Calgary Junior High Mathematics Competition 2015/16

NAME: ______________________________ GENDER: ________

PLEASE PRINT (First name Last name)

SCHOOL: ______________________________ GRADE: ________

(9,8,7,...)

- You have 90 minutes for the examination. The test has two parts: PART A — short answer; and PART B — long answer. The exam has 9 pages including this one.
- Each correct answer to PART A will score 5 points. You must put the answer in the space provided. No part marks are given. PART A has a total possible score of 45 points.
- Each problem in PART B carries 9 points. You should show all your work. Some credit for each problem is based on the clarity and completeness of your answer. You should make it clear why the answer is correct. PART B has a total possible score of 54 points.
- You are permitted the use of rough paper. Geometry instruments are not necessary. References including mathematical tables and formula sheets are **not** permitted. Simple calculators without programming or graphic capabilities **are** allowed. Diagrams are not drawn to scale: they are intended as visual hints only.
- When the teacher tells you to start work you should read all the problems and select those you have the best chance to do first. You should answer as many problems as possible, but you may not have time to answer all the problems.
- Hint: Read all the problems and select those you have the best chance to solve first. You may not have time to solve all the problems.

MARKERS' USE ONLY	
PART A ________ ×5	
B1	
B2	
B3	
B4	
B5	
B6	
TOTAL (max: 99)	

BE SURE TO MARK YOUR NAME AND SCHOOL AT THE TOP OF THIS PAGE.

THE EXAM HAS 9 PAGES INCLUDING THIS COVER PAGE.

Please return the entire exam to your supervising teacher at the end of 90 minutes.

Part A

A1 A rectangle with integer length and integer width has area 13 cm^2. What is the perimeter of the rectangle in cm?

A1

A2 A nice fact about the current year is that 2016 is equal to the sum $1+2+3+\cdots+63$ of the first 63 positive integers. When Richard told this to his grandmother, she said: *Interesting! I was born in a year which is also the sum of the first X positive integers, where X is some positive integer.* In what year was Richard's grandmother born? (You may assume that Richard's grandmother is less than 100 years old.)

A2

A3 Suppose you reduce each of the following 64 fractions to lowest terms:

$$\frac{1}{64}, \frac{2}{64}, \frac{3}{64}, \ldots, \frac{64}{64}.$$

How many of the resulting 64 reduced fractions have a denominator of 8?

A3

A4 Peppers come in four colours: green, red, yellow and orange. In how many ways can you make a bag of six peppers so that there is at least one of each colour?

A4

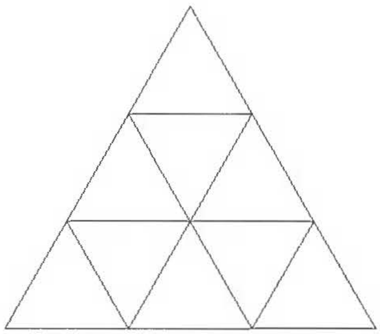
A5 In the following addition of four-digit numbers, X, Y and Z stand for digits (not necessarily different). If the addition is correct, what digit does Y stand for?

$$\begin{array}{rcccc} & 2 & X & X & Y \\ + & 3 & X & Y & Z \\ \hline & Z & Y & X & X \end{array}$$

A5

A6 How many equilateral triangles of any size are there in the figure below?

A6



A7 A number was decreased by 20%, and the resulting number increased by 20%. What percentage of the original number is the final result?

A7

A8 A group of grade 7 students and grade 9 students are at a banquet. The average height of the grade 9 students is 180 cm. The average height of the grade 7 students is 160 cm. If the average height of all students at the banquet is 168 cm and there are 72 grade 9 students, how many grade 7 students are there?

A8

A9 If the straight-line distance from one corner of a cube to the opposite corner (i.e., the length of the long diagonal or body-diagonal of a cube) is 9 cm, what is the area (in cm^2) of one of its faces?

A9

Part B

B1 When the Cookie Monster visits the cookie jars, he takes from as many jars as he likes, but always takes the same number of cookies from each of the jars that he does select.

(i) Suppose that there are four jars containing 11, 5, 4 and 2 cookies. Then, for example, he might take 4 from each of the first three jars, leaving 7, 1, 0 and 2; then 2 from the first and last, leaving 5, 1, 0 and 0, and he will need two more visits to empty all the jars. Show how he could have emptied these four cookie jars in less than four visits.

(ii) Suppose instead that the four jars contained a, b, c and d cookies, respectively, with $a \geq b \geq c \geq d$. Show that if $a = b + c + d$, then three visits are enough to empty all the jars.

B2 The number 102564 has the property that if the last digit is moved to the front, the resulting number, namely 410256, is 4 times bigger than the original number:

$$410256 = 4 \times 102564.$$

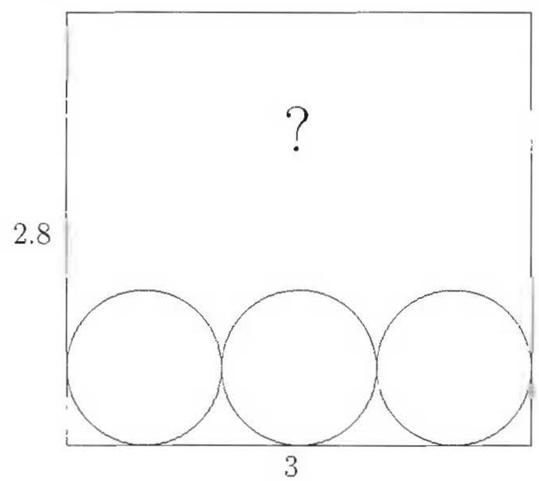
Find a six-digit number whose last digit is 9 and which becomes 4 times bigger when we move this 9 to the front.

B3 In a sequence, each term after the first is the sum of squares of the digits of the previous term. For example, if the first term is 42 then the next term is $4^2+2^2=20$. The next term after 20 is then $2^2+0^2=4$, followed by $4^2=16$, which is then followed by $1^2+6^2=37$, and so on, giving the sequence 42, 20, 4, 16, 37, and so on.

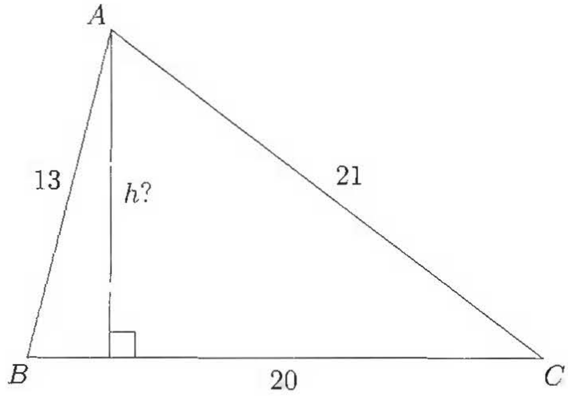
(a) If the first term is 44, what is the 2016th term?

(b) If the first term is 25, what is the 2016th term?

B4 Is it possible to pack 8 balls of diameter 1 into a 1 by 3 by 2.8 box? Explain why or why not.



B5 The triangle ABC has edge-lengths $BC = 20$, $CA = 21$, and $AB = 13$. What is its height h shown in the figure?



B6 Find all positive integer solutions a, b, c, with $a \le b \le c$ such that

$$\frac{6}{7} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$$

and show that there are no other solutions.

Solutions (Part A)

A1 A rectangle with integer length and integer width has area 13 cm^2. What is the perimeter of the rectangle in cm?

A1
28

A2 A nice fact about the current year is that 2016 is equal to the sum $1+2+3+\cdots+63$ of the first 63 positive integers. When Richard told this to his grandmother, she said: *Interesting! I was born in a year which is also the sum of the first X positive integers, where X is some positive integer.* In what year was Richard's grandmother born? (You may assume that Richard's grandmother is less than 100 years old.)

A2
1953

A3 Suppose you reduce each of the following 64 fractions to lowest terms:

$$\frac{1}{64}, \frac{2}{64}, \frac{3}{64}, \ldots, \frac{64}{64}.$$

How many of the resulting 64 reduced fractions have a denominator of 8?

A3
4

A4 Peppers come in four colours: green, red, yellow and orange. In how many ways can you make a bag of six peppers so that there is at least one of each colour?

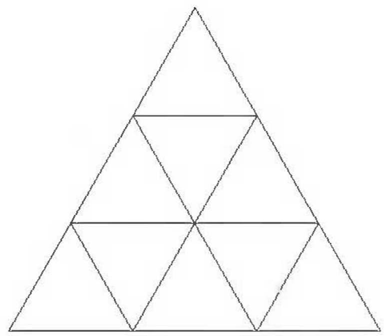
A4
10

A5 In the following addition of four-digit numbers, X, Y and Z stand for digits (not necessarily different). If the addition is correct, what digit does Y stand for?

$$\begin{array}{cccc} & 2 & X & X & Y \\ + & 3 & X & Y & Z \\ \hline & Z & Y & X & X \end{array}$$

A5
9

A6 How many equilateral triangles of any size are there in the figure below?



A6

13

A7 A number was decreased by 20%, and the resulting number increased by 20%. What percentage of the original number is the final result?

A7

96

A8 A group of grade 7 students and grade 9 students are at a banquet. The average height of the grade 9 students is 180 cm. The average height of the grade 7 students is 160 cm. If the average height of all students at the banquet is 168 cm and there are 72 grade 9 students, how many grade 7 students are there?

A8

108

A9 If the straight-line distance from one corner of a cube to the opposite corner (i.e., the length of the long diagonal or body-diagonal of a cube) is 9 cm, what is the area (in cm^2) of one of its faces?

A9

27

Solutions (Part B)

B1 When the Cookie Monster visits the cookie jars, he takes from as many jars as he likes, but always takes the same number of cookies from each of the jars that he does select.

(i) Suppose that there are four jars containing 11, 5, 4 and 2 cookies. Then, for example, he might take 4 from each of the first three jars, leaving 7, 1, 0 and 2; then 2 from the first and last, leaving 5, 1, 0 and 0, and he will need two more visits to empty all the jars. Show how he could have emptied these four cookie jars in less than four visits.

Solution. Take 5 from each of the first two jars, leaving 6, 0, 4, 2; then 4 from the first and third; and finally 2 from the first and last; and he has done it in three visits.

(ii) Suppose instead that the four jars contained a, b, c and d cookies, respectively, with $a \geq b \geq c \geq d$. Show that if $a = b + c + d$, then three visits are enough to empty all the jars.

Solution. Take b from each of the first two jars, leaving $c + d$, 0, c, d; then c from the first and third; and finally d from the first and last; and he has done it in three visits.

B2 The number 102564 has the property that if the last digit is moved to the front, the resulting number, namely 410256, is 4 times bigger than the original number:

$$410256 = 4 \times 102564.$$

Find a six-digit number whose last digit is 9 and which becomes 4 times bigger when we move this 9 to the front.

Solution 1. We must find a, b, c, d, e so that

$$\begin{array}{cccccc} a & b & c & d & e & 9 \\ \times & & & & & 4 \\ \hline 9 & a & b & c & d & e \end{array}$$

Multiplying gives $e = 6$, $d = 7$, $c = 0$, $b = 3$ and $a = 2$. Thus, a six-digit number whose last digit is 9 and which becomes 4 times bigger when we move this 9 to the front is 230769.

Solution 2. Consider a six-digit number whose last digit is 9: $a\,b\,c\,d\,e\,9$.
Letting $x = a\,b\,c\,d\,e$ gives

$$900000 + x = 4(10x + 9)$$

$$39x = 899964$$

$$3x = 69228$$

$$x = 23076$$

Thus, the number is 230769.

B3 In a sequence, each term after the first is the sum of squares of the digits of the previous term. For example, if the first term is 42 then the next term is $4^2+2^2=20$. The next term after 20 is then $2^2+0^2=4$, followed by $4^2=16$, which is then followed by $1^2+6^2=37$, and so on, giving the sequence 42, 20, 4, 16, 37, and so on.

(a) If the first term is 44, what is the 2016th term?

(b) If the first term is 25, what is the 2016th term?

Solution.

(a) Starting with 44 gives

$4^2+4^2=16+16=32 \quad \to \quad 3^2+2^2=9+4=13 \quad \to \quad 1^2+3^2=1+9=10$
$\to \quad 1^2+0^2=1+0=1 \quad \to \quad 1^2=1 \quad \to \quad 1^2=1 \quad \cdots$

The sequence is then

$$\{44, 32, 13, 10, 1, 1, 1, \ldots\}$$

with 2016th term equal to 1.

(b) Starting with 25 gives

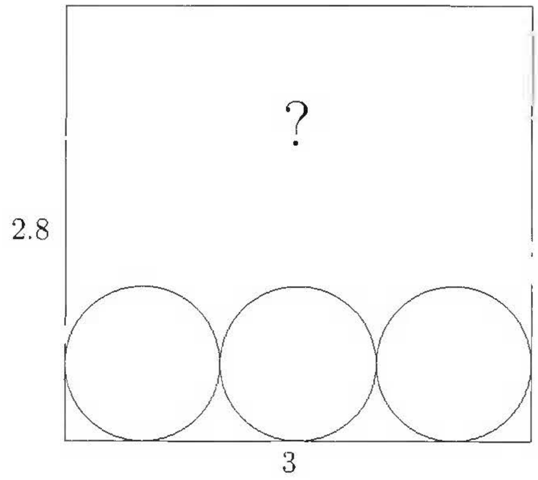
$2^2+5^2=4+25=29 \quad \to \quad 2^2+9^2=4+81=85 \quad \to \quad 8^2+5^2=64+25=89$
$\to \quad 8^2+9^2=64+81=145 \quad \to \quad 1^2+4^2+5^2=1+16+25=42$
$\to \quad 4^2+2^2=16+4=20 \quad \to \quad 2^2+0^2=4+0=4 \quad \to \quad 4^2=16$
$\to \quad 1^2+6^2=1+36=37 \quad \to \quad 3^2+7^2=9+49=58 \quad \to \quad 5^2+8^2=25+64=89$

The sequence is then

$$\{25, 29, 85, \mathbf{89}, \mathbf{145}, \mathbf{42}, \mathbf{20}, \mathbf{4}, \mathbf{16}, \mathbf{37}, \mathbf{58}, 89, \ldots\}$$

and repeats with a period of 8. Since $2013 = 251 \times 8 + 5$, the 2016th term in the sequence is 4.

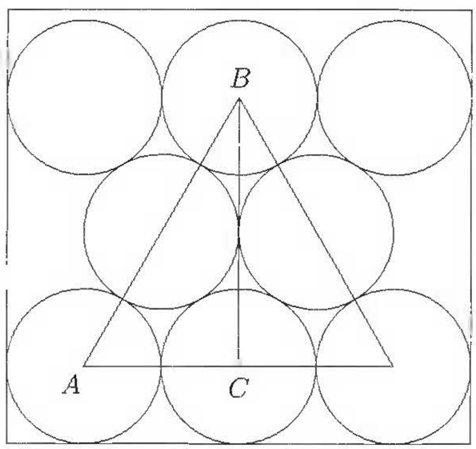
B4 Is it possible to pack 8 balls of diameter 1 into a 1 by 3 by 2.8 box? Explain why or why not.



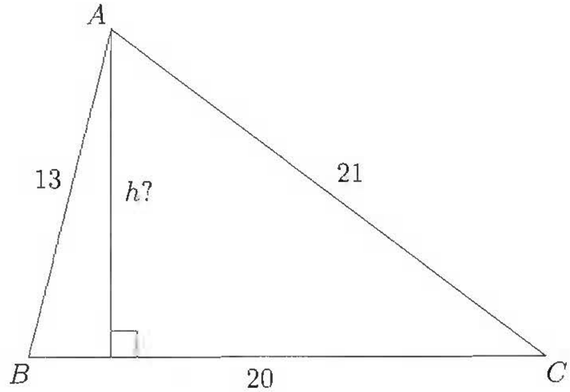
Solution.

Yes. it is possible to pack 8 balls.

The triangle ABC has edge-lengths $AB = 2$ and $AC = 1$. Using Pythagoras's theorem. $BC = \sqrt{3}$. Thus, the distance from the bottom of the bottom row of balls to the top of the top row of balls is $\frac{1}{2} + \sqrt{3} + \frac{1}{2} = 1 + \sqrt{3} = 2.732\ldots < 2.8$.



B5 The triangle ABC has edge-lengths $BC = 20$, $CA = 21$, and $AB = 13$. What is its height h shown in the figure?



Solution 1. Using Pythagoras's theorem we have

$\sqrt{13^2 - h^2} + \sqrt{21^2 - h^2} = 20$
$20 - \sqrt{13^2 - h^2} = \sqrt{21^2 - h^2}$
$400 - 40\sqrt{13^2 - h^2} + 13^2 - h^2 = 21^2 - h^2$
$128 = 40\sqrt{13^2 - h^2}$
$3.2 = \sqrt{13^2 - h^2}$
$h^2 = 13^2 - 3.2^2 = 9.8 \times 16.2$
$h = 12.6$

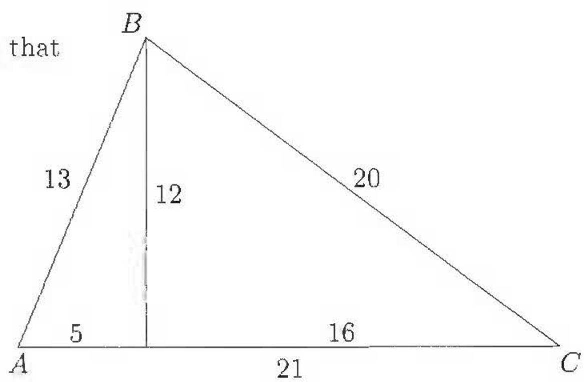
Solution 2. Heron's formula states that the area of a triangle whose sides have lengths a, b, and c is

$$\text{Area} = \sqrt{s(s-a)(s-b)(s-c)},$$

where $s = (a+b+c)/2$ is the semiperimeter of the triangle. Then $a = 13$, $b = 21$ and $c = 20$ gives $s = (13+21+20)/2 = 27$. Thus, the triangle has area $\sqrt{27 \cdot 7 \cdot 6 \cdot 14} = 3^2 \cdot 7 \cdot 2 = 126$. Using the base of the triangle as 20, we have $126 = \frac{1}{2}(20)h$ implying $h = 126/(\frac{1}{2}20) = 12.6$.

Solution 3. Turn the triangle over and note that it is made up of two Pythagorean triangles.

It is then immediate that the area is 126, and division by half the base, 10, gives 12.6.



36 Find all positive integer solutions a, b, c, with $a \leq b \leq c$ such that

$$\frac{6}{7} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$$

and show that there are no other solutions.

Solution. Note that

$$\frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \frac{3}{4} < \frac{6}{7}.$$

hence $a \leq 3$. If $a = 3$, then $b = 3$ since $a \leq b$ and

$$\frac{1}{3} + \frac{1}{4} + \frac{1}{4} = \frac{5}{6} < \frac{6}{7}.$$

This implies $c = 27/4$ which is not an integer, thus $a = 2$. Now

$$\frac{1}{2} + \frac{1}{6} + \frac{1}{6} = \frac{5}{6} < \frac{6}{7}$$

implies $b \leq 5$. This gives four cases.
If $b = 2$, then $c = -7$.
If $b = 3$, then $c = 42$.
If $b = 4$, then $c = 28/3$.
If $b = 5$, then $c = 70/11$.
The only solution consisting of positive integers is $(a, b, c) = (2, 3, 42)$.