Trigonometry Workshop for High School

Rosalind Carson

For the Love of Triangles

This approach to visualizing trigonometry is based on how the great mathematicians learned to measure chords in circles and use these measurements to accurately map the celestial objects in the sky. It is my experience (in formal classrooms and tutoring) in teaching Grades 10, 11 and 12 that students have an easier time understanding and applying the trigonometry concepts when given the opportunity to see and explore the origins of this mathematics. This is how I begin with the students: "Tell me all that you know about triangles."

The story begins with the man responsible for today's theme, "For the Love of Triangles." His name was Thales (640–546 BCE or 636–546 BCE). In written history, he is the first recorded mathematician. Of course there were earlier ones—Babylonian math was recorded on stone tablets 1,200 years before Thales, but there are no individual names. Thales was Greek and lived in Miletus, which was a prosperous colony on the coast of Asia Minor, a thriving trading port. Of the many stories told of Thales, his quest to understand things himself was most prevalent. A famous saying from these tales is "in order to seek wisdom, you must know thyself." He was prized for his logic and mathematical contributions by later Greek generations and was honoured with the title of being one of Seven Wise Men of Antiquity. Thales was the only mathematician to have this honour.

In one of the many tales of how Thales used his knowledge to be wise, he became a rich man. A poor friend was complaining about how the poor man always stays poor. Thales disagreed strongly and told the man that if one sets one's mind to a task, he can become rich. Thales told his friend to give him six months to prove his point. At this time, there was a drought and many poor olive farmers. Thales put his observations to use, talked to old farmers and predicted a good year. Due to current economics, many farmers were willing to sell their olive presses at very low prices. When the rain came, the farmers had to rent the presses from Thales in order to handle the crop, and Thales prospered. It is told that Thales was not in this for the money. Eventually he sold the presses back to the farmers and moved on to other pursuits.

Discoveries Made by Thales

Below are the mathematical contributions that Thales made:

- A circle is bisected by its diameter.
- Angles at the base of an isosceles triangle are equal.
- When two straight lines cut each other (intersect), the vertically opposite angles are equal.
- The inscribed angle (triangle) in a semicircle is (has) a right angle.
- The sides of similar triangles are proportional.
- Two triangles are congruent if they have two angles and a respective side congruent (ASA).

Thales was the first mathematician to have a specific mathematical discovery named after him, Thales's Theorem: The triangle inscribed in a semicircle is a right triangle.

Similar Triangles Activity

Thales travelled and studied in Babylon and in Egypt. In Babylonia, he learned astronomical measurement techniques. In Egypt, he learned about the geometric surveying techniques, re-establishing the land boundaries after every flood of the Nile River. One of the stories states that when admiring the Great Pyramid of Giza, he asked the locals what the pyramid's height was. He was surprised that people did not know the height of such a great structure, so he quickly calculated it (to his spectator's surprise) by using similar triangles. He used indirect measurement and these simple tools: his shadow, his staff, knowledge of similar triangles and the sun. How did he do it?

This is where I leave the students to think about how Thales performed this calculation.

Thales' achievement was a small step for trigonometry, the science of triangles, and a giant leap for mankind. By deducing a measurement logically from the intrinsic properties of a shape he was thinking differently from the Egyptians, who had shown remarkable skills in practical activities like pyramid-building but whose mathematical knowledge was essentially limited to rules of thumb and triangles that actually existed. Thales' calculation involved a triangle that was an

> abstraction of reality, made by the sun's rays. His insights marked the beginning of Greek rational thought, which we generally consider the foundation of Western mathematics, philosophy and science. (Bellos 2014, 59)

Thales also knew that any triangle can be split into two right triangles. Did you know that? In fact, Greeks were obsessed with right triangles because of this fact. The Greeks learned the word *hypotenuse* from the Egyptians. In Egypt, a land surveyor would hire three slaves to stretch the special rope knotted in the designated lengths of what we know as Pythagorean Triples, typically the 3, 4, 5 triangle. This surveyor was called harpedonopta, literally meaning rope stretcher. The Greek word *hypotenuse* means stretched against.

Three centuries later, Eratosthenes used similar triangles and logic to get a good abstract measurement for the circumference of the earth.

Pythagoras

According to various resources, Pythagoras was born between 560 and 580 BCE on the island of Samos in the Aegean Sea. He travelled most of his life as far as India, learning from the Babylonians and Egyptians. It has been said that Thales was one of his teachers. The Babylonian point of study is an interesting one because of the fact that 1,200 years earlier the Babylonians knew the relationship of the sides in a right triangle (Pythagorean triples) because stone tablets have been found written in Cuneiform (base 60) with lists of these triples.

If you want to create right triangles (side lengths that actually make right triangles), use the following:

Let p and q be whole numbers; where $p > q > 0$, p and q have no common divisor (save 1), p and q are both not odd, then the following produce Pythagorean Triples:

$x = p^2 - q^2$; $y = 2pq$; $z = p^2 + q^2$

Google Plimpton 322 Babylonian Stone Tablet to get many results on how to read and work in base 60, which is the reason why we use 60 as a base measurement for degrees, time and so on.

Pythagoras returned to Samos and started his school teachings at the age of 50. The society (that it became) was strict. There were many levels of teachings. Entry level was for learners and above who contemplated and discussed mathematics, trying to explain the universe with numbers—rational numbers since they only could work with whole numbers and factors—that is, any number that can be multiplied and divided, hence their version of fractions. Those who followed his teachings did not eat meat or beans, did not drink wine nor wear wool or touch a white rooster! Pythagoras died after refusing to run through a bean field when an angry mob was chasing him. The mob caught up and killed him.

Here is another story about root 2. The Pythagoreans explored mathematics to describe nature by writing in the sand or using pebbles. When they explored a square of side length one, it was said that the diagonal was impossible to determine, as a ratio (of the diagonal to side length) of whole numbers could not be written to represent that diagonal. Hippasus tried to convince Pythagoras that there must be another type of number, an irrational number. This was so absurd that they threw him overboard to his end. This was the first sign of irrational behaviour.

As the Pythagoreans studied harmony with the universe, they studied astronomy, music, numbers (number theory) and geometry. They believed that all matter was composed of four elements: fire, water, air and earth. These elements were closely associated with the numbers assigned. All four together were the holy tetractys whose sum $1 + 2 + 3 + 4 = 10$, the number for the universe. Two represented female, three was male, five was marriage, four was justice (a square deal) and six was the creation of the universe, and is a perfect number (perfect numbers are those that are the sum of their proper factors). The pentagram was held sacred for all the hidden pentagrams within the structure, and all the ratios of sides that represent the Golden Ration (not that they knew a number for that because the Golden Ration is an irrational number!) but that they knew this ratio is aesthetically pleasing, as it was used in all current and ancient art and architecture.

Pythagoras (or one of his students) created the word *mathematics*, which means science.

Handout: Pythagorean Theorem Discovery—Chinese way.

Show picture on page 22 in* Euclid's Window, *by Leonard Mlodinow or* Bedside Book of Geometry, *by Mike Askew and Sheila Ebbutt.

Invention of Trigonometry: Hipparchus of Rhodes

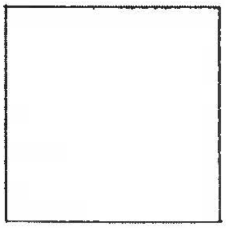
Hipparchus of Rhodes (Greece, 170–126 BCE) created trigonometry (not called that until 1600s) because the main focus was the mapping and movement of the celestial objects. Trigonometry was used for astronomy—plain and simple. It was not used for surveying until the 1600s (first noted in a book in 1595). He used similar triangles and the chord length in a circle to measure the movement of stars or distance between celestial objects.

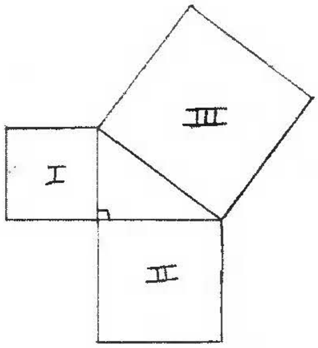
There are many different accounts of why we use 360 degrees for a full rotation of a circle. Basically, the Babylonian method has stood the test of time! They used a base 60 number system. The word *degrees* refers to the measurement of movement. Dividing the arc length of a circle by 60 leaves 6 equilateral triangles. Angle measurements were arc length measurements and a degree was divided into 60 smaller units each called *pars minuta prima*, meaning the first minute part. Those were then divided into 60 units, each called *pars minuta secunda*, meaning second minute part and translated into seconds, hence the basis of time keeping. The Chinese originally divided the circle into 365 ¼ parts making the sun movement one degree per day—nice! But that made other calculations difficult, so we stuck with the Babylonians.

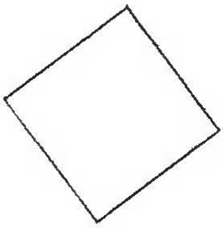
Hipparchus created huge tables of values of degrees and chord lengths that were readily used by all astronomers. In the second century, Ptolemy compiled a table of chords for a circle with radius = 60 units and every half degree angle. These tables were invaluable to western astronomers.

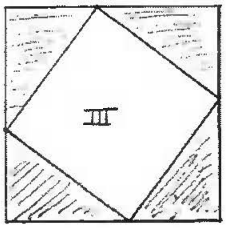
Astronomy was expanding in India because they (like the Babylonians) had a place value number system allowing large and small calculations. Indian place value was base 10. They took the knowledge of chords further and created tables of half-chord lengths. By using half-chords, there was the need to utilize the exciting and popular relationship of the sides of a right triangle (Babylonian

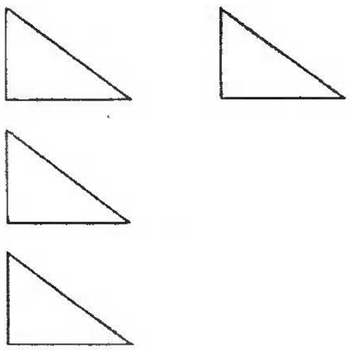
Pythagorean Theorem Discovery—Chinese Way











Theorem—oops!—Pythagorean Theorem). The Arabians learned from the Indians and translated the Sanskrit word *jya-ardha* (string-half) to *jiba* in Arabian, which sounded the same but was meaningless. Since *jiba* looked like an Arab word *jaib* (meaning bosom or bay), that was the word recorded for the half-chord length. Only when Fibonacci studied in Arabian countries, learning and hence promoting the use of Arabian numerals, zero and the line that separates the numerator and denominator in fractions, did the Europeans finally adopted the half-chord knowledge. The Latin translation for *jaib* was sinus (meaning the fold of cloth of a toga over the woman's chest), which became sine.

It was Georg Joachim Rheticus (1550) who pushed the idea (that stuck) of focusing the vision of trigonometry to be just a close look at right triangles. Rheticus also established *sine*, *cosine* and *tangent* as the names for the ratio of the sides of triangles that we now memorize and apply.

Show photo on page 90 of **A Mathematical Mystery Tour,** ***by A K Dewdney, to visualize the 3-D sense of mapping stars and how we then transform that image onto a 2-D plane = the coordinate plane. Explore the length of the chord to measure the location of the stars.***

We can easily see the sine, cosine and tangent of an angle when the stretched side (hypotenuse) = 1.

Now let us look at similar triangles: (draw all examples and how that transcends into five sections of workbook or textbook topics).

Let us look at triangles without right angles. Remember, all triangles can be split into two right triangles. Let us do that in the triangle we drew! Voilà! Law of Cosines.

Draw another triangle that is not a right triangle. Split it into two right triangles. Label with sines. Voilà! Law of Sines!

Mapping using trigonometry: You just learned how to do all the calculations your phone does to find the nearest coffee shop! You are a genius! (Pages 74–76 of **The Grapes of Math,** ***by Alex Bellos.)***

Questions to ponder:

1. Kevin argues that there is no need for knowing trigonometry. He states that all you have to do is use a ruler to find the side lengths. Do you agree or disagree? Explain.
2. The sine of an angle in a right triangle is quite large. What might the two acute angles be? Explain.
3. The cosine of an angle in the right triangle is approximately 0.707. What might the dimensions of the triangle be?
4. Jason said, "The sine of angle A in a right triangle cannot be larger than the cosine of angle B in the same triangle." Do you agree? Explain.
5. The cosine of angle A (think of side length of triangle) is a lot more than the sine of that angle. What do you know about angle A (in a right triangle)?

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