculum states that students be introduced to the metaphor of balance in the Patterns and Relations outcome number 4: "Describe equality as a balance and inequality as an imbalance, concretely and pictorially (0 to 20)." Teachers need to be cautious as students' understanding of weight may interfere with their ability to consider whether or not two sets or expressions balance. When two lengths or two volumes are being compared, the metaphor "balance" has no meaning. We really use balance when we're talking about equations that are balanced. In order to address all aspects of equality, teachers need to use a variety of models and/or metaphors. When using the balance metaphor, we want to ensure students understand that we use the term as an adjective to describe the state of equality rather than a verb.

Note: The *Alberta K–9 Mathematics Achievement Indicators* (Alberta Learning 2016) suggest that students using a balance

- construct two equal sets, using the same objects (same shape and mass), and demonstrate their equality of number, using a balance (limited to 20 elements); and
- construct two unequal sets, using the same objects (same shape and mass), and demonstrate their inequality of number, using a balance (limited to 20 elements).

The balance metaphor is one way to develop strategies for setting up and solving equations. Working with balances (as in Figure 7) allows students to practise number facts as they learn what it means to "balance equations."

Where would you start to decide what numbers are missing on the right-hand side of the balance in Figure 7?

Student A: I am thinking 28 = 12 + something × 2

Student B: I am thinking subtract the 12 on the right and 12 from 28 and you will still have equal. I think I would write that 28 – 12 = something × 2.

Consider Figure 8. What would you do?

Ask your students, "How would you represent Figure 8 as an equation?" What do you think about this challenge?

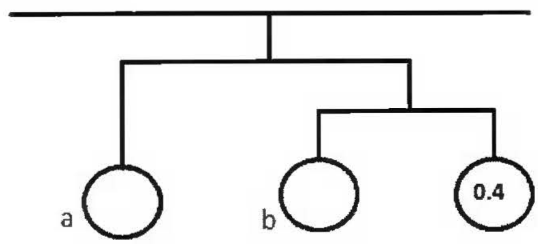


Figure 8.

The revisions to the Alberta curriculum in 2007 introduced the topic of equality in Grade 1. Equality was not part of the Grade 1 outcomes prior to this point. The intent of this article was to draw the reader's focus to the importance of the equal sign indicating a relationship. Students need to understand the equal sign thoroughly as a relationship before ever using it to write equations.

The examples to this point have only dealt with equality as a relationship between quantities: how much or how many? But equal is a relationship that

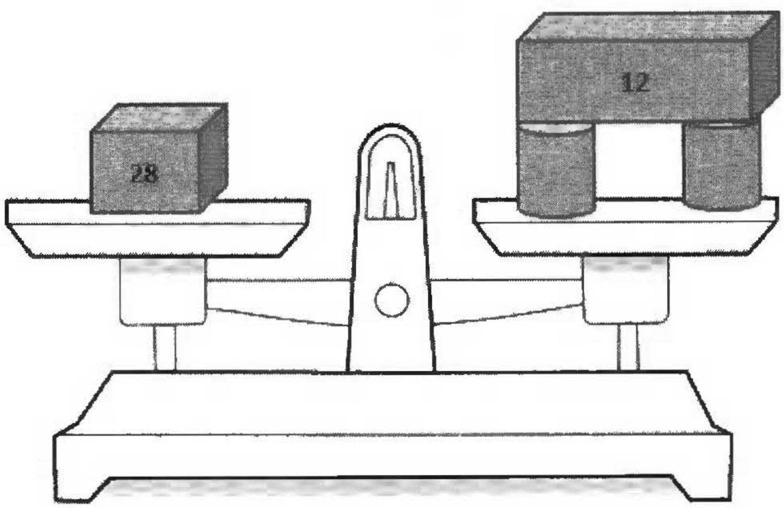


Figure 7.

applies to a comparison of any measure: equal length; equal distance around; equal weight, equivalent units as in cm→m and so on.

Where is equality specifically identified in the Alberta Program of Studies?

Grade 1:

- Describe equality as a balance and inequality as an imbalance, concretely and pictorially. (0 to 20) [*C, CN, R, V*]
- Record equalities, using the equal symbol. [*C, CN, PS, V*]

Grade 2:

- Demonstrate and explain the meaning of equality and inequality, concretely and pictorially. [*C, CN, R, V*]
- Record equalities and inequalities symbolically, using the equal symbol or the not equal symbol. [*C, CN, R, V*]

Grade 6:

- Demonstrate and explain the meaning of preservation of equality, concretely and pictorially. [*C, CN, PS, R, V*]

However, understanding equals forms the basis for success with all operations in K–12 mathematics.

During the 2015/16 school year, the Elementary Mathematics Professional Learning Opportunities (EMPLO) website was launched. A grant from Alberta Education provided an opportunity for collaborative teams to provide access to free resources as part of a coordinated effort to ensure that K–6 education stakeholders share common understandings around the expectations of the Alberta Mathematics Program of Studies.

During the 2016/17 year, this website will continue to morph and grow through the addition of resources, activities, research and evidence of learner understanding. Once you visit, check back often. If you are on Twitter, follow us at @EMPL_AB in order to receive update alerts and contest announcements.

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Alberta High School Mathematics Competition 2015/16

Part 1

Question 1.
How many three digit numbers have the product of the three digits equal to 5?

(a) 1 (b) 2 (c) 3 (d) 5 (e) 6

Question 2.
Let m, n be positive integers such that $2^{30}3^{30} = 8^m9^n$. Determine the value of $m + n$.

(a) 15 (b) 20 (c) 25 (d) 30 (e) 35

Question 3.
The x-intercept, y-intercept, and slope of a certain straight line are three nonzero real numbers. The number of *negative* numbers among these three numbers is:

(a) 0 or1 (b) 1 or 2 (c) 2 or 3 (d) 0 or 2 (e) 1 or 3

Question 4.
The length of a certain rectangle is increased by 20% and its width is increased by 30%. Then its area is increased by:

(a) 25% (b) 48% (c) 50% (d) 56% (e) 60%

Question 5.
Each of Alan, Bailey, Clara and Diane has a number of candies. Compared with the average of the number of candies each person has, Alan has 6 more than the average, Bailey has 2 more than the average, Clara has 10 fewer than the average and Diane has k candies more than the average. Determine k.

(a) 1 (b) 2 (c) 3 (d) 4 (e) not uniquely determined

Question 6.
Ellie wishes to choose three of the seven days (Monday, Tuesday,. . . , Sunday) on which to wash her hair every week, so that she will never wash her hair on consecutive days. The number of ways she can choose these three days is:

(a) 6 (b) 7 (c) 8 (d) 10 (e) 14

Question 7.
How many different sets of two or more consecutive whole numbers have sum 55?

(a) 2 (b) 3 (c) 4 (d) 5 (e) none of these

Question 8.
There are 5 boys and 6 girls in a class. A committee of three students is to be made such that there is a boy and a girl on the committee. In how many different ways can the committee be selected?

(a) 100 (b) 135 (c) 145 (d) 155 (e) 165

Question 9.
In a class with 20 students, 14 wear glasses, 15 wear braces, 17 wear ear-rings and 18 wear wigs. What is the minimum number of students in this class who wear all four items?

(a) 4 (b) 6 (c) 7 (d) 9 (e) 10

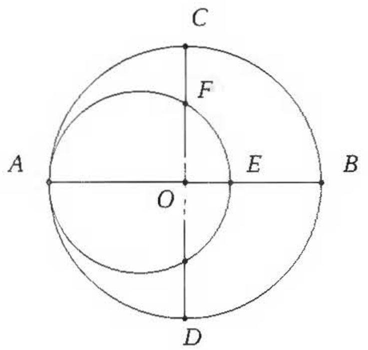
Question 10.
Each person has two legs. Some are sitting on three-legged stools while the others are sitting on four-legged chairs such that all the stools and chairs are occupied. If the total number of legs is 39, how many people are there?

(a) 5 (b) 6 (c) 7 (d) 8 (e) 9

Question 11.
The number of integers n for which the fraction $\frac{2^{2015}}{5n+1}$ is an integer is

(a) 503 (b) 504 (c) 1006 (d) 1007 (e) 1008

Question 12.
In the diagram below, which is not drawn to scale, the circles are tangent at A, the centre of the larger circle is at O and the lines AB and CD are perpendicular.



If $EB = 3$ and $FC = 2$ then the radius of the smaller circle is

(a) 4/3 (b) 5/3 (c) 5/2 (d) 3 (e) 7/2

Question 13.
Consider the expansion

$$\left(1+x+x^2+\cdots+x^{50}\right)^3 = c_0 + c_1x + c_2x^2 + \cdots + c_{150}x^{150}.$$

The value of the coefficient c_{50} is

(a) 1274 (b) 1275 (c) 1326 (d) 1378 (e) none of these

Question 14.
A 1000 digit number has the property that every two consecutive digits form a number that is a product of four prime numbers. The digit in the 500th position is

(a) 2 (b) 4 (c) 5 (d) 6 (e) 8

Question 15
Points E and F are on the sides BC and respectively CD of the parallelogram $ABCD$ such that $\frac{EB}{EC} = \frac{2}{3}$ and $\frac{FC}{FD} = \frac{1}{4}$. Let M be the intersection of AE and BF. The value of $\frac{AM}{ME}$ is equal to

(a) 11 (b) $11\frac{1}{2}$ (c) 12 (d) $12\frac{1}{2}$ (e) $12\frac{3}{4}$

Question 16.
Each of Alvin, Bob and Carmen spent five consecutive hours composing problems. Alvin started alone, and was later joined by Bob. Carmen joined in before Alvin stopped. When one person was working alone 4 problems were composed per hour. When two people were working together, each only composed 3 problems per hour. When all three were working, each composed only 2 problems per hour. No coming or going occurs during the composition of any problem. At the end, 46 problems were composed. How many were composed by Bob?

(a) 14 (b) 15 (c) 16 (d) 17 (e) 18

Solutions (Part 1)

Question 1.
How many three digit numbers have the product of the three digits equal to 5?

(a) 1 (b) 2 (c) 3 (d) 5 (e) 6

Solution:
The numbers are $115, 151, 511$. The answer is **(c)**.

Question 2.
Let m, n be positive integers such that $2^{30}3^{30} = 8^m 9^n$. Determine the value of $m+n$.

(a) 15 (b) 20 (c) 25 (d) 30 (e) 35

Solution:
The equation can be written as $2^{30-3m} = 3^{2n-30}$ hence $m = 10, n = 15$, thus $m+n = 25$. The answer is **(c)**.

Question 3.
The x-intercept, y-intercept, and slope of a certain straight line are three nonzero real numbers. The number of *negative* numbers among these three numbers is:

(a) 0 or1 (b) 1 or 2 (c) 2 or 3 (d) 0 or 2 (e) 1 or 3

Solution:
The slope of the line with the x- intercept at $(a, 0)$ and y-intercept at $(b, 0)$ is $m = -\frac{b}{a}$. If a, b are of the same sign, m is negative and if they are of opposite sign m is positive. Hence the number of negative numbers among a, b, m is 1 or 3. The answer is **(e)**.

Question 4.
The length of a certain rectangle is increased by 20% and its width is increased by 30%. Then its area is increased by:

(a) 25% (b) 48% (c) 50% (d) 56% (e) 60%

Solution:
If l and w denote the length and width of the rectangle then its area is $A = l \cdot w$ while the area of the increased rectangle is

$$(l + \frac{20l}{100}) \cdot (w + \frac{30w}{100}) = l \cdot w \cdot \frac{156}{100} = A \cdot \frac{156}{100} = A + A \cdot \frac{56}{100}.$$

Thus the area of the rectangle is increased by 56%. The answer is **(d)**.

Question 5.
Each of Alan, Bailey, Clara and Diane has a number of candies. Compared with the average of the number of candies each person has, Alan has 6 more than the average, Bailey has 2 more than the average, Clara has 10 fewer than the average and Diane has k candies more than the average. Determine k.

(a) 1 (b) 2 (c) 3 (d) 4 (e) not uniquely determined

Solution:
Let m be the average in question. Then the four people have a total of $4m$ candies. The Alan, Bailey, Clara has $m+6, m+2, m-10$ candies, which totals $3m-2$ candies. Therefore, Diane has $4m-(3m-2)=m+2$ and thus has 2 more candies than the average.

An alternate approach: since the total of the differences from the average must be zero, Diane should have just 10-6-2=2 candies more than the average.

The answer is **(b)**.

Question 6.
Ellie wishes to choose three of the seven days (Monday, Tuesday,..., Sunday) on which to wash her hair every week, so that she will never wash her hair on consecutive days. The number of ways she can choose these three days is:

(a) 6 (b) 7 (c) 8 (d) 10 (e) 14

Solution:
Ellie can choose on of the following triplets: (M, W, F), (M, W, Sa), (M, R, Sa), (T, R, Sa), (T, R, Su), (T, F, Su), (W, F, Su). There are seven possibilities.

Here is an alternate solution: In any such choice of three wash days, exactly one of them must be followed by two non-wash days. The choice of this day will determine the other two wash days. There are seven possibilities and thus the answer is **(b)**.

Question 7.
How many different sets of two or more consecutive whole numbers have sum 55?

(a) 2 (b) 3 (c) 4 (d) 5 (e) none of these

Solution:
The sum of k positive consecutive integers is

$$a+(a+1)+\cdots(a+k-1)=ka+\frac{k(k-1)}{2}=\frac{k(2a+k-1)}{2}.$$

and thus $k(2a+k-1)=110=2\cdot 5\cdot 11$. The solutions (k,a) are $(2,27),(5,9)$, and $(10,1)$ for which

$$55=27+28=9+10+11+12+13=1+2+\cdots+10.$$

If one consider the set of whole numbers $W=\{1,2,3,...\}$ then there are three sets of consecutive whole numbers having the sum 55. However, if $W=\{0,1,2,3,...\}$ then also

$$55=0+1+2+\cdots+10.$$

and we find four sets with the required property.

The answer is **(b)** or **(c)**.

Question 8.
There are 5 boys and 6 girls in a class. A committee of three students is to be made such that there is a boy and a girl on the committee. In how many different ways can the committee be selected?

(a) 100 (b) 135 (c) 145 (d) 155 (e) 165

Solution:

The number of committees of 3 students made with 11 students is $\binom{11}{3} = 165$. The number of committees of three girls or three boys is $\binom{6}{3} + \binom{5}{3} = 20 + 10 = 30$. The number of requested committees is 165-30=135. The answer is **(b)**.

Question 9.

In a class with 20 students, 14 wear glasses, 15 wear braces, 17 wear ear-rings and 18 wear wigs. What is the minimum number of students in this class who wear all four items?

(a) 4 (b) 6 (c) 7 (d) 9 (e) 10

Solution:

We have 6 students not wearing glasses, 5 students not wearing braces, 3 students not wearing ear-rings and 2 students not wearing wigs. Even if these are 6+5+3+2=16 different students, we still have $20 - 16 = 4$ students wearing all four items. The answer is **(a)**.

Question 10.

Each person has two legs. Some are sitting on three-legged stools while the others are sitting on four-legged chairs such that all the stools and chairs are occupied. If the total number of legs is 39, how many people are there?

(a) 5 (b) 6 (c) 7 (d) 8 (e) 9

Solution:

Let the number of stools be m and the number of chairs be n. Then $5m + 6n = 39$. Hence m is a multiple of 3 but not a multiple of 6. Moreover, $5m < 39$ so that $m \leq 7$. It follows that $m = 3$ and $n = (39 - 3 \times 5) \div 6 = 4$, so that the number of people is 3+4=7. The answer is **(c)**.

Question 11.

The number of integers n for which the fraction $\frac{2^{2015}}{5n-1}$ is an integer is

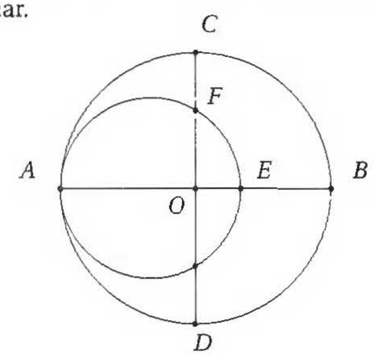
(a) 503 (b) 504 (c) 1006 (d) 1007 (e) 1008

Solution:

We should have $5n + 1 = \pm 2^k$ where $0 \leq k \leq 2015$. If $5n + 1 = 2^k$, then $5 | (2^k - 1)$, which happens if $k = 0, 4, 8 \cdots$, hence $k = 4s, 0 \leq s \leq 503$. For these values of k we obtain 504 nonnegative integers n. If $5n + 1 = -2^k$ then $5 | (2^k + 1)$, which happens if $k = 2, 6, 10, \cdots$, hence $k = 4s + 2$, $0 \leq s \leq 503$. For these values of k we obtain 504 negative integers n. Therefore there are 1008 convenient values for n. The answer is **(e)**.

Question 12.

In the diagram below, which is not drawn to scale, the circles are tangent at A, the centre of the larger circle is at O and the lines AB and CD are perpendicular.



If $EB = 3$ and $FC = 2$ then the radius of the smaller circle is

(a) 4/3 (b) 5/3 (c) 5/2 (d) 3 (e) 7/2

Solution:

Let R, r denote the lengths of the large and respectively the small radius and $EB = a, FC = b$. First $a = 2R - 2r$ so that $R = r + a/2$. If O' denotes the centre of the smaller circle then $O'F^2 = O'O^2 + OF^2$ hence $r^2 = \frac{a^2}{4} + (r + \frac{a}{2} - b)^2$. Solving for r we get $r = \frac{(2b-a)^2 + a^2}{4(2b-a)}$. Taking $a = 3, b = 2$ we get $r = \frac{5}{2}$. The answer is **(c)**.

Question 13.

Consider the expansion

$$\left(1 + x + x^2 + \cdots + x^{50}\right)^3 = c_0 + c_1 x + c_2 x^2 + \cdots + c_{150} x^{150}.$$

The value of the coefficient c_{50} is

(a) 1274 (b) 1275 (c) 1326 (d) 1378 (e) none of these

Solution:

$$\left(1 + x + x^2 + \cdots + x^{50}\right)^3$$
$$= \left(1 + x + x^2 + \cdots + x^{50}\right) \cdot \left(1 + x + x^2 + \cdots + x^{50}\right) \cdot \left(1 + x + x^2 + \cdots + x^{50}\right)$$

The coefficient of x^{50} is just the number of $x^a x^b x^c = x^{a+b+c}$ with $a + b + c = 50, a, b, c \in \{0, 1, \cdots, 50\}$. If $a = 0$ the equation $b + c = 50$ has 51 solutions, namely $(0,50), (1,49), \cdots, (50,0)$. Also, if $a = 1$, the equation $b + c = 49$ has 50 solutions and so on. The number of all solutions is

$$51 + 50 + \cdots + 1 = 1326$$

Hence $c_{50} = 1326$. The answer is **(c)**.

Question 14.

A 1000 digit number has the property that every two consecutive digits form a number that is a product of four prime numbers. The digit in the 500th position is

(a) 2 (b) 4 (c) 5 (d) 6 (e) 8

Solution:

The numbers of two digits which are written as product of four prime numbers are the following: $2 \cdot 2 \cdot 2 \cdot 2 = 16$, $2 \cdot 2 \cdot 2 \cdot 3 = 24$, $2 \cdot 2 \cdot 2 \cdot 5 = 40$, $2 \cdot 2 \cdot 2 \cdot 7 = 56$, $2 \cdot 2 \cdot 2 \cdot 11 = 88$, $2 \cdot 2 \cdot 3 \cdot 3 = 36$, $2 \cdot 2 \cdot 3 \cdot 5 = 60$, $2 \cdot 2 \cdot 3 \cdot 7 = 84$, $2 \cdot 3 \cdot 3 \cdot 3 = 54$, $2 \cdot 3 \cdot 3 \cdot 5 = 90$, $3 \cdot 3 \cdot 3 \cdot 3 = 81$.

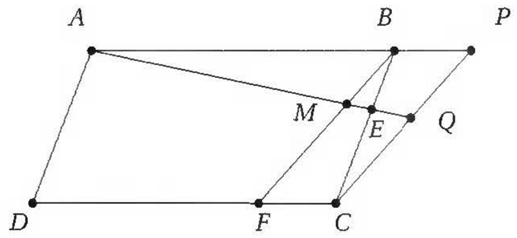
We conclude that the number that satisfies the conditions in the problem should have all its digits equal to 8. The answer is **(e)**

Question 15

Points E and F are on the sides BC and respectively CD of the parallelogram $ABCD$ such that $\frac{EB}{EC} = \frac{2}{3}$ and $\frac{FC}{FD} = \frac{1}{4}$. Let M be the intersection of AE and BF. The value of $\frac{AM}{ME}$ is equal to

(a) 11 (b) $11\frac{1}{2}$ (c) 12 (d) $12\frac{1}{2}$ (e) $12\frac{3}{4}$

Solution:



The parallel line to FB through C intercepts AB at P and AM intercepts CP at Q. Then

$$\frac{MQ}{ME} = \frac{ME + EQ}{ME} = 1 + \frac{EQ}{ME} = 1 + \frac{EC}{EB} = 1 + \frac{3}{2} = \frac{5}{2}$$

and

$$\frac{AM}{MQ} = \frac{AB}{BP} = \frac{DC}{FC} = \frac{DF + FC}{FC} = 1 + \frac{DF}{FC} = 4 + 1 = 5$$

Hence $\frac{AM}{ME} = \frac{AM}{MQ} \cdot \frac{MQ}{ME} = 5 \cdot \frac{5}{2} = 12.5$. The answer is **(d)**.

Question 16.
Each of Alvin, Bob and Carmen spent five consecutive hours composing problems. Alvin started alone, and was later joined by Bob. Carmen joined in before Alvin stopped. When one person was working alone 4 problems were composed per hour. When two people were working together, each only composed 3 problems per hour. When all three were working, each composed only 2 problems per hour. No coming or going occurs during the composition of any problem. At the end, 46 problems were composed. How many were composed by Bob?

(a) 14 (b) 15 (c) 16 (d) 17 (e) 18

Solution:
The total work period may be divided into five intervals by comings and goings. The respective numbers of people working during these intervals are 1, 2, 3, 2 and 1 respectively. Note that the total length of any three consecutive intervals is 5 hours. Hence the fourth interval has the same length as the first and the fifth interval has the same length as the second. During each of the second, third and fourth interval, the number of problems composed was 6 per hour since 3+3=6=2+2+2. Hence $5 \times 6 = 30$ problems were composed when Bob was working. The number of problems composed when Alvin or Carmen was working alone was 46-30=16. It follows that the total length of these two intervals is equal to $16 \div 4 = 4$ hours. Hence the total length of the second and the fourth interval is also 4 hours, so that the length of the third interval is $5 - 4 = 1$ hour. Thus the number of problems composed by Bob was $1 \times 2 + 4 \times 3 = 14$. The answer is **(a)**.

Part 2

Problem 1.
Find all linear polynomials $f(x) = ax + b$, where a and b are real constants, satisfying $f(f(3)) = 3f(3)$ and $f(f(4)) = 4f(4)$.

Problem 2.

(a) Alya adds the following sequence of numbers together, one number at a time: $1, 2, -3, -4, 5, 6, -7, -8, 9\ldots$, where the first two numbers are positive, the next two negative, the next two positive, and so on. Thus she gets the totals $1, 1+2, 1+2-3, 1+2-3-4, 1+2-3-4+5$, and so on. Prove that she will get zero infinitely often.

(b) Suppose instead Alya adds together the numbers $1, 2, 3, -4, -5, -6, 7, 8\ldots$ where the first three numbers are positive, the next three negative, the next three positive, and so on. Prove that she will **never** get zero as a sum.

Problem 3.
In a rectangle of area 12 are placed 16 polygons, each of area 1. Show that among these polygons there are at least two which overlap in a region of area at least $\frac{1}{30}$.

Problem 4.
Find all finite sets M of real numbers such that, whenever a number x is in M, then the number $x^2 - 3|x| + 4$ is also in M. (Note that $|x|$ denotes the absolute value of the real number x.)

Problem 5.
In $\triangle ABC$, $\widehat{A}$ is the largest angle and M, N are points on the sides $[AB]$ and respectively $[AC]$ such that $\frac{MB}{MA} = \frac{NA}{NC}$. Show that there is a point P on the side $[BC]$ such that $\triangle PMN$ and $\triangle ABC$ are similar.

Solutions (Part 2)

Problem 1.

Find all linear polynomials $f(x) = ax + b$, where a and b are real constants, satisfying $f(f(3)) = 3f(3)$ and $f(f(4)) = 4f(4)$.

Solution:

The condition $f(f(3)) = 3f(3)$ becomes $a(3a+b)+b = 3(3a+b)$ which simplifies to

$$3a^2 + ab = 9a + 2b, \tag{1}$$

and similarly the condition $f(f(4)) = 4f(4)$ simplifies to

$$4a^2 + ab = 16a + 3b. \tag{2}$$

Subtracting (1) and (2) we get $b = a^2 - 7a$, which when plugged into (1) gives $a^3 - 6a^2 + 5a = 0$. Factoring, we get $a(a-1)(a-5) = 0$, so $a = 0, 1, 5$. These give respectively $b = 0, -6, -10$, so the solutions for f are

$$f(x) = 0, \quad f(x) = x - 6, \quad f(x) = 5x - 10.$$

Problem 2.

(a) Alya adds the following sequence of numbers together, one number at a time: $1, 2, -3, -4, 5, 6, -7, -8, 9 \ldots$, where the first two numbers are positive, the next two negative, the next two positive, and so on. Thus she gets the totals $1, 1+2, 1+2-3, 1+2-3-4, 1+2-3-4+5$, and so on. Prove that she will get zero infinitely often.

(b) Suppose instead Alya adds together the numbers $1, 2, 3, -4, -5, -6, 7, 8 \ldots$ where the first *three* numbers are positive, the next three negative, the next three positive, and so on. Prove that she will **never** get zero as a sum.

Solution:

(a) Break the sequence into blocks of size four. Each block is of the form $1+4k, 2+4k, -(3+4k), -(4+4k)$ for some non-negative integer k, and the sum of each such block is always -4. Thus if Alya adds up p blocks, she gets a sum of $-4p$. The next three numbers in the sequence will be $4p+1, 4p+2$, and $-(4p+3)$, so the sum after adding these in will be respectively $-4p+4p+1 = 1, 1+(4p+2) = 4p+3$, and $(4p+3)-(4p+3) = 0$. So Alya will get a sum of zero whenever she stops adding at a number of the form $-(4p+3)$, for any value of p.

(b) Break the sequence into blocks of size six. Each block is of the form $1+6k, 2+6k, 3+6k, -4-6k, -5-6k, -6-6k$ for some non-negative integer k, and the sum of each such block is always -9. Thus if Alya adds up p blocks, she gets a sum of $-9p$. The next five numbers in the sequence will be$1+6p, 2+6p, 3+6p, -4-6p$, and $-5-6p$ so the sum after adding these in will be respectively $-9p+(1+6p) = 1-3p, 1-3p+(2+6p) = 3p+3, (3p+3)+(3+6p) = 9p+6, 9p+6+(-4-6p) = 3p+2$, and $3p+2+(-5-6p) = -3p-3$. None of these sums can equal zero for any nonnegative integer value of p, so Alya will never get a sum of zero.

Problem 3.

In a rectangle of area 12 are placed 16 polygons, each of area 1. Show that among these polygons there are at least two which overlap in a region of area at least $\frac{1}{30}$.

Solution:

Let $P_1, P_2, ..., P_{16}$ denote the given polygons of area 1. Assume that any two of them intersect in a region of area $< \frac{1}{30}$. Then, the area of the part of P_2 that is not covered by P_1 is of area $> 1 - \frac{1}{30}$, the area of the part of P_3 that is not covered by P_1 and P_2 is of area $> 1 - \frac{2}{30}$ and so on, the part of P_{16} that is not covered by $P_1, ...P_{15}$ is of area $> 1 - \frac{15}{30}$. Therefore, the area of the region obtained by considering the union of all 16 polygons would be $> 1 + (1 - \frac{1}{30}) + \ldots + (1 - \frac{15}{30}) = 16 - \frac{15 \cdot 16}{60} = 12$, that is a contradiction.

Problem 4.

Find all finite sets M of real numbers such that, whenever a number x is in M, then the number $x^2 - 3|x| + 4$ is also in M. (Note that $|x|$ denotes the absolute value of the real number x.)

Solution:

The empty set satisfies the requested condition. So now we assume that M contains at least one element. For any real number x, let $f(x) = x^2 - 3|x| + 4$, and notice the following:

(i) $f(x) = (|x| - 3/2)^2 + 7/4 \geq 7/4$, in particular $f(x)$ cannot equal 1 for any x.

(ii) If $f(x) = 2$ then $x^2 - 3|x| + 2 = 0$ which means $(|x| - 1)(|x| - 2) = 0$ which means $|x| = 1$ or $|x| = 2$. Conversely, if $|x| = 1$ or $|x| = 2$ then $f(x) = 2$.

From (i), if M is nonempty then M must contain at least one element which is $\geq 7/4$. In fact, we claim that 1 and 2 are the only possible positive elements of M. For assume that $a_0 \in M$ where $a_0 > 0$, $a_0 \neq 1$ and $a_0 \neq 2$. Since a_0 is in M, so is $a_1 = f(a_0) = a_0^2 - 3a_0 + 4 = (a_0 - 2)^2 + a_0 > a_0$. Also, from (i) $a_1 \neq 1$, and from (ii) $a_1 \neq 2$. Similarly $a_2 = f(a_1) \in M$, $a_2 \neq 1$, $a_2 \neq 2$ and $a_2 > a_1 > a_0$, and so on indefinitely, hence M is not finite. Consequently 1 and 2 are the only possible positive elements of M, as claimed.

Now if $b \in M$ and $b \leq 0$ then $f(b) > 0$ (from (i)) and $f(b) \in M$, hence $f(b) = 1$ or $f(b) = 2$. From (i), $f(b) = 1$ is impossible, so $f(b) = 2$. From (ii), $|b| = 1$ or 2, so $b = -1$ or $b = -2$.

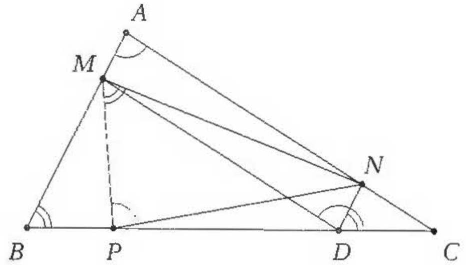
From the above, any nonempty set M satisfying the condition must contain 2, and may contain any subset of $\{-2, -1, 1\}$. Therefore the set M is one of the following sets:

$$\emptyset, \{2\}, \{-2, 2\}, \{-1, 2\}, \{1, 2\}, \{-2, -1, 2\}, \{-2, 1, 2\}, \{-1, 1, 2\}, \{-2, -1, 1, 2\}.$$

Problem 5.

In $\triangle ABC$, $\widehat{A}$ is the largest angle and M, N are points on the sides $[AB]$ and respectively $[AC]$ such that $\frac{MB}{MA} = \frac{NA}{NC}$. Show that there is a point P on the side $[BC]$ such that $\triangle PMN$ and $\triangle ABC$ are similar.

Solution:



Let ND be parallel to AB, $D \in [BC]$. Since $\frac{DB}{DC} = \frac{NA}{NC} = \frac{MB}{MA}$ it follows that $DM \parallel AC$, hence $MAND$ is a parallelogram and thus $\widehat{A} = \widehat{MDN}$.

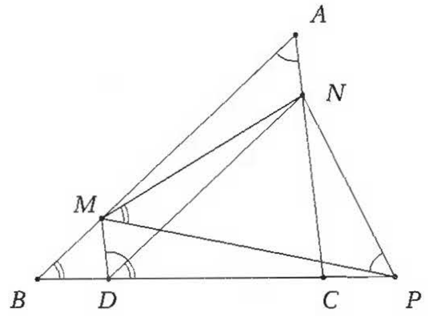
We remark that at least one of the inequalities $\widehat{BMN} > \widehat{B}$ and $\widehat{MNC} > \widehat{C}$ is true. Indeed, if we assume to the contrary that $\widehat{BMN} \leq \widehat{B}$ and $\widehat{MNC} \leq \widehat{C}$ then

$$360° = \widehat{B} + \widehat{C} + \widehat{BMN} + \widehat{MNC} \leq 2(\widehat{B} + \widehat{C}) \iff 180° \leq \widehat{B} + \widehat{C}$$

which is a contradiction. We may assume that $\widehat{BMN} > \widehat{B}$ and let $(MP$ be the ray inside $\widehat{BMN}$, with P on the ray $(BC$ such that $\widehat{PMN} = \widehat{B}$.

If $P \in [BC]$ (as in the above diagram), then $\widehat{PMN} = \widehat{B} = \widehat{NDC}$, hence the quadrilateral $MNDP$ is cyclic and therefore $\widehat{MDN} = \widehat{MPN}$. Consequently $\widehat{A} = \widehat{MPN}$ and thus $\triangle ABC$ is similar to $\triangle PMN$ and P is the requested point.

Notice that if P is such that C is between B and P (as in the diagram below), then similarly as above, we obtain $\widehat{A} = \widehat{MPN}$. However, this is not possible since it leads to $\widehat{C} > \widehat{CPN} > \widehat{MPN} = \widehat{A}$, a contradiction.



Calgary Junior High Mathematics Competition 2015/16

NAME: ________________________________ GENDER: _______

PLEASE PRINT (First name Last name)

SCHOOL: ________________________________ GRADE: _______

(9,8,7,...)

- You have 90 minutes for the examination. The test has two parts: PART A — short answer; and PART B — long answer. The exam has 9 pages including this one.
- Each correct answer to PART A will score 5 points. You must put the answer in the space provided. No part marks are given. PART A has a total possible score of 45 points.
- Each problem in PART B carries 9 points. You should show all your work. Some credit for each problem is based on the clarity and completeness of your answer. You should make it clear why the answer is correct. PART B has a total possible score of 54 points.
- You are permitted the use of rough paper. Geometry instruments are not necessary. References including mathematical tables and formula sheets are **not** permitted. Simple calculators without programming or graphic capabilities **are** allowed. Diagrams are not drawn to scale: they are intended as visual hints only.
- When the teacher tells you to start work you should read all the problems and select those you have the best chance to do first. You should answer as many problems as possible, but you may not have time to answer all the problems.
- Hint: Read all the problems and select those you have the best chance to solve first. You may not have time to solve all the problems.

MARKERS' USE ONLY	
PART A _______ ×5	
B1	
B2	
B3	
B4	
B5	
B6	
TOTAL (max: 99)	

BE SURE TO MARK YOUR NAME AND SCHOOL AT THE TOP OF THIS PAGE.

THE EXAM HAS 9 PAGES INCLUDING THIS COVER PAGE.

Please return the entire exam to your supervising teacher at the end of 90 minutes.

Part A

A1 A rectangle with integer length and integer width has area 13 cm^2. What is the perimeter of the rectangle in cm?

A1

A2 A nice fact about the current year is that 2016 is equal to the sum $1+2+3+\cdots+63$ of the first 63 positive integers. When Richard told this to his grandmother, she said: *Interesting! I was born in a year which is also the sum of the first X positive integers, where X is some positive integer.* In what year was Richard's grandmother born? (You may assume that Richard's grandmother is less than 100 years old.)

A2

A3 Suppose you reduce each of the following 64 fractions to lowest terms:

$$\frac{1}{64}, \frac{2}{64}, \frac{3}{64}, \ldots, \frac{64}{64}.$$

How many of the resulting 64 reduced fractions have a denominator of 8?

A3

A4 Peppers come in four colours: green, red, yellow and orange. In how many ways can you make a bag of six peppers so that there is at least one of each colour?

A4

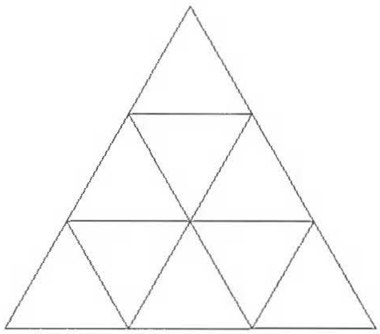
A5 In the following addition of four-digit numbers, X, Y and Z stand for digits (not necessarily different). If the addition is correct, what digit does Y stand for?

$$\begin{array}{r} 2\ X\ X\ Y \\ +\ 3\ X\ Y\ Z \\ \hline Z\ Y\ X\ X \end{array}$$

A5

A6 How many equilateral triangles of any size are there in the figure below?

A6



A7 A number was decreased by 20%, and the resulting number increased by 20%. What percentage of the original number is the final result?

A7

A8 A group of grade 7 students and grade 9 students are at a banquet. The average height of the grade 9 students is 180 cm. The average height of the grade 7 students is 160 cm. If the average height of all students at the banquet is 168 cm and there are 72 grade 9 students, how many grade 7 students are there?

A8

A9 If the straight-line distance from one corner of a cube to the opposite corner (i.e., the length of the long diagonal or body-diagonal of a cube) is 9 cm, what is the area (in cm^2) of one of its faces?

A9

Part B

B1 When the Cookie Monster visits the cookie jars, he takes from as many jars as he likes, but always takes the same number of cookies from each of the jars that he does select.

(i) Suppose that there are four jars containing 11, 5, 4 and 2 cookies. Then, for example, he might take 4 from each of the first three jars, leaving 7, 1, 0 and 2; then 2 from the first and last, leaving 5, 1. 0 and 0, and he will need two more visits to empty all the jars. Show how he could have emptied these four cookie jars in less than four visits.

(ii) Suppose instead that the four jars contained a, b, c and d cookies, respectively, with $a \geq b \geq c \geq d$. Show that if $a = b + c + d$, then three visits are enough to empty all the jars.

B2 The number 102564 has the property that if the last digit is moved to the front, the resulting number, namely 410256, is 4 times bigger than the original number:

$$410256 = 4 \times 102564.$$

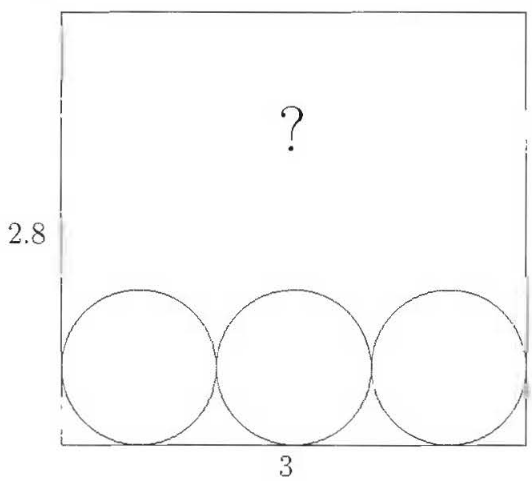
Find a six-digit number whose last digit is 9 and which becomes 4 times bigger when we move this 9 to the front.

B3 In a sequence, each term after the first is the sum of squares of the digits of the previous term. For example, if the first term is 42 then the next term is $4^2+2^2=20$. The next term after 20 is then $2^2+0^2=4$, followed by $4^2=16$, which is then followed by $1^2+6^2=37$, and so on, giving the sequence 42, 20, 4, 16, 37, and so on.

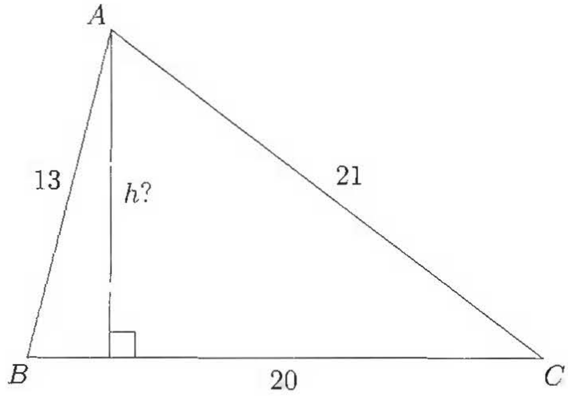
(a) If the first term is 44, what is the 2016th term?

(b) If the first term is 25, what is the 2016th term?

B4 Is it possible to pack 8 balls of diameter 1 into a 1 by 3 by 2.8 box? Explain why or why not.



B5 The triangle ABC has edge-lengths $BC = 20$, $CA = 21$, and $AB = 13$. What is its height h shown in the figure?



B6 Find all positive integer solutions a, b, c, with $a \leq b \leq c$ such that

$$\frac{6}{7} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$$

and show that there are no other solutions.

Solutions (Part A)

A1 A rectangle with integer length and integer width has area 13 cm^2. What is the perimeter of the rectangle in cm?

A1
28

A2 A nice fact about the current year is that 2016 is equal to the sum $1+2+3+\cdots+63$ of the first 63 positive integers. When Richard told this to his grandmother, she said: *Interesting! I was born in a year which is also the sum of the first X positive integers, where X is some positive integer.* In what year was Richard's grandmother born? (You may assume that Richard's grandmother is less than 100 years old.)

A2
1953

A3 Suppose you reduce each of the following 64 fractions to lowest terms:

$$\frac{1}{64}, \frac{2}{64}, \frac{3}{64}, \ldots, \frac{64}{64}.$$

How many of the resulting 64 reduced fractions have a denominator of 8?

A3
4

A4 Peppers come in four colours: green, red, yellow and orange. In how many ways can you make a bag of six peppers so that there is at least one of each colour?

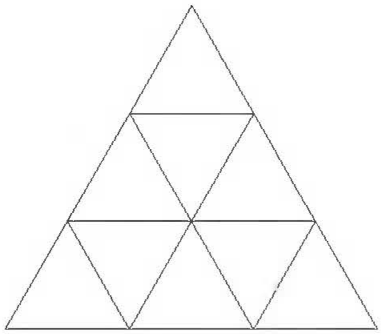
A4
10

A5 In the following addition of four-digit numbers, X, Y and Z stand for digits (not necessarily different). If the addition is correct, what digit does Y stand for?

$$\begin{array}{r} 2\;X\;X\;Y \\ +\;3\;X\;Y\;Z \\ \hline Z\;Y\;X\;X \end{array}$$

A5
9

A6 How many equilateral triangles of any size are there in the figure below?



A6
13

A7 A number was decreased by 20%, and the resulting number increased by 20%. What percentage of the original number is the final result?

A7
96

A8 A group of grade 7 students and grade 9 students are at a banquet. The average height of the grade 9 students is 180 cm. The average height of the grade 7 students is 160 cm. If the average height of all students at the banquet is 168 cm and there are 72 grade 9 students, how many grade 7 students are there?

A8
108

A9 If the straight-line distance from one corner of a cube to the opposite corner (i.e., the length of the long diagonal or body-diagonal of a cube) is 9 cm, what is the area (in cm^2) of one of its faces?

A9
27

Solutions (Part B)

B1 When the Cookie Monster visits the cookie jars. he takes from as many jars as he likes, but always takes the same number of cookies from each of the jars that he does select.

(i) Suppose that there are four jars containing 11, 5. 4 and 2 cookies. Then. for example, he might take 4 from each of the first three jars. leaving 7. 1. 0 and 2; then 2 from the first and last, leaving 5, 1. 0 and 0. and he will need two more visits to empty all the jars. Show how he could have emptied these four cookie jars in less than four visits.

Solution. Take 5 from each of the first two jars, leaving 6. 0, 4. 2: then 4 from the first and third; and finally 2 from the first and last: and he has done it in three visits.

(ii) Suppose instead that the four jars contained a. b, c and d cookies, respectively, with $a \geq b \geq c \geq d$. Show that if $a = b + c + d$. then three visits are enough to empty all the jars.

Solution. Take b from each of the first two jars, leaving $c + d$, 0, c. d; then c from the first and third; and finally d from the first and last: and he has done it in three visits.

B2 The number 102564 has the property that if the last digit is moved to the front, the resulting number, namely 410256, is 4 times bigger than the original number:

$$410256 = 4 \times 102564.$$

Find a six-digit number whose last digit is 9 and which becomes 4 times bigger when we move this 9 to the front.

Solution 1. We must find a, b, c, d, e so that

$$\begin{array}{cccccc} a & b & c & d & e & 9 \\ \times & & & & & 4 \\ \hline 9 & a & b & c & d & e \end{array}$$

Multiplying gives $e = 6$, $d = 7$, $c = 0$, $b = 3$ and $a = 2$. Thus, a six-digit number whose last digit is 9 and which becomes 4 times bigger when we move this 9 to the front is 230769.

Solution 2. Consider a six-digit number whose last digit is 9: $a\,b\,c\,d\,e\,9$.
Letting $x = a\,b\,c\,d\,e$ gives

$$900000 + x = 4(10x + 9)$$

$$39x = 899964$$

$$3x = 69228$$

$$x = 23076$$

Thus, the number is 230769.

B3 In a sequence, each term after the first is the sum of squares of the digits of the previous term. For example, if the first term is 42 then the next term is $4^2+2^2=20$. The next term after 20 is then $2^2+0^2=4$, followed by $4^2=16$, which is then followed by $1^2+6^2=37$, and so on, giving the sequence 42, 20, 4, 16, 37, and so on.

(a) If the first term is 44, what is the 2016th term?

(b) If the first term is 25, what is the 2016th term?

Solution.

(a) Starting with 44 gives

$4^2+4^2=16+16=32 \quad\to\quad 3^2+2^2=9+4=13 \quad\to\quad 1^2+3^2=1+9=10$
$\to\quad 1^2+0^2=1+0=1 \quad\to\quad 1^2=1 \quad\to\quad 1^2=1 \quad\cdots$

The sequence is then

$$\{44, 32, 13, 10, 1, 1, 1, \ldots\}$$

with 2016th term equal to 1.

(b) Starting with 25 gives

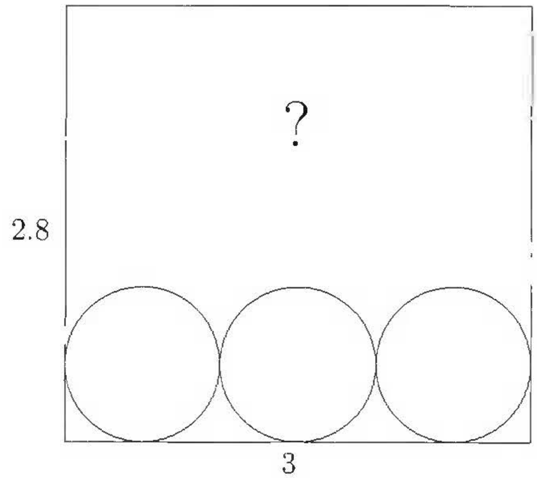
$2^2+5^2=4+25=29 \quad\to\quad 2^2+9^2=4+81=85 \quad\to\quad 8^2+5^2=64+25=89$
$\to\quad 8^2+9^2=64+81=145 \quad\to\quad 1^2+4^2+5^2=1+16+25=42$
$\to\quad 4^2+2^2=16+4=20 \quad\to\quad 2^2+0^2=4+0=4 \quad\to\quad 4^2=16$
$\to\quad 1^2+6^2=1+36=37 \quad\to\quad 3^2+7^2=9+49=58 \quad\to\quad 5^2+8^2=25+64=89$

The sequence is then

$$\{25, 29, 85, \mathbf{89}, \mathbf{145}, \mathbf{42}, \mathbf{20}, \mathbf{4}, \mathbf{16}, \mathbf{37}, \mathbf{58}, 89, \ldots\}$$

and repeats with a period of 8. Since $2013 = 251 \times 8 + 5$, the 2016th term in the sequence is 4.

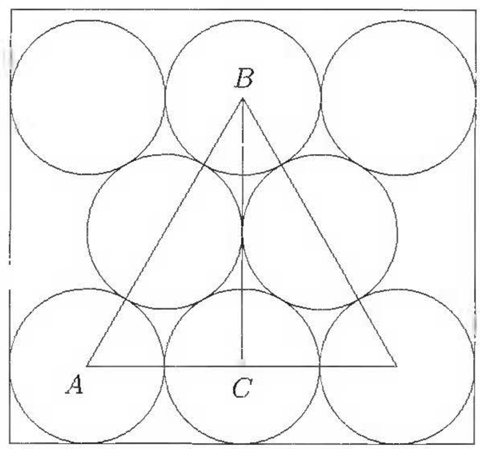
B4 Is it possible to pack 8 balls of diameter 1 into a 1 by 3 by 2.8 box? Explain why or why not.



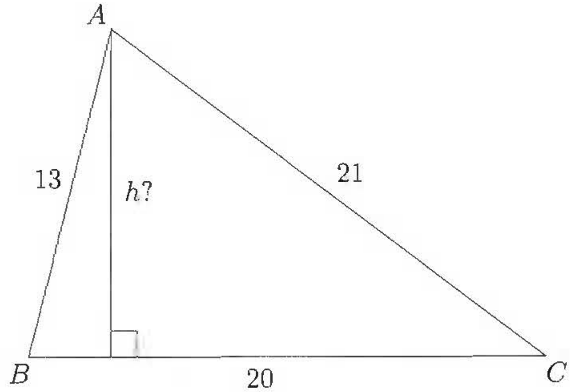
Solution.

Yes, it is possible to pack 8 balls.

The triangle ABC has edge-lengths $AB = 2$ and $AC = 1$. Using Pythagoras's theorem, $BC = \sqrt{3}$. Thus, the distance from the bottom of the bottom row of balls to the top of the top row of balls is $\frac{1}{2} + \sqrt{3} + \frac{1}{2} = 1 + \sqrt{3} = 2.732\ldots < 2.8$.



B5 The triangle ABC has edge-lengths $BC = 20$. $CA = 21$. and $AB = 13$. What is its height h shown in the figure?



Solution 1. Using Pythagoras's theorem we have

$\sqrt{13^2 - h^2} + \sqrt{21^2 - h^2} = 20$
$20 - \sqrt{13^2 - h^2} = \sqrt{21^2 - h^2}$
$400 - 40\sqrt{13^2 - h^2} + 13^2 - h^2 = 21^2 - h^2$
$128 = 40\sqrt{13^2 - h^2}$
$3.2 = \sqrt{13^2 - h^2}$
$h^2 = 13^2 - 3.2^2 = 9.8 \times 16.2$
$h = 12.6$

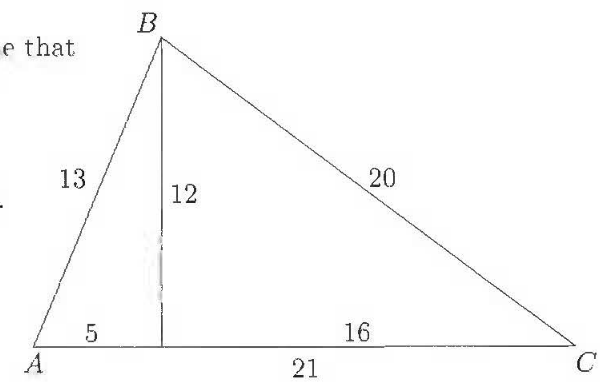
Solution 2. Heron's formula states that the area of a triangle whose sides have lengths a, b, and c is

$$\text{Area} = \sqrt{s(s-a)(s-b)(s-c)}.$$

where $s = (a+b+c)/2$ is the semiperimeter of the triangle. Then $a = 13$. $b = 21$ and $c = 20$ gives $s = (13+21+20)/2 = 27$. Thus, the triangle has area $\sqrt{27 \cdot 7 \cdot 6 \cdot 14} = 3^2 \cdot 7 \cdot 2 = 126$. Using the base of the triangle as 20. we have $126 = \frac{1}{2}(20)h$ implying $h = 126/(\frac{1}{2}\,20) = 12.6$.

Solution 3. Turn the triangle over and note that it is made up of two Pythagorean triangles.

It is then immediate that the area is 126, and division by half the base. 10. gives 12.6.



36 Find all positive integer solutions a, b, c, with $a \leq b \leq c$ such that

$$\frac{6}{7} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$$

and show that there are no other solutions.

Solution. Note that

$$\frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \frac{3}{4} < \frac{6}{7}.$$

hence $a \leq 3$. If $a = 3$, then $b = 3$ since $a \leq b$ and

$$\frac{1}{3} + \frac{1}{4} + \frac{1}{4} = \frac{5}{6} < \frac{6}{7}.$$

This implies $c = 27/4$ which is not an integer. thus $a = 2$. Now

$$\frac{1}{2} + \frac{1}{6} + \frac{1}{6} = \frac{5}{6} < \frac{6}{7}$$

implies $b \leq 5$. This gives four cases.
If $b = 2$, then $c = -7$.
If $b = 3$, then $c = 42$.
If $b = 4$. then $c = 28/3$.
If $b = 5$. then $c = 70/11$.
The only solution consisting of positive integers is $(a.b, c) = (2, 3, 42)$.

Edmonton Junior High Mathematics Competition 2015/16

Part A: Multiple Choice: Each correct answer is worth 4 points. Each unanswered question is worth 2 points to a maximum of 3 unanswered questions.

1. On a contest with 30 questions:

   - Each correct answer is awarded 6 points;
   - Each incorrectly answered question is penalized 2 points;
   - Each unanswered question is penalized 1 point.

   Abby answered 70% of the questions. The ratio of correct answers to incorrect answers to unanswered question is 5:2:3. How many points did she receive?

   A. 45 B. 126 C. 63 D. 81 E. 69

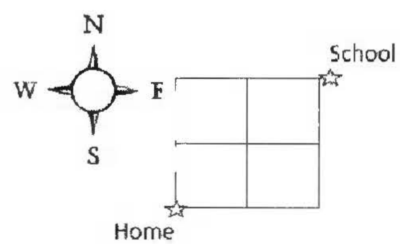
2. Working at a constant rate, if Ben can cut a metal pipe into 5 pieces in 30 minutes, how many minutes would it take to cut a similar pipe into 15 pieces?

   A. 210 B. 170 C. 105 D. 90 E. 65

3. Each of the first 6 prime numbers is written on the 6 different faces of a red cube. Each of the first 6 composite numbers is written on the 6 different faces of a blue cube. The two cubes are tossed/rolled once. What is the probability that the sum of the numbers rolled is 13?

   A. $\frac{1}{18}$ B. $\frac{1}{12}$ C. $\frac{1}{9}$ D. $\frac{1}{6}$ E. $\frac{1}{3}$

4. The distance from home to school is a total of 4 blocks. Jane must stay on the pathways and walk either east or north, as shown in the diagram.

   How many ways are there in total for Jane to walk from home to school and back again, if she must return using a different path?

   A. 5 B. 6 C. 11 D. 30 E. 36

5. Two rectangles with integer dimensions each have an area of 216 cm^2. The length of the first rectangle is 30 cm greater than that of the second rectangle. However, the width of the first rectangle is 5 cm less than that of the second rectangle. What is the difference in the perimeters in centimetres, of the two rectangles?

A. 25 B. 35 C. 50 D. 60 E. 70

6. A family agreed to share the total cost of buying a $2268 (including GST) gaming computer.

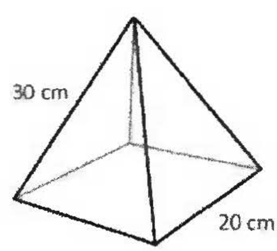
- Dad paid for $\frac{1}{6}$ of the total cost.
- Then the oldest daughter, Kylee, paid for $\frac{1}{5}$ of the remaining cost.
- Then the son, Shawn, paid for $\frac{1}{4}$ of the remaining cost.
- Then the middle daughter, Erin, paid for $\frac{1}{3}$ of the remaining cost.
- Then the youngest daughter, Kassidy, paid for $\frac{1}{2}$ of the remaining cost.

If Mom paid for the remaining amount, then approximately how much more did the children pay compared to the amount paid by the parents?

A. $300 B. $450 C. $600 D. $750 E. $900

7. The net of the square pyramid shown at the right would consist of 4 congruent isosceles triangles (each with sides measuring 30 cm, 30 cm, 20 cm) and one square base.

What is the total surface area, to the nearest whole cm^2, of the pyramid?



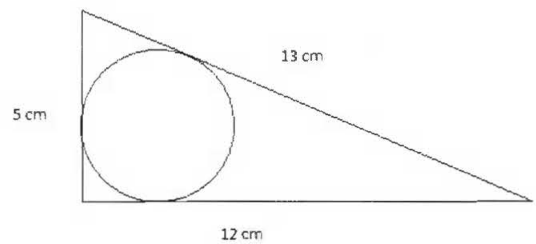
A. 1531 B. 1585 C. 1600 D. 1665 E. 2800

Part B: Short Answer: Place the answer in the blank provided on the answer sheet. Each correct answer is worth 5 points.

8. Two different digits from 1 to 9 are chosen randomly to form a 2-digit number. By reversing the order of the digits, a second 2-digit number is formed. What is the probability that the sum of these two numbers will be a multiple of the sum of the digits?

9. How many positive proper fractions in lowest terms have a denominator of 90?

10. A circle is inscribed in a right triangle as shown. What is the radius of this circle?



11. A carpenter stores his nails in a metal box in the shape of a rectangular prism with a square base that measures 5 cm by 5 cm by 9 cm. To the nearest **<u>tenths</u>** of a cm, what is the length of the longest nail that can be stored inside the box?

12. What is the difference between the sum of all multiples of 3 less than 50 and the sum of all multiples of 4 less than 50?

13. How many 9s are in the product 999 999 999 x 20 162 016?

Part C: Short Answer: Place the answer in the blank provided on the answer sheet. Each correct answer is worth 7 points.

14. Three people are coloring the same piece of $8\frac{1}{2}\ by\ 14$ paper.

Abby starts on the left side and painted $\frac{1}{2}$ of the paper red.
Ben starts on the right side and painted $\frac{3}{4}$ of the entire paper green.
Cathy starts in the middle and painted $\frac{1}{3}$ of the entire paper, evenly on either side of the centre line, using blue color.

What fraction of the paper has all three color painted on it?

15. $\overline{AB}$ is a diameter of the base of a cylinder and T is point on the opposite base of the cylinder directly above B. M is the midpoint of $\overline{TB}$. If $\overline{MB}$ = 8 and $\overline{AB}$ = $\frac{30}{\pi}$, what is the shortest distance between A and M along the curved surface of the cylinder?

16. The fraction $\frac{1}{7}$,when expressed as a repeating decimal, is equal to $0.142857142857\ldots$

Note that 1 is in the tenths place, 4 is in the hundredths place and so on. Let m be the 20th digit to the right of the decimal point and n be the 104th digit to the right of the decimal point, find n – m.

17. There were 11 baskets of Easter eggs, containing 14, 15, 19, 20, 22, 23, 24, 26, 27, 34 and 40 eggs respectively. John and Mary took all but one basket, each getting several baskets. John had twice as many eggs as Mary at this point. Mary then gave one of her baskets to John and now John had three times as many eggs as Susan. Which basket, indicated by the number of eggs above, did Mary give to John?

18. How many copies of the digit 0 are there among the digits of the first 2016 positive integers?

19. E and F are the respective midpoints of the sides $\overline{AB}$ and $\overline{BC}$ of a rectangle ABCD of area 2016. G is the point of intersection of $\overline{AF}$ and $\overline{CE}$. What is the area of the quadrilateral AGCD?

Solutions

Part A: Multiple Choice: Each correct answer is worth 4 points. Each unanswered question is worth 2 points to a maximum of 3 unanswered questions.

1. On a contest with 30 questions:

   - Each correct answer is awarded 6 points;
   - Each incorrectly answered question is penalized 2 points;
   - Each unanswered question is penalized 1 point.

   Abby answered 70% of the questions. The ratio of correct answers to incorrect answers to unanswered question is 5:2:3. How many points did she receive?

   A. 45 B. 126 C. 63 D. 81 E. (69)

Solution:

Abby answered 0.70 x 30 = 21 questions. With the ratio 5:2:3, it translates to 15 questions correct, 6 questions incorrect, 9 questions unanswered. The total points earned is 15(6) – 6(2) – 9(1) = 69.

2. Working at a constant rate, if Ben can cut a metal pipe into 5 pieces in 30 minutes, how many minutes would it take to cut a similar pipe into 15 pieces?

   A. 210 B. 170 C. (105) D. 90 E. 65

Solution:

5 pieces requires 4 cuts (in 30 min). Since 15 pieces requires 14 cuts, then:

Minutes : Cuts = 30 : 4 = m : 14. Therefore $m = \frac{30(14)}{4} = 105$ min.

3. Each of the first 6 prime numbers is written on the 6 different faces of a red cube. Each of the first 6 composite numbers is written on the 6 different faces of a blue cube. The two cubes are tossed/rolled once. What is the probability that the sum of the numbers rolled is 13?

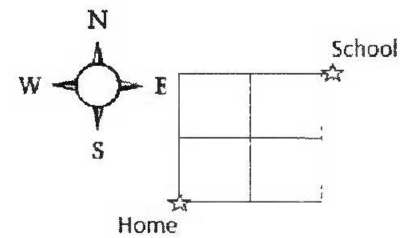
   A. $\frac{1}{18}$ B. ($\frac{1}{12}$) C. $\frac{1}{9}$ D. $\frac{1}{6}$ E. $\frac{1}{3}$

Solution:

Red = {2, 3, 5, 7, 11, 13} Blue = {4, 6, 8, 9, 10, 12} Favourable outcomes = {(3,10), (5,8), (7,6)}

$P(\text{Sum} = 13) = \frac{3}{6(6)} = \frac{1}{12}$

4. The distance from home to school is a total of 4 blocks. Jane must stay on the pathways and walk either east or north, as shown in the diagram.



How many ways are there in total for Jane to walk from home to school and back again, if she must return using a different path?

A. 5 B. 6 C. 11 D. 30 E. 36

Solution:

There are a total of 6 different pathways from Home to School.

Therefore, there are 6 - 1 = 5 different pathways back.
Therefore, the total different pathways for the return trip = 6(5) = 30.

5. Two rectangles with integer dimensions each have an area of 216 cm^2. The length of the first rectangle is 30 cm greater than that of the second rectangle. However, the width of the first rectangle is 5 cm less than that of the second rectangle. What is the difference in the perimeters, in centimetres, of the two rectangles?

A. 25 B. 35 C. 50 D. 60 E. 70

Solution:

Possible Dimensions:

L	216	108	72	54	36	27	24	18
W	1	2	3	4	6	8	9	12

The only two possibilities for the rectangles' dimensions with lengths differing by 30 and widths differing by 5 are:
Rectangle #1: 24 x 9 (with perimeter= 66 cm) and Rectangle #2: 4 x 54 (with perimeter= 116 cm).
Therefore, the difference in their perimeters = 116 – 66 = 50 cm

Alternate solution:
Let dimensions of Rectangle #2 be: Length = L; and Width = W. Therefore: Perimeter = 2L + 2W.
Therefore, dimensions of Rectangle #1 are:
Length = (L + 30); and Width = (W – 5). Therefore: P = 2(L + 30) + 2(W – 5) = 2L + 2W + 50
Therefore, the Difference in perimeters is: (2L + 2W + 50) – (2L + 2W) = 50 cm.

6. A family agreed to share the total cost of buying a $2268 (including GST) gaming computer.

- Dad paid for $\frac{1}{6}$ of the total cost.
- Then the oldest daughter, Kylee, paid for $\frac{1}{5}$ of the remaining cost.
- Then the son, Shawn, paid for $\frac{1}{4}$ of the remaining cost.

- Then the middle daughter, Erin, paid for $\frac{1}{3}$ of the remaining cost.
- Then the youngest daughter, Kassidy, paid for $\frac{1}{2}$ of the remaining cost.

If Mom paid for the remaining amount, then approximately how much more did the children pay compared to the amount paid by the parents?

A. \$300 B. \$450 C. \$600 D. (\$750) E. \$900

Solution:

Dad's Share = $\frac{2268}{6}$ = \$378 Kylee's Share = $\frac{2268-378}{5}$ = \$378 Shawn's Share = $\frac{1890-378}{4}$ = \$378

Erin's Share = $\frac{1512-378}{3}$ = \$378 Kassidy's Share = $\frac{1134-378}{2}$ = \$378 Mom's Share = \$378

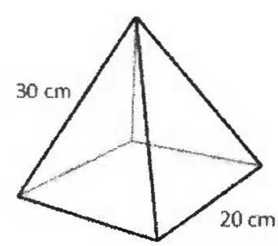
Note: Each person paid \$378! Since there were 4 children and 2 parents:

The 4 children paid a total of (4 – 2)(378) = 2(378) = \$756 more than what the parents paid together.

Therefore, the closest answer is \$750.

7. The net of the square pyramid shown at the right would consist of 4 congruent isosceles triangles (each with sides measuring 30 cm, 30 cm, 20 cm) and one square base.

What is the total surface area, to the nearest whole cm^2, of the pyramid?



A. (1531) B. 1585 C. 1600 D. 1665 E. 2800

Solution:

Using the Pythagorean Theorem, the altitude of each isosceles triangle = $\sqrt{900-100} = \sqrt{800}$ cm.

Therefore, SA = 4 triangles + Square base = $\frac{4(\sqrt{800})(20)}{2} + (20)(20) = 40(\sqrt{800}) + 400 = 1531.37 \approx 1531$

Part B: Short Answer: Place the answer in the blank provided on the answer sheet. Each correct answer is worth 5 points.

8. Two different digits from 1 to 9 are chosen randomly to form a 2-digit number. By reversing the order of the digits, a second 2-digit number is formed. What is the probability that the sum of these two numbers will be a multiple of the sum of the digits?

Solution:

Let the units digit be U, and the tens digit be T. Therefore the value of the 2-digit number = 10T + U.
By reversing the digits, the value of that 2-digit number = 10U + T.
Therefore, the sum of the two 2-digit numbers = (10T + U) + (10U + T) = 11U + 11T = 11(U + T).
Therefore, the sum must be eleven times the sum of the two digits. Therefore, the Probability = 100%.

9. How many positive proper fractions in lowest terms have a denominator of 90?

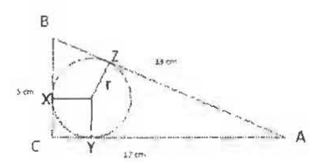
Solution:

The numerator of a basic fraction (with a denominator of 90) must be either 1, or any prime number less than 90 that is not a factor of 90. (The prime factors of 90 are: 2, 3, and 5. The quantity of possible numerators can be determined without listing all of them; just eliminate all multiples of these three prime numbers, as follows:

- To begin with, there are 90 numbers from 1 to 90, inclusive.
- Of these 90 numbers, eliminate all multiples of 2. That leaves 45 remaining possibilities (all are odd).
- Now eliminate all odd multiples of 3: (3, 9, 15, 21, 27, 33, 39, 45, 51, 57, 63, 69, 75, 81, 87). There are 15 of them. Therefore, the remaining number of possible numerators = 45 – 15 = 30 numbers.
- Now eliminate all remaining odd multiples of 5: (5, 25, 35, 55, 65, 85). There are only 6 of them. Therefore, the remaining number of possible numerators = 30 – 6 = 24 numbers!
- If you would like the list of all 24 possible numerators, they are: {1, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 49, 53, 59, 61, 67, 71, 73, 77, 79, 83, 89}.

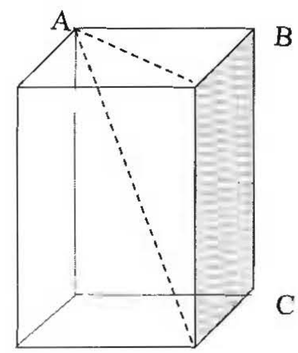
10. A circle is inscribed in a right triangle as shown. What is the radius, in cm, of this circle?

Solution:



AY = AZ = 12-r
BX = BZ = 5-r
AB = AZ + BZ
13 = (12 – r) + (5 – r)
2r = 12 + 5 – 13
Therefore, r = 2 cm

11. A carpenter stores his nails in a metal box in the shape of a rectangular prism with a square base that measures 5 cm by 5 cm by 9 cm. To the nearest **<u>tenths</u>** of a cm, what is the length of the longest nail that can be stored inside the box?



Solution:

The longest nail is represented by AC.

AB = $\sqrt{25 + 25} = \sqrt{50}$ cm.

$AC = \sqrt{50 + 81} = \sqrt{131} = 11.4$ cm

12. What is the difference between the sum of all multiples of 3 less than 50 and the sum of all multiples of 4 less than 50?

Solution:

$$(3 + 6 + 9 + \ldots + 48) - (4 + 8 + 12 + \ldots + 48)$$
$$= 8(51) - 6(52) = 408 - 312 = 96$$

13. How many 9s are in the product 999 999 999 x 20 162 016?

Solution:

(1000000000 – 1) x 20162016 = 20162016000000000 – 20162016 = 20162015979837983.
Therefore, three 9's.

Part C: Short Answer: Place the answer in the blank provided on the answer sheet. Each correct answer is worth 7 points.

14. Three people are coloring the same piece of $8\frac{1}{2}$ by 14 paper.

Abby starts on the left side and painted $\frac{1}{2}$ of the paper red.
Ben starts on the right side and painted $\frac{3}{4}$ of the entire paper green.
Cathy starts in the middle and painted $\frac{1}{3}$ of the entire paper, evenly on either side of the centre line, using blue color.

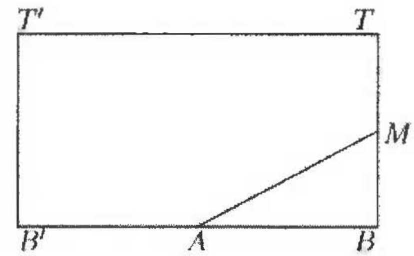
What fraction of the paper has all three color painted on it?

Solution:
The least common multiple of 2, 4 and 3 is twelve. We could divide the paper into twelve equal sections and label each color accordingly.

Red	Red	Red	Red	Red	Red						
			Green	Green	Green	Green	Green	Green	Green	Green	Green
				Blue	Blue	Blue	Blue				

From the diagram we could see that two of the vertical sections contain all three color, $\frac{2}{12} = \frac{1}{6}$.

15. $\overline{AB}$ is a diameter of the base of a cylinder and T is point on the opposite base of the cylinder directly above B. M is the midpoint of $\overline{TB}$. If $\overline{MB}$ = 8 and $\overline{AB}$ = $\frac{30}{\pi}$, what is the shortest distance between A and M along the curved surface of the cylinder?



Solution: Cut the cylinder along $\overline{TB}$ and unrolled it into the rectangle TBB'T'. Then A is the midpoint of $\overline{BB'}$. Since the radius of the base of the cylinder is $\frac{15}{\pi}$, $\overline{BB'}$ =2 $\pi\left(\frac{15}{\pi}\right)$ = 30 so that $\overline{AB}$ = 15 in the rectangle. The shortest path between A and M along the curved surface of the cylinder becomes the segment AM in the rectangle. By Pythagoras' Theorem, AM = $\sqrt{\overline{MB}^2 + \overline{AB}^2}$= 17.

16. The fraction $\frac{1}{7}$,when expressed as a repeating decimal, is equal to 0.142857142857 ...

Note that 1 is in the tenths place, 4 is in the hundredths place and so on. Let m be the 20th digit to the right of the decimal point and n be the 104th digit to the right of the decimal point, find n – m.

Solution

Any one of the digits (1, 4, 2, 8, 5 and 7) will reappear six places later. Note that 20 + 6(14) = 20 + 84 = 104. This means the two digits are the same; hence, n – m = 0.

17. There were 11 baskets of Easter eggs, containing 14, 15, 19, 20, 22, 23, 24, 26, 27, 34 and 40 eggs respectively. John and Mary took all but one basket, each getting several baskets. John had twice as many eggs as Mary at this point. Mary then gave one of her baskets to John and now John had three times as many eggs as Susan. Which basket, indicated by the number of eggs above, did Mary give to John?

Solution

There are 264 Easter eggs in total. Firstly, John had twice as many eggs as Mary. This means the total number of eggs between them is divisible by 3. After the exchange, John had three times as many eggs as Mary. This also means the total number of eggs between them is divisible by 4.

Let J, M be the number of Easter eggs that John and Mary have. L be the number of Easter eggs left behind. Since (J + M) + L = 264, we have both (J + M) and 264 divisible by 3 and 4, it follows that the basket not taken is also a multiple of 3 and of 4. The only basket that is divisible by 3 and 4 is 24.

Between John and Mary, they share 264 – 24 = 240 Easter eggs. At start, John would have 160 eggs and Mary 80 eggs. After the exchange, John would have 180 eggs and Mary 60 eggs. Therefore, Mary gave away the basket that contains 20 eggs.

18. How many copies of the digit 0 are there among the digits of the first 2016 positive integers?

Solution:

Among the units digits, the copies of 0s are in 10, 20, ..., 2010, so that there are 201 of them. Among the tens digits, the copies of 0s are in 100 to 109, 200 to 209, ..., 2000 to 2009, so that there are 200 of them. Among the hundreds digits, the copies of 0s are in 1000 to 1099 and 2000 to 2016, so that there are 117 of them. The total is 201+ 200 + 117 = 518.

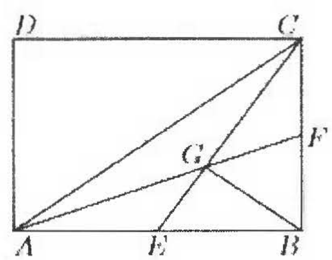
19. E and F are the respective midpoints of the sides $\overline{AB}$ and $\overline{BC}$ of a rectangle ABCD of area 2016. G is the point of intersection of $\overline{AF}$ and $\overline{CE}$. What is the area of the quadrilateral AGCD?

Solution:

Denote the area of the polygon P by [P]. Then [ABCD] = 2016. Let [GEB] = x and [GFB] = y. Since AE = EB and BF = FC, [AGE] = x and [CGF] = y.

Now $2x + y = [ABF] = \frac{1}{2}[ABC] = [CBE] = x + 2y$.

Hence x = y. Since [ABC] = 1008, x = y = 168, so that [AGCD] = [ABCD] − 2x − 2y = 1344



The Math Olympian by Richard Hoshino Friesen Press, 2015

Reviewed by Marc Chamberland

The book opens with an atypical scene: a teenage girl waits nervously as she is about to participate in the Canadian Mathematical Olympiad. Moments later, the reader is presented with a set of five math problems. As someone who has delighted in math contest problems since I was a student, my attention was immediately ensnared. Indeed, I stopped reading and plunged into solving the problems! After some success, it dawned on me that the author probably didn't expect his readers to put down his novel, pull out a pad of paper and start doing math. Or did he?

This is the fictional story of a teenage girl named Bethany who, after seeing a news report about national and international math contests, realizes that there is a community of far-flung young people who, like her, unabashedly love mathematics. She wants to be one of them: a math olympian. From the outset, the context reveals that she will succeed, but her path is anything but clear. How does one become a math olympian? The author Richard Hoshino would know, having competed for Canada in the International Math Olympiad in 1996.

Hoshino has written a human story about Bethany's journey from being a naive girl who knows little about math or herself to someone who comes to know both much better. The chapters are structured around the five problems that Hoshino presents at the beginning of the book.

Hoshino does a masterful job of weaving the mathematics into the narrative. For each of the five main contest problems, he gently motivates the techniques that will be helpful, develops the components of the solution and lastly offers an economical, polished write-up. Hoshino makes the problem-solving process accessible by having the characters ask questions, explore and surmount dead ends, and express their mathematical epiphanies. We witness Bethany's thought processes, an informative lesson for both students and teachers alike.

Don't be misled, however, into thinking that this is principally a problem-solving book in disguise. It is a novel detailing how an esteem-challenged teenager, through diligence, support and serendipity, becomes a star at math contests. One reads not only about how the world of mathematics is opened up to her, but also about Bethany's normal experiences as a young person, including the complexity of teenage relationships (with friends, enemies, lovers, coaches and parents), overcoming her anxieties, and overall, navigating the challenges of growing up.

Hoshino also does not limit the mathematics to the problems that are solved. While one encounters standard material—always presented in an engaging way—such as summing consecutive integers to produce the triangular numbers, telescoping series, the use of symmetry to solve algebraic equations and some classical planar geometry, the reader is also exposed to Newcomb's Law (usually called Benford's Law), the idea of groups via Rubik's Cube (even a mention of Burnside's Problem) and dimensional analysis. Hoshino presents mathematics as it should be seen for the uninitiated: a taste of fascinating mathematical ideas without too much symbolic clutter.

As a Canadian who has lived outside the country for almost 20 years, it was also a treat to "tour" Canada. Bethany is from Sydney, Nova Scotia, and, until her involvement with math contests, had never left her home province. She soon brings us along to Ottawa and for a longer visit to Hoshino's current province, British Columbia. It has been said that an author's first novel is usually autobiographical, and there is enough evidence to suggest that *The Math Olympian* is no exception. Hoshino takes us to his current institution (Quest University), escorts us into the world of competitive math contests and, I suspect, shows us other elements of his personal life, all embodied in Bethany's experience. If you're looking for a young person's embrace of mathematics as she navigates the uneven road of growing up, you'll find an enjoyable read here. And some math problems to solve along the way

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Marc Chamberlain is author of Single Digits—In Praise of Small Numbers, *which was reviewed in* CMS Notes, *Volume 47, Number 6.*

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Math Munch

www.mathmunch.org

Lorelei Boschman

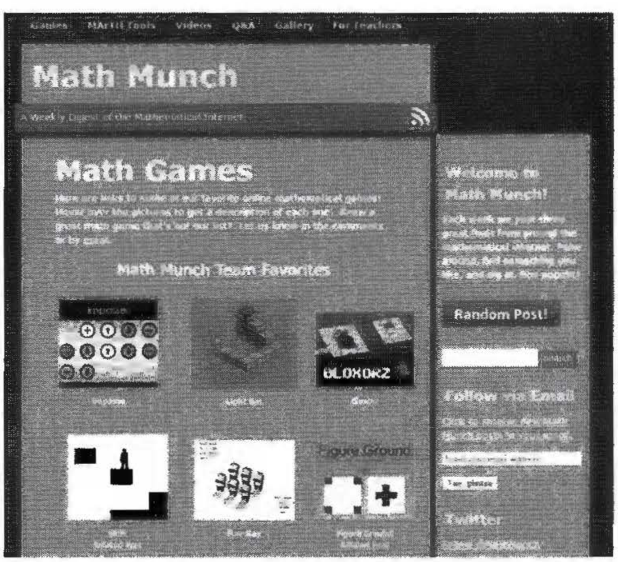
Math Munch is a tremendous resource that combines games, puzzles and tools, math art and videos, all stemming from the creators' love of mathematics and their desire to share this with others.

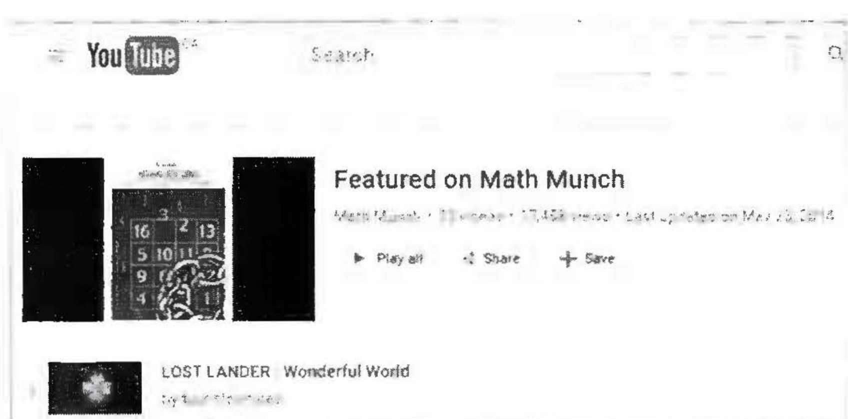
The math games are engaging and relate to many math levels. They present many different branches of mathematics to fit in all curricular areas.

The math art tools are fabulous and will captivate your students.

Contributors' backgrounds and stories are highlighted—this helps to make the person and the math real. Submissions are encouraged!

Watch their TEDx Talk to hear about Math Munch and the creators' goals.





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MCATA Mission Statement

Providing leadership to encourage the continuing enhancement of teaching, learning and understanding mathematics.

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