Problems requiring Short Answers

1. Ace must receive at least \(20 + \left(\frac{500-20}{2} + 1\right) = 261\) of the remaining 500 votes. **Answer: 261**

2. When Peter runs 5 laps, Steven runs 3 laps for a difference of 2 laps. For a difference of 4 laps, Peter must run 10 laps and Steven 6. **Answer: 6**

3. In order for \(n + 13\) to be a multiple of 5, we must have \(n = 2, 7, 12, \ldots\). Now 2 - 13 is not divisible by 6, and while 7 - 13 is, it is not a positive multiple of 6. The least common multiple of 5 and 6 is 30, and 30 + 7 = 37 is the desired minimum. **Answer: 37.**

4. We have \(\frac{w}{z} = \frac{2}{3}\). Hence the smallest positive integers satisfying the given relation are \(w = 5^3, x = 5^2 \times 8, y = 5 \times 8^2\) and \(z = 8^3\), yielding \(w + x + y + z = 1157\). **Answer: 1157**

5. The last digit of \(1^3 - 2^3 + \cdots - 10^3\) is the same as that of \(11^3 - 12^3 + \cdots - 20^3\), and so on. It follows that the last digit of \(1^3 - 2^3 + \cdots - 2000^3\) must be 0, so that the last digit of \(1^3 - 2^3 + 3^3 - \cdots + 2001^3\) is 1. **Answer: 1**

6. Let \(v\) be the amount of viruses produced per minute and \(a\) be the amount of viruses destroyed by each antivirus per minute. Then \(40(2a - v) = 16(4a - v)\), which simplifies to \(\frac{a}{v} = \frac{3}{2}\). Let \(n\) be the desired number of antiviruses. Then \(40(2a - v) = 10(na - v)\) which is equivalent to \(8v = (\frac{3n}{2} - 1)v\). Hence \(n = 6\). **Answer: 6**

7. In order for the four-digit number to be divisible by 11, the sum of the first and the third digits must be 5. They may either be 1 and 4 or 2 and 3. Moreover, each pair can be permuted internally, yielding a total of \(2 \times 2 \times 2 = 8\) such numbers. **Answer: 8**

8. We have \(217 = (m - n)^2 + 13n^2\). If \(n \geq 5\), then \((m - n)^2\) will be negative, which is impossible. If \(n = 1\), then \((m - n)^2 = 204\) has no integral solutions. Hence \(n = 3\) so that \((m - 3)^2 = 100\), which yields \(m = 13\). **Answer: 13**

9. We have \(xy = \frac{1}{(\sqrt{3} + \sqrt{2})(\sqrt{3} - \sqrt{2})} = 1\) while \(x + y = \frac{\sqrt{3} - \sqrt{2} + \sqrt{3} + \sqrt{2}}{(\sqrt{3} + \sqrt{2})(\sqrt{3} - \sqrt{2})} = 2\sqrt{3}\). It follows that \(x^2 + y^2 = (x + y)^2 - 2xy = 10\). **Answer: 10**

10. We have \(x^4 + x^3 + xy^3 + y^4 = x^3(x + y) + y^3(x + y) = (x + y)^2(x^2 - xy + y^2) = 36\). **Answer: 36**

11. The area of triangle \(GAD\) is \(\frac{1}{4} \times 24 = 6\). The area of triangle \(AGE\) is \(\frac{1}{2} \times \frac{1}{4} \times 24 = 3\). The area of triangle \(BEF\) is \(\frac{1}{2} \times \frac{3}{4} \times \frac{1}{3} \times 24 = 3\). The area of triangle \(CFG\) is \(\frac{1}{2} \times \frac{1}{2} \times \frac{3}{8} \times 24 = 3\). Hence the area of triangle \(EF\) is \(24 - 6 - 3 - 3 - 4 = 8\). **Answer: 8**

12. Let \(BC\) intersect \(DE\) at \(H\). We have \(\angle BGE = 75^\circ\) and \(75^\circ + \angle E = \angle CHD = 180^\circ - \angle C - \angle D\). Hence \(\angle C + \angle D + \angle E = 105^\circ\). Similarly, \(\angle A + \angle B + \angle F = 105^\circ\), so that the sum of these six angles is \(210^\circ\). **Answer: 210^\circ**

13. By symmetry, \(AP = CP\). By Pythagoras' Theorem, \(CP^2 = AP^2 = AD^2 + PD^2 = 16 + (8 - CP)^2\). This yields \(CP = 5\). Hence the area of triangle \(CAP\) is \(\frac{1}{2} CP \cdot AD = 10\). **Answer: 10**
14. The ratio of the areas of triangles $BPD$ and $CPD$ is 2:3. By symmetry, triangles $CPD$ and $CPE$ have the same area. Hence the ratio of the areas of triangles $PCB$ and $PCE$ is 5:3. Since the area of triangle $ABC$ is $\frac{25}{2}$, the area of triangle $BAP$ is $\frac{25}{2} \times \frac{4}{5} \times \frac{5}{8} = \frac{25}{8}$. Answer: $\frac{25}{8}$

15. Let the extensions of $DM$ and $CB$ meet at $N$, and let the foot of perpendicular from $D$ to $BC$ be $P$. Since $M$ is the midpoint of $AB$, we have $BN = AD = 2$. Also, $CP = \frac{BC - AD}{2} = 3$. Note that triangles $CDP$ and $DNP$ are similar to each other, so that $\frac{CP}{DP} = \frac{DP}{NP}$. This yields $DP = \sqrt{21}$ so that the area of $ABCD$ is $\frac{2 + 8}{2} \sqrt{21} = 5\sqrt{21}$. Answer: $5\sqrt{21}$